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The monoid representation of upho posets and total positivity

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Abstract. In this paper, we establish a bijection between locally finite colored upho posets and left-cancellative invertible-free monoids. This bijection maps $\mathbb{N}$-graded colored upho posets to left-cancellative homogeneous monoids. We use this and the new concept of semi-upho posets to prove that every totally positive power series is the rank-generating function of some upho poset, resolving a conjecture of Gao et al.

Keywords: upho posets, left-cancellative monoids, totally positive, log-concavity

1 Introduction

A poset is called *upper homogeneous*, abbreviated as *upho*, if each principal order filter is isomorphic to the poset itself. This concept was introduced by Richard Stanley during his research on the enumeration properties of Stern's triangle and its poset [11, 12].

The study of Stern's poset, particularly regarding enumeration problems, has been a focal point of numerous research efforts [13, 8]. Upho posets, as a generalization of Stern's poset, preserve the attribute of self-similarity, and hence exhibit many intriguing structural and enumeration properties. For example, in [3], Gao et al. give a concise characterization of $\mathbb{N}$-graded planar upho posets using their rank-generating functions; in [4], Hopkins proves that the characteristic generating functions of upho posets are the inverse of their rank-generating functions. Moreover, the breadth of applications for upho posets spans several domains, including lattice theory [4, 5], commutative algebra [5, Theorem 1.6] [3, Conjecture 1.1], and finite geometry [5, Theorem 1.7].

For simplicity, we say a formal power series is an *upho function* if it is the rank-generating function of some upho poset. A fundamental problem is:

Is there a criterion to determine if a formal power series is an upho function?

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In [3], Gao et al. prove that there are uncountably many different upho functions, and a straightforward corollary is that almost all upho functions are not rational functions. Hence, the complete characterization of upho functions is anticipated to be challenging.

In this paper, we prove the main conjecture proposed by Gao et al. in [3].

Theorem 1 ([3, Conjecture 3.3])**.** *A formal power series $f(x) \in 1 + x\mathbb{Z}_{\geq 0}[[x]]$ is the rank-generating function of an upho poset P whose Ehrenborg quasi-symmetric function is a Schur-positive symmetric function if and only if $f(x)$ is totally positive.*

In Section 3, we introduce the notion of *colored upho posets* and establish the following correspondence, which is explained in detail later.

Theorem 2. *There is a bijection between locally finite colored upho posets and left-cancellative invertible-free monoids. Moreover, this bijection maps $\mathbb{N}$-graded colored upho posets to left-cancellative homogeneous monoids, and maps finite type $\mathbb{N}$-graded colored upho posets to finitely generated left-cancellative homogeneous monoids.*

This correspondence plays an important role in understanding the self-similarity of upho posets: On one hand, we can do concrete calculations on upho posets using monoids from an algebraic perspective; on the other hand, we can get an intuition for the enumeration problems of left-cancellative monoids from a combinatorial perspective.

In Section 4, we explore the relationship between upho functions and totally positive formal power series.

Definition 1 ([7])**.** *A formal power series $\sum_{n=0}^{\infty} a_n x^n$ is* totally positive *if all finite minors of the infinite Toeplitz matrix*

$$\begin{pmatrix} a_0 & 0 & 0 & 0 & \cdots \\ a_0 & a_1 & 0 & 0 & \cdots \\ a_0 & a_1 & a_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

are nonnegative.

Our working definition of totally positive formal power series follows from a special case of the Aissen-Edrei-Schoenberg-Whitney theorem.

Theorem 3 ([7])**.** *A formal power series $f(x) \in 1 + x\mathbb{Z}_{\geq 0}[[x]]$ is totally positive if and only if $f(x)$ is of the form of $\frac{g(x)}{h(x)}$, where $g(x), h(x) \in 1 + x\mathbb{Z}[x]$ such that all the complex roots of $g(x)$ are real and negative, and all the complex roots of $h(x)$ are real and positive.*

By employing Theorem 2 and analyzing the rank-generating function of the newly defined concept of *semi-upho posets*, we prove the following theorem.

Theorem 4. *Let $f(x) \in 1 + x\mathbb{Z}_{\geq 0}[[x]]$ be totally positive. Then $f(x)$ is an upho function.*

And finally, we show that Theorem 1 is a corollary of Theorem 4.

The paper is structured as follows: Section 2 provides necessary background on upho posets and introduces the concept of semi-upho posets. Section 3 and Section 4 are dedicated to the exposition and proof of the aforementioned results.

2 Background on Upho Posets

2.1 Upho Posets

In this paper, we employ the standard terminology used in order theory. For a more detailed exposition, readers are referred to [10, Chapter 3]. In this subsection, we introduce several concepts related to upho posets, along with some illustrative examples.

Definition 2. *A poset P is* upper homogeneous, *abbreviated as* upho, *if for every $s \in P$ we have $V_{P,s} \cong P$, where $V_{P,s} := \{p \in P \mid s \leq_P p\}$ is the* principal order filter *generated by s.*

Note that each principal order filter has a unique minimal element, so every upho poset P has a unique minimal element, denoted $\hat{0}_P$. We abbreviate $V_{P,s}$ to V_s and $\hat{0}_P$ to $\hat{0}$ when the poset P referred to is clear.

A poset P is said to be $\mathbb{N}$-*graded* if P can be written as a disjoint union $P = P_0 \sqcup P_1 \sqcup P_2 \sqcup \dots$ such that every maximal chain has the form $p_0 \lessdot_P p_1 \lessdot_P p_2 \lessdot_P \cdots$, where $p_i \in P_i$ for all $i \in \mathbb{N}$. The *rank function* $\rho : P \to \mathbb{N}$ of P is defined by $\rho(p) = i$ for all $p \in P_i$. We refer to P_i as the *i-th layer* of P.

An $\mathbb{N}$-graded poset P is said to be of *finite type* if $|P_i|$ is finite for all $i \in \mathbb{N}$. The *rank-generating function* of a finite type $\mathbb{N}$-graded poset P is defined to be $F_P(x) := \sum_{k=0}^{\infty} |P_k| \, x^k$.

The *Ehrenborg quasi-symmetric function* [2] of a finite type $\mathbb{N}$-graded poset P is defined to be $E_P := \sum_{n\geq 0} E_{P,n}$, where $E_{P,0} = 1$, and

$$E_{P,n}\left(x_1, x_2, \cdots, x_n\right) := \sum_{\substack{\hat{0}=t_0 \leq_P t_1 \leq_P \cdots \leq_P t_{k-1} <_P t_k \\ \rho(t_k)=n}} x_1^{\rho(t_1)-\rho(t_0)} x_2^{\rho(t_2)-\rho(t_1)} \cdots x_k^{\rho(t_k)-\rho(t_{k-1})},\ n \geq 1.$$

In this paper, we define the following two finiteness conditions.

Definition 3. *A poset P is* Noetherian *if its principal order filters satisfy* ascending chain condition, *that is, for every element $s \in P$, there is no infinite strictly ascending chain $V_s \subseteq V_{s_1} \subseteq V_{s_2} \subseteq \cdots$ (or equivalently, $s >_P s_1 >_P s_2 >_P \cdots$).*

Definition 4. *The* height *of an element s in a poset P is the maximal length of chains in P with s as its maximum. A poset P is* locally finite *if every element in P has finite height.*

All $\mathbb{N}$-graded posets are locally finite, and all locally finite posets are Noetherian. Moreover, in a Noetherian upho poset P, a locally finite upho poset $P' \subseteq P$ can be obtained by selecting all elements of finite height in P, with P' inheriting the order of P.

In a poset P, we define $\mathcal{E}_P := \{(r,s) \mid r,s \in P, r \lessdot s\}$, which corresponds to the edges in the Hasse diagram of P if P is locally finite. If a poset P has a unique minimum $\hat{0}$, we define $\mathcal{A}_P := \{s \in P \mid \hat{0} \lessdot s\}$, and elements in $\mathcal{A}_P$ are called *atoms* of P. In a Noetherian poset, each maximal chain includes exactly one atom. However, both $\mathcal{A}_P$ and $\mathcal{E}_P$ can be empty if P is non-Noetherian. See Example 1.

A formal power series $f(x) \in 1 + x\mathbb{Z}[[x]]$ is said to be an *upho function* if it is the rank-generating function of an upho poset. An important property of upho function is:

Lemma 1 ([3, Lemma 2.3])**.** *Let P and Q be upho posets. Then $P \times Q$ is an upho poset. Furthermore, $F_{P\times Q} = F_P F_Q$, and $E_{P\times Q} = E_P E_Q$.*

We list below some examples of upho posets and upho functions.

Example 1. *Nonnegative real numbers $\mathbb{R}_{\geq 0}$ with usual order is a non-Noetherian upho poset, and there are no atoms in $\mathbb{R}_{\geq 0}$. Let $\mathbb{R}_{\geq 0} \times \mathbb{N}$ be the poset product of $\mathbb{R}_{\geq 0}$ and $\mathbb{N}$, both with usual order. Then $\mathbb{R}_{\geq 0} \times \mathbb{N}$ is also a non-Noetherian upho poset, and $(0,1)$ is its only atom.*

Example 2. *The poset $\mathbb{N} \times \mathbb{N}$ with lexicographical order forms a Noetherian upho poset which is not locally finite. For every $(m,n) \in \mathbb{N} \times \mathbb{N}$, the isomorphism $\tau : V_{(m,n)} \cong \mathbb{N} \times \mathbb{N}$ is given by $\tau(m, n+s) = (0,s), \tau(m+t, s) = (t,s)$ for all $s \in \mathbb{N}, t \in \mathbb{N}_{>0}$.*

Example 3. *The upho poset P_M is defined by the following data: The elements in P_M are those in the monoid $M = \langle x_1, x_2 \mid x_1^3 = x_2 x_1 \rangle$; The partial order of P_M is defined by left divisibility in M, that is, $a \leq_{P_M} b$ if and only if there exists $c \in M$ such that $ac = b$ in M.*

It can be shown that P_M is a locally finite upho poset, yet not $\mathbb{N}$-graded.

Example 4. *The upho poset P_{M_1} and P_{M_2} are defined as follow: P_{M_1} consists of elements in the free monoid M_1 generated by $[0,1] \subseteq \mathbb{R}$, with a partial order defined by left divisibility. Similarly, P_{M_2} consists of elements in the free commutative monoid M_2 generated by $[0,1] \subseteq \mathbb{R}$, with its partial order also defined by left divisibility.*

Both P_{M_1} and P_{M_2} are $\mathbb{N}$-graded upho posets that are not of finite type. The d-th layer of P_{M_1} can be thought of as a cube of dimension d, while the d-th layer of P_{M_2} can be thought of as a simplex of dimension d.

Example 5. *Figure 1 shows the Hasse diagrams of $\mathbb{N}$ (with usual order), full binary tree, Stern's poset, and bowtie poset from left to right. Their rank-generating functions are $\frac{1}{1-x}$, $\frac{1}{1-2x}$, $\frac{1}{(1-x)(1-2x)}$, and $\frac{1+x}{1-x}$, respectively. All of them are $\mathbb{N}$-graded upho posets of finite type.*

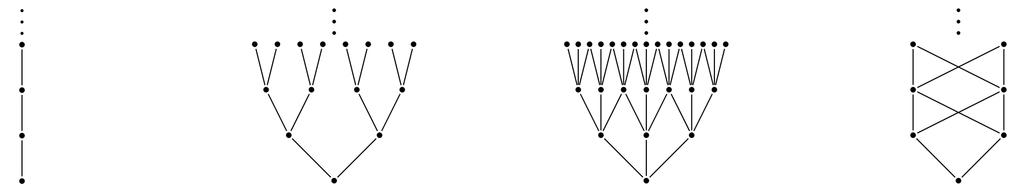

Figure 1: The Hasse diagrams of Example 5.

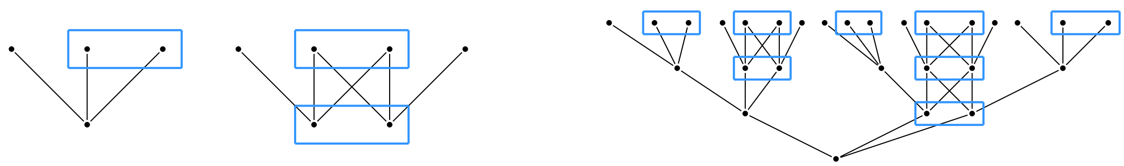

Figure 2: The generating rules and the first several layers of Fei's poset.

Example 6. *Fei's poset $\mathcal{F}$ is the upho poset that satisfies the generating rules depicted in the left two components of Figure 2. This poset is a finite type $\mathbb{N}$-graded upho poset, with its rank-generating function given by $F_{\mathcal{F}}(x) = \frac{1-x}{1-2x-x^2}$. The first three layers of the Hasse diagram of Fei's poset are shown in the rightmost component of Figure 2.*

On one hand, Example 6 shows that not all upho functions are log-concave (since $|\mathcal{F}_1| \cdot |\mathcal{F}_3| = 51 > 49 = |\mathcal{F}_2|^2$). On the other hand, it illustrates potential connections between upper homogeneity and linear recurrence, inspiring our proof of Proposition 4.

2.2 Semi-upho Posets

In this subsection, we define *semi-upho posets* which have partial self-similarity. The motivation for introducing semi-upho posets is to explore formal power series $g(x)$ that, when multiplied by any upho function $f(x)$, yield an upho function. As shown in Lemma 1, upho functions are optimal choices for $g(x)$. Moreover, in Theorem 6, we generalize this result to the rank-generating functions of *tree-like semi-upho posets*.

Given posets P' and P with unique minima $\hat{0}_{P'}$ and $\hat{0}_P$ respectively, an injection $\eta : P' \hookrightarrow P$ is said to be an *induced saturated order embedding*, abbreviated as *isoembedding*, if $\eta(\hat{0}_{P'}) = \hat{0}_P$, and furthermore, for every chain C in P' with given maximum a and minimum b, C is a maximal chain with given maximum a and minimum b if and only if $\eta(C)$ is a maximal chain with maximum $\eta(a)$ and minimum $\eta(b)$.

Definition 5. *A poset S is* semi-upho *if for every $s \in S$, there exists an isoembedding $V_s \hookrightarrow S$.*

Upho posets are semi-upho posets. Moreover, a semi-upho poset can be thought of as an upho poset with some parts cut off. Figure 3 is an example of *tree-like semi-upho posets*, defined as locally finite semi-upho posets whose Hasse diagrams are trees.

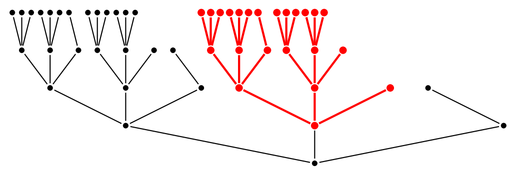

Figure 3: A tree-like semi-upho poset. The red part is a principal order filter that can be isoembedded into the poset itself.

3 Monoid Representation

In this section, we establish a bijection between locally finite *colored upho posets* and *left-cancellative invertible-free monoids*, and associate the bijection to upho posets.

3.1 Colored Upho Posets

The motivation for introducing *colored upho posets* is to designate a unique isomorphism between each given principal order filter and the upho poset itself.

Definition 6. *A colored upho poset $\tilde{P}$ consists of the data (P, col_P): The poset P is an upho poset, and the color mapping $\mathrm{col}_P : \mathcal{E}_P \to \mathcal{A}_P$ satisfies the following conditions:*

- $\mathrm{col}_P(\hat{0}, t) = t$ *for all* $t \in \mathcal{A}_P$*;*
- *For every $s \in P$, there exists an isomorphism $\phi_s : V_s \xrightarrow{\sim} P$ such that $\mathrm{col}_P(u, v) = \mathrm{col}_P(\phi_s(u), \phi_s(v))$ for all $(u, v) \in \mathcal{E}_{V_s}$ (in fact such a ϕ_s is unique for a given $s \in P$).*

Locally finite colored upho posets can be conceptualized as structures in which each edge extending upward from the same vertex in their Hasse diagrams is assigned a distinct color. Moreover, there exists a unique isomorphism between each principal order filter and the poset itself, which maps edges to identically colored ones.

Similarly, we introduce the concept of colored semi-upho posets.

Definition 7. *A colored semi-upho poset $\tilde{S}$ consists of the data (S, col_S): The poset S is a semi-upho poset, and the color mapping $\mathrm{col}_S : \mathcal{E}_S \to \mathcal{A}_S$ satisfies the following conditions:*

- $\mathrm{col}_S(\hat{0}, t) = t$ *for all* $t \in \mathcal{A}_S$*;*
- *For every $s \in S$, there exists an isoembedding $\psi_s : V_s \hookrightarrow S$ such that $\mathrm{col}_S(u, v) = \mathrm{col}_S(\psi_s(u), \psi_s(v))$ for all $(u, v) \in \mathcal{E}_{V_s}$ (in fact such a ψ_s is unique for a given $s \in S$).*

3.2 Correspondence with Monoids

In this subsection, we build the bijection between locally finite colored upho posets and *left-cancellative invertible-free monoids* explicitly. First, we recall some terminology of monoids, and readers may see [6] for a more detailed exposition.

The *identity element* of a monoid M, denoted e, is an element satisfying $ex = xe = x$ for every $x \in M$. A *zero element* of a monoid M, denoted 0, is an element satisfying $0x = x0 = 0$ for every $x \in M$. An element $x \in M$ is said to be *left-invertible* if there exists an element $y \in M$ such that $yx = e$, and *right-invertible* if there exists an element $y \in M$ such that $xy = e$. An element $x \in M$ is said to be *inveritible* if it is both left-invertible and right-invertible. A monoid M is said to be *invertible-free* if it has no left-invertible or right-invertible elements other than e; equivalently, $ab = e$ implies $a = b = e$. An element a of M is said to be *irreducible* if it is not invertible and is not the product of any two non-invertible elements. The set of all irreducible elements of M is denoted $\mathcal{I}_M$. A monoid M is said to be *left-cancellative* if for every $a, x, y \in M$, $ax = ay$ implies $x = y$. An invertible-free monoid M is said to be *homogeneous* if for every $a_1 a_2 \cdots a_n = b_1 b_2 \cdots b_m \in M$, where $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m \in \mathcal{I}_M$, we have $n = m$. We define the *length* of this element to be n, and denote the set of all distinct elements of length n by W_n^M.

We define a mapping $\mathcal{M}$ which maps locally finite colored upho posets to left-cancellative invertible-free monoids (abbreviated as *LCIF monoids*) by the following rule. For a given locally finite colored upho poset $\tilde{P} = (P, \mathrm{col}_P)$, the elements of $\mathcal{M}(\tilde{P})$ are the elements of P. For every $s, t \in P$, we define the multiplication st by $st := \phi_s^{-1}(t)$. It can be verified that such $\mathcal{M}(\tilde{P})$ is a well-defined LCIF monoid, and $\mathcal{A}_{\mathcal{P}} = \mathcal{I}_{\mathcal{M}(\tilde{P})}$.

Conversely, we define a mapping $\tilde{\mathcal{P}} = (\mathcal{P}, \mathcal{C})$ which maps LCIF monoids to locally finite colored upho posets by the following rule. The elements of $\mathcal{P}(M)$ are the elements of M. The partial order $\leq_P$ in $\mathcal{P}(M)$ is defined by the left divisibility in M, as is explained in Example 3. Then we have for every $a, b \in M$, $a \lessdot_P b$ if and only if there exists $c \in \mathcal{I}_M$ such that $ac = b$. So the unique minimum in $\mathcal{P}(M)$ is e, and $\mathcal{A}_{\mathcal{P}(M)} = \mathcal{I}_M$. For every ordered pair $(a, b) \in \mathcal{E}_{\mathcal{P}(M)}$, define $\mathcal{C}(M)(a, b)$ to be $c \in \mathcal{I}_M$ such that $ac = b$ (such c is unique by the left cancellative property of M). It can be verified that such $\tilde{\mathcal{P}}(\mathcal{M}) := (\mathcal{P}(M), \mathcal{C}(M))$ is a well-defined locally finite colored upho posets.

Furthermore, $\tilde{\mathcal{P}}(\mathcal{M}(\tilde{P})) \cong \tilde{P}$, and $\mathcal{M}(\tilde{\mathcal{P}}(M)) \cong M$. Hence we have:

Theorem 5. *The mutually inverse mappings $\mathcal{M}$ and $\tilde{\mathcal{P}}$ give a bijection between locally finite colored upho posets and left-cancellative invertible-free monoids.*

Restrict to $\mathbb{N}$-graded colored upho posets, we have:

Corollary 1. *The mutually inverse maps $\mathcal{M}$ and $\tilde{\mathcal{P}}$ give a bijection between $\mathbb{N}$-graded colored upho posets and left-cancellative homogeneous monoids (abbreviated as* LCH monoids*). Moreover, finite type $\mathbb{N}$-graded colored upho posets correspond to finitely generated LCH monoids.*

Furthermore, the bijection can be generalized to semi-upho posets.

An *LCIF 0-monoid* is defined to be an LCIF monoid with an additional zero element 0 and some relations $X_i = 0$, where X_i is an element of the original LCIF monoid. We denote these relations $X_i = 0$ as *0-defining relations*. Similarly, an *LCH 0-monoid* is defined to be an LCH monoid added by a zero element 0 and some 0-defining relations. It is

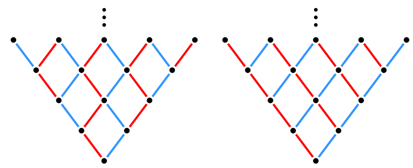

Figure 4: The forgetful mapping is not injective.

worth mentioning that, in general, LCIF 0-monoids are not left-cancellative, and LCH 0-monoids are neither left-cancellative nor homogeneous. The mutually inverse mappings $\mathcal{M}_0$ and $\tilde{\mathcal{S}}$ between locally finite colored semi-upho posets and LCIF 0-monoids are defined similarly to $\mathcal{M}$ and $\tilde{\mathcal{P}}$, respectively. The only difference is that the elements in the posets now correspond to *non-zero* elements in the monoids.

Corollary 2. *The mutually inverse mappings $\mathcal{M}_0$ and $\tilde{\mathcal{S}}$ give a bijection between locally finite colored semi-upho posets and LCIF 0-monoids. Moreover, $\mathbb{N}$-graded colored semi-upho posets correspond to LCH 0-monoids, and finite type $\mathbb{N}$-graded colored semi-upho posets correspond to finitely generated LCH 0-monoids.*

3.3 The Forgetful Mapping and Regularity of Upho Posets

Through the *forgetful mapping*, we establish an association between monoids and locally finite colored upho posets with locally finite upho posets.

The *forgetful mapping* $\mathfrak{F}$ maps a locally finite colored semi-upho poset $\tilde{S} = (S, \mathrm{col}_S)$ to S. This mapping is well-defined on upho posets since upho posets are semi-upho. It can be conceptualized as forgetting the colors in the colored semi-upho posets.

Figure 4 demonstrate that $\mathfrak{F}$ is not injective. Moreover, readers could verify that $\langle x_1, x_2 \mid x_1^2 = x_2^2 \rangle$ is the LCH monoid corresponding to the left colored upho poset, while $\langle x_1, x_2 \mid x_1 x_2 = x_2 x_1 \rangle$ corresponds to the right. These monoids are not isomorphic.

Another point of inquiry is the surjectivity of the forgetful mapping, which remains an open question.

Now we define *regular semi-upho posets* to be the semi-upho posets in $\mathbf{im}\mathfrak{F}$, and *regular upho posets* to be the upho posets in $\mathbf{im}\mathfrak{F}$. Moreover, an upho function is said to be *regular upho* if it is the rank-generating function of a regular upho poset.

By the correspondence established in Section 3.2, on one hand, the properties of upho posets can be used to address enumeration problems in left-cancellative monoids. For instance, in an upho poset P, it is straightforward to show that $|P_k| = |P_{k+1}|$ implies $|P_n| = |P_k|$ for all $n \geq k$. Converting this fact into monoids, we then prove that in an LCH monoid M, $|W_k^M| = |W_{k+1}^M|$ implies $|W_n^M| = |W_k^M|$ for all $n \geq k$.

On the other hand, we can use monoids to construct a variety of well-defined upho posets and semi-upho posets, as illustrated in Example 3 and Example 4. Furthermore, our proof of Theorem 4 in Section 4 is entirely based on this method.

4 Totally Positive Upho Functions

In this section, we use Theorem 3 to characterize total positivity, and we split our proof of Theorem 4 into three parts. In Section 4.1, we address the case where the numerator is 1 and the denominator has two roots that are not less than 1. In Section 4.2, we address the case where the numerator is 1 and the denominator has only one root not less than 1. In Section 4.3, we prove the remaining parts of Theorem 4.

4.1 Type I Unitary Totally Positive Functions

We first give a simple yet useful recast of semi-upho posets.

Lemma 2. *A poset P is semi-upho if and only if for every atom $s \in \mathcal{A}_P$, there exists an isoembedding $V_s \hookrightarrow P$.*

In the following text, we say a formal power series $g(x) = \sum b_i x^i$ is *log-concave* if the coefficient sequence $\{b_i\}_{i=0}^{\infty}$ consists solely of nonnegative numbers and contains no internal zeros, moreover, $b_{i+1}^2 \geq b_i b_{i+2}$ for every $i \in \mathbb{N}$. Using Lemma 2, we have:

Proposition 1. *Let $g(x) \in 1 + x\mathbb{Z}_{\geq 0}[[x]]$ be a log-concave formal power series, then there exists a tree-like semi-upho poset Q whose rank-generating function equals $g(x)$.*

Sketch of proof. Given a log-concave formal power series $g(x) = \sum_{i=0}^{\infty} b_i x^i$, where $b_0 = 1$, we construct the semi-upho poset layer by layer. We construct the $(k+1)$-th layer from the first k layers by first constructing a canonical maximum case, and then deleting points from right to left. It can be shown that $b_{k+1}^{max} \geq \frac{b_k^2}{b_{k-1}} \geq b_{k+1}$. □

Example 7. *Given $g(x) = 1 + 3x^2 + 7x^3 + 13x^3 + b_4^{max} x^4$, Figure 3 is the Hasse diagram of the poset we construct in the procedure above.*

We further use monoids to show that multiplying a regular upho function by the rank-generating function of a tree-like semi-upho poset yields a regular upho function.

Proposition 2. *Let $M_1 = \langle \mathcal{I}_{M_1} \mid R_{M_1} \rangle$ be a finitely generated LCH monoid. Let $M_2 = \langle \{0\} \cup \mathcal{I}_{M_2} \mid R_{M_2} \rangle$ be a finitely generated LCH 0-monoid, where R_{M_2} only has 0-defining relations. Define $M := \langle \mathcal{I}_{M_1} \cup \mathcal{I}_{M_2} \mid R_M \rangle$, where R_M consists of the following relations:*

- *$R_{M_1} \subseteq R_M$;*
- *If $y_i \in \mathcal{I}_{M_2}, Y_j \in M_2$, and $y_i Y_j = 0$ is in R_{M_2}, then $y_i Y_j = x_1 Y_j$ is in R_M;*
- *$y_i x_j = x_1 x_j \in R_M$ for all elements $x_j \in \mathcal{I}_{M_1}, y_i \in \mathcal{I}_{M_2}$.*

Then M is a finitely generated LCH monoid. Moreover, $F_{\mathcal{P}(M)} = F_{\mathcal{P}(M_1)} F_{\mathcal{P}(M_2)}$.

Figure 5 depicts how to "convolve" a tree-like semi-upho poset and an upho poset. Rewriting Proposition 2 in terms of formal power series yields the following result:

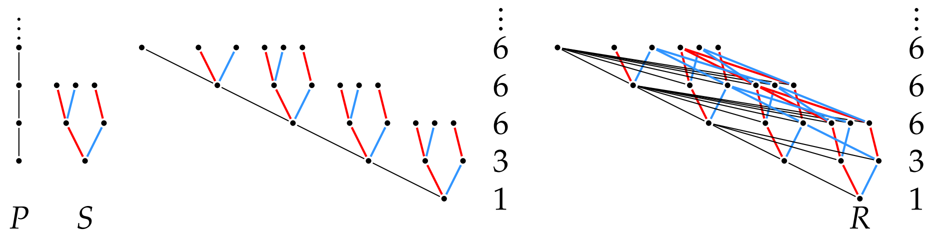

Figure 5: Construction of an upho poset R by "convolving" a regular upho poset P and a tree-like semi-upho poset S.

Theorem 6. *Let $f(x) \in 1 + x\mathbb{Z}_{\geq 0}[[x]]$ be regular upho and $g(x) \in 1 + x\mathbb{Z}_{\geq 0}[[x]]$ be the rank-generating function of a tree-like semi-upho poset, then $f(x)g(x)$ is regular upho.*

By employing Proposition 1 and Theorem 6, we obtain the following corollary:

Corollary 3. *Let $f(x) \in 1 + x\mathbb{Z}_{\geq 0}[[x]]$ be regular upho and $g(x) \in 1 + x\mathbb{Z}_{\geq 0}[[x]]$ be log-concave, then $f(x)g(x)$ is regular upho.*

Notice that multiplication preserves log-concavity [9, Proposition 2], hence we have:

Proposition 3. *If a formal power series $f(x) \in 1 + x\mathbb{Z}_{\geq 0}[[x]]$ is of type I, that is:*

$$f(x) = \frac{1}{h(x)} = \prod_{i=1}^{n} \frac{1}{(1 - \lambda_i x)},\ 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n,$$

where $h(x)$ is irreducible over $\mathbb{Z}[X]$, $\deg h(x) \geq 2$, and $1 \leq \lambda_{n-1} \leq \lambda_n$, then $f(x)$ is a regular upho function.

Proof. Note that $\frac{1}{1-\lambda_i x}$ for $1 \leq i \leq n-2$ and $\frac{1-x}{(1-\lambda_{n-1}x)(1-\lambda_n x)}$ are log-concave, and $\frac{1}{1-x}$ is a regular upho function, so by Lemma 1 and Corollary 3, $f(x)$ is an upho function. □

4.2 Type II Unitary Totally Positive Functions

In this subsection, we consider formal power series $f(x) \in 1 + x\mathbb{Z}_{\geq 0}[[x]]$ of type II:

$$f(x) = \frac{1}{h(x)} = \frac{1}{1 + \sum_{i=1}^{\infty} h_i x^i} = \prod_{i=1}^{n} \frac{1}{(1 - \lambda_i x)} = 1 + \sum_{i=1}^{\infty} c_i x^i,\ 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n,$$

where $h(x)$ is irreducible over $\mathbb{Z}[X]$, $\deg h(x) \geq 2$, and λ_{n-1}, λ_n satisfies $\lambda_{n-1} < 1 \leq \lambda_n$. We prove that $f(x)$ of type II is regular upho by explicitly constructing an LCH monoid whose rank-generating function equals $f(x)$. A technical lemma we use is:

Lemma 3. *There exist $l_i \in \mathbb{Z}$ for $1 \leq i \leq n$ with $l_1 \geq l_2 \geq \cdots l_n \geq 0$ such that*

$$c_i = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 & \cdots & l_n \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}^i \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Example 8. *For $n = 5$, we have $l_1 = h_1 - 4$, $l_2 = h_1 - h_5 - 4$, $l_3 = h_1 - h_4 + 3h_5 - 3$, $l_4 = h_1 - h_3 + 2h_4 - 3h_5 - 2$ and $l_5 = h_1 - h_2 + h_3 - h_4 + h_5 - 1$.*

Proposition 4. *A formal power series $f(x)$ of type II is a regular upho function.*

Sketch of proof. Let M be a monoid generated by $x_i^j, 1 \leq j \leq n, 1 \leq i \leq r_j$, where $r_1 = l_1$ and $r_i = 1$ for $i > 1$. And we let the defining relations of M be:

$$x_1^j x_k^1 = x_1^1 x_k^1, \quad 2 \leq j \leq n, 1 \leq k \leq l_1 - l_j;$$
$$x_1^j x_1^i = x_1^1 x_1^i, \quad 2 \leq i < j \leq n.$$

By Proposition 2 and Lemma 3, we then prove that M is a well-defined LCH monoid and it rank-generating function equals $f(x)$. □

4.3 Total Positivity Implies Upho

In this subsection, we use the results obtained earlier to prove Theorem 4 and Theorem 1.

We first divide the totally positive functions into the "denominator part" and the "numerator part" using Theorem 3. The result on the "denominator part" can be obtained directly from combining Lemma 1, Proposition 3, Proposition 4, and the fact that $\frac{1}{1-nx}$ is a regular upho function for all $n \in \mathbb{N}$.

Proposition 5. *If a formal power series $f(x) \in 1 + x\mathbb{Z}_{\geq 0}[[x]]$ is of the form $f(x) = \frac{1}{h(x)}$, where all the complex roots of $h(x) \in 1 + x\mathbb{Z}_{\geq 0}[x]$ are real and positive, then $f(x)$ is regular upho.*

To deal with the "numerator part", we just need to use the lemma below.

Lemma 4. *$g(x) \in 1 + x\mathbb{Z}[x]$ is log-concave if all its complex roots are real and negative.*

Proof of Theorem 4. Combine Corollary 3, Theorem 3, Proposition 5, and Lemma 4. □

Before proving Theorem 1, we state the Thoma–Kerov-Vershik theorem as follows.

Theorem 7 ([1]). *A formal power series $f(x) \in 1 + x\mathbb{Z}_{\geq 0}[[x]]$ is totally positive if and only if $f(t_1)f(t_2)\cdots$ is Schur positive.*

Proof of Theorem 1. For a finite type $\mathbb{N}$-graded upho poset P, according to [3, Lemma 2.2], the Ehrenborg quasi-symmetric function $E_P(x_1, x_2, \dots) = \prod_{i=1}^{\infty} F_P(x_i)$. Combining Theorem 4 and Theorem 7, the proof of Theorem 1 is completed. □

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