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Charmed roots and the Kroweras complement

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Abstract. Although both noncrossing partitions and nonnesting partitions are uniformly enumerated for Weyl groups, the exact relationship between these two sets of combinatorial objects remains frustratingly mysterious. In this abstract, we give a precise combinatorial answer in the case of the symmetric group using a new definition of *charmed roots*.

Résumé. Les partitions non-croisées et les partitions non-emboîtées soient uniformément énumérées pour les groupes de Weyl, la relation exacte entre ces deux ensembles d'objets combinatoires reste frustrante. Dans cet abstrait, nous donnons une réponse combinatoire précise dans le cas du groupe symétrique en utilisant une nouvelle définition de *racines charmées*.

Keywords: Catalan combinatorics, noncrossing, nonnesting, Kreweras complement

1 Introduction

1.1 Noncrossing and nonnesting partitions

Let $W \subseteq \mathrm{GL}(V)$ be a finite complex reflection group acting in its reflection representation on a complex vector space V of dimension r with reflections T [10, 13]. Our results will mostly concern the symmetric group $W = S_n$, where the set of reflections is the set of all transpositions (i, j). The ring of W-invariants $\mathbb{C}[V]^W$ is a polynomial ring generated by invariants of degrees $d_1 \leq d_2 \leq \cdots \leq d_r$. The *Coxeter number* of a well-generated W (that is, W is generated by r reflections) is $h = d_r$ and the *W-Catalan number* is

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$$\mathrm{Cat}(W) := \prod_{i=1}^{r} \frac{h+d_i}{d_i}. \tag{1.1}$$

The *c-noncrossing partition lattice* $\mathrm{NC}(W,c)$ is the interval $[e,c]_T$ in the absolute order. For S_n, the absolute length of a permutation $w \in S_n$ is n minus the number of cycles of w; the cycles of $w \in \mathrm{NC}(W,c)$ are the blocks of a noncrossing partition. Until very recently, the number of noncrossing partitions had only been computed case-by-case; a uniform proof was found in the case of real W in [8]. For W a well-generated finite complex reflection group with Coxeter element c, $|\mathrm{NC}(W,c)| = \mathrm{Cat}(W)$.

Let now W be a Weyl group (a crystallographic real reflection group), with positive roots Φ^+. In this abstract, we will systematically replace positive roots by their corresponding reflections. The positive root poset is the partial order on Φ^+ defined by $\alpha \leq \beta$ iff $\beta - \alpha$ is a nonnegative sum of positive roots; for S_n, Φ^+ is the partial order defined on the transpositions (i,j) with covering relations $(i+1,j) \lessdot (i,j) \lessdot (i,j+1)$ (see Figure 1 for an example). The *nonnesting partitions* $\mathrm{NN}(W)$ are the order ideals in the positive root poset [16, Remark 2]. There are uniform proofs that $|\mathrm{NN}(W)| = \mathrm{Cat}(W)$.

Despite the fact that they are both counted by $\mathrm{Cat}(W)$, there are at least two Incongruities between $\mathrm{NC}(W,c)$ and $\mathrm{NN}(W)$:

1. $\mathrm{NC}(W,c)$ is defined for well-generated complex reflection groups, while $\mathrm{NN}(W)$ is only defined for Weyl groups;

2. the definition of $\mathrm{NC}(W,c)$ requires the choice of a Coxeter element, while $\mathrm{NN}(W)$ has no such dependence;

The exact relationship between noncrossing and nonnesting partitions remains frustratingly mysterious, and finding a uniform "natural" bijection is perhaps the biggest open question in Coxeter–Catalan combinatorics. To our taste, there are two approaches to this problem: the first approach is based on the case-by-case combinatorial models available in the classical types A, B, D [9, 19, 6, 11, 18, 3]; the second approach was pioneered in [2] based on observations in [14, 4], and uses a mysterious coincidence of two cyclic actions to induce a bijection. Our main theorem refines both of these approaches in the special case of the symmetric group S_n.

1.2 Cyclic actions

For W a well-generated complex reflection group and c a Coxeter element, the *c-Kreweras complement* on the noncrossing partition lattice is the anti-automorphism of $\mathrm{NC}(W,c)$ defined by $\mathrm{Krew}_c(\pi) = \pi^{-1}c$ [1, Section 4.2], [12]. Since $\mathrm{Krew}_c^2(\pi) = c^{-1}\pi c$ and c has order h, Krew_c has order h if $-1 \in W$, and $2h$ otherwise.

For W a Weyl group, *rowmotion* on nonnesting partitions is the map $\mathrm{Row}(p) = \min_{\Phi^+}\{\alpha \mid \alpha \not\leq \beta \text{ for any } \beta \in p\}$. Panyushev conjectured that the order of Row on $\mathrm{NN}(W)$ was h if $-1 \in W$ and $2h$ otherwise [14], and Bessis and Reiner refined Panyushev's conjecture by observing that Row had the same orbit structure on $\mathrm{NN}(W)$ as Krew on $\mathrm{NC}(W,c)$ [4]. This was proven by Armstrong, Stump, and Thomas [2].

But Incongruity (2) remains—while the definitions of $\mathrm{NC}(W,c)$ and Krew_c depend on the choice of a Coxeter element, the set $\mathrm{NN}(W)$ and its action Row do not depend on any such choice. We address this lack of dependence on the Coxeter element c by modifying the definition of Row [20, 21]. It is well-known that rowmotion can be written as a sequence of local moves as follows [5, 17]. A *toggle* $\mathrm{tog}_\alpha(p)$ of a nonnesting partition p at a positive root α either adds α to p (when $\alpha \notin p$) or removes α from p (when $\alpha \in p$), provided that the result is again a nonnesting partition. For nonnesting partitions, Row can be computed by toggling each root of the root poset in order of height (or by *row*). It is natural to modify the order of these toggles.

For c a standard Coxeter element and c a particular choice of reduced word for c, the *c-sorting word* for the long element $w_\circ$ is the leftmost reduced word in simple reflections for $w_\circ$ in c^∞. Write $\mathsf{w}_\circ(\mathsf{c}) = [r_1, r_2, \ldots, r_N]$, with each $r_i \in S$ and define the inversion sequence $\mathrm{inv}(\mathsf{w}_\circ(\mathsf{c})) = [t_1, t_2, \ldots, t_N]$, where $t_a := (r_1 r_2 \cdots r_{a-1}) r_a (r_1 r_2 \cdots r_{a-1})^{-1}$. Then $\mathrm{inv}(\mathsf{w}_\circ(\mathsf{c}))$ totally orders the reflections of W.

We can now address Incongruity (2) by defining a modification of rowmotion to accomodate a Coxeter element c:

$$\begin{aligned} \mathrm{Krow}_c : \mathrm{NN}(W) &\to \mathrm{NN}(W) \\ p &\mapsto \left(\mathrm{tog}_{t_N} \circ \cdots \circ \mathrm{tog}_{t_2} \circ \mathrm{tog}_{t_1}\right)(p). \end{aligned} \tag{1.2}$$

We call this map the *c-Kroweras complement*.

1.3 Main Theorem

Our main theorem uses the c-Kreweras and c-Kroweras complements to relate noncrossing and nonnesting partitions. Recall that the *support* of a noncrossing partition $\pi \in \mathrm{NC}(W,c)$ is the set $\mathrm{Supp}(\pi)$ of simple reflections required to write a reduced word in simple reflections for π; similarly, the *support* of a nonnesting partition $p \in \mathrm{NN}(W)$ is the set $\mathrm{Supp}(p)$ of simple roots that lie in p (as an order ideal of Φ^+).

Theorem 1. *Let S_n be the symmetric group, and fix a standard Coxeter element $c \in S_n$. Then there is a unique bijection $\mathrm{Charm}_c\colon \mathrm{NC}(S_n,c) \to \mathrm{NN}(S_n)$ satisfying $\mathrm{Charm}_c \circ \mathrm{Krew}_c = \mathrm{Krow}_c \circ \mathrm{Charm}_c$ and $\mathrm{Supp} = \mathrm{Supp} \circ \mathrm{Charm}_c$.*

In particular, for any standard Coxeter element $c \in S_n$, the order of Krow_c on $\mathrm{NN}(W)$ is $2h$. The statement of the main theorem in Armstrong-Stump-Thomas [2] can be obtained from the statement of our Theorem 1 by replacing Krow_c by Row, and replacing the symmetric group by any finite Weyl group; the main difference from the result in [2] is that we resolve Incongruity (2), constructing truly different bijections between noncrossing and nonnesting partitions for each Coxeter element.

2 Coxeter elements and charmed roots

Recall that a *(standard) Coxeter element* c is a product of the simple reflections in any order. To ease notation, we reserve the symbols r_i to refer to a fixed ordering $\mathsf{c} := [r_1, r_2, \ldots, r_{n-1}]$ of S and define $c := r_1 r_2 \cdots r_{n-1}$. We also reserve the symbol $s_k = r_1$ for the first simple reflection in the chosen reduced word, and write $c' = r_2 \cdots r_{n-1} r_1$.

It is easy to show that the cycle notation of any Coxeter element in the symmetric group has a particularly simple form: $c \in S_n$ consists of a single cycle with an initial increasing subsequence starting at 1 and ending at n, followed by a decreasing sequence of the remaining unused entries. Let c be a Coxeter element with cycle notation $(w_1, w_2, \ldots, w_m, w_{m+1} \ldots, w_n)$, where $1 = w_1 < w_2 < \cdots < w_m = n$ and $n = w_m > w_{m+1} > \cdots > w_n > w_1 = 1$. Write

$$L_c := \{w_2, \ldots, w_{m-1}\} \text{ and } R_c := \{w_{m+1}, \ldots, w_n\}.$$

Definition 1. *For $1 < i < j < n$, we say that a root (i, j) is c-charmed if $i \in L_c$ and $j \in R_c$ or if $i \in R_c$ and $j \in L_c$ and c-ordinary otherwise. We write $\heartsuit_c$ for the set of c-charmed roots.*

In figures, we depict c-charmed roots with a ♥ and ordinary roots by a circle. The root poset of type A_8 is illustrated in Figure 1.

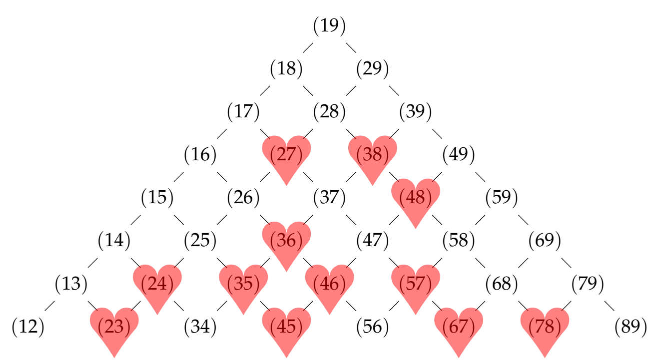

Figure 1: The Hasse diagram of the positive root poset Φ^+ of type A_8. For $c = s_2 s_1 s_3 s_6 s_5 s_4 s_8 s_7$, the c-charmed roots from Definition 1 are marked using hearts.

Example 1. *Consider the Coxeter element* $c = s_2s_1s_3s_6s_5s_4s_8s_7$ *in* S_9*. The element* c *has cycle notation* $(1,3,4,7,9,8,6,5,2)$*, so that*

$$L_c = \{3,4,7\} \text{ and } R_c = \{8,6,5,2\}.$$

We visualize the cycle notation of c by drawing it as points labeled $w_1, w_2, \ldots, w_n$ counter-clockwise around a circle. We visualize a root (i,j) by connecting the vertices labeled i and j by a line segment. For any $a,b,c,d \in [n]$, we say that (a,b) *crosses* (c,d) if and only if (a,b) and (c,d) are crossing in their interior (note that (a,b) does not cross itself). For $1 < i < j < n$, it is easy to check that a root (i,j) is c-charmed if and only if (i,j) crosses $(i-1,j+1)$. Figure 2 illustrates this visualization.

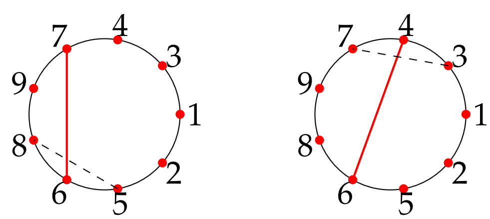

Figure 2: The visualization of the cycle notation $(1,3,4,7,9,8,6,5,2)$ of $c = s_2s_1s_3s_6s_5s_4s_8s_7$. *Left:* the initial c-charmed simple root $(6,7)$ (in red) intersecting the root $(5,8) = (6-1,7+1)$ (dashed). *Right:* the c-charmed root $(4,6)$ (in red) intersecting the root $(3,7) = (4-1,6+1)$ (dashed).

3 Charmed bijections

In this section, we define a general family of *charmed bijections* between *balanced pairs* of subsets and nonnesting partitions. Our charmed bijections depend on a choice of decoration of the roots in $\Phi^+ = \Phi^+(A_{n-1})$, and use certain *intimate* families of lattice paths as intermediate objects. Specializing to the c-charmed roots coming from a Coxeter element c, we obtain our *c-charmed bijections* between $\mathrm{NC}(S_n, c)$ and $\mathrm{NN}(S_n)$.

3.1 Balanced pairs and noncrossing partitions

Definition 2. *Say that a pair of sets* (O, I) *with* $O, I \subseteq [n]$ *is* balanced *if* $|O| = |I|$ *and* $|O \cap [k]| \geq |I \cap [k]|$ *for all* $1 \leq k \leq n$*. Write* $\mathrm{Bal}(n)$ *for all balanced pairs of subsets of* $[n]$*.*

We first show that balanced pairs are naturally in bijection with c-noncrossing partitions. Let $\pi \in \mathrm{NC}(S_n, c)$. We define $O(\pi)$ to be the set of integers i for which there exists an $j > i$ in the same block as i, and we define the set $I(\pi)$ to be the set of integers j such that there exists an $i < j$ in the same block as j. It is immediate from the definition that $(O, I) \in \mathrm{Bal}(n)$.

Proposition 1. *The map*[1] $\pi \mapsto (O(\pi), I(\pi))$ *is a bijection between* $\mathrm{NC}(S_n, c)$ *and* $\mathrm{Bal}(n)$.

Proof. We construct its inverse. For a given pair $(O, I) \in \mathrm{Bal}(n)$, we can construct a $\pi \in \mathrm{NC}(S_n, c)$ with $O(\pi) = O$ and $I(\pi) = I$ as follows. The *closed singletons* of π are the integers that are neither in O nor I. We place each integer in $O \setminus I$ in its own block and call these blocks *open*. Then we add iteratively the integers in I to the open blocks, starting with the smallest integer, such that the intermediate partition is always noncrossing. This is achieved by adding an integer x to the first open block we visit when walking from x towards n via 1 in the cycle notation of c. If an integer in $I \setminus O$ is added to a block we call this block *closed* and thereafter do not add any integers to it. By construction we have $O(\pi) = O$ and $I(\pi) = I$. □

The bijection of Proposition 1 is illustrated in Figure 3.

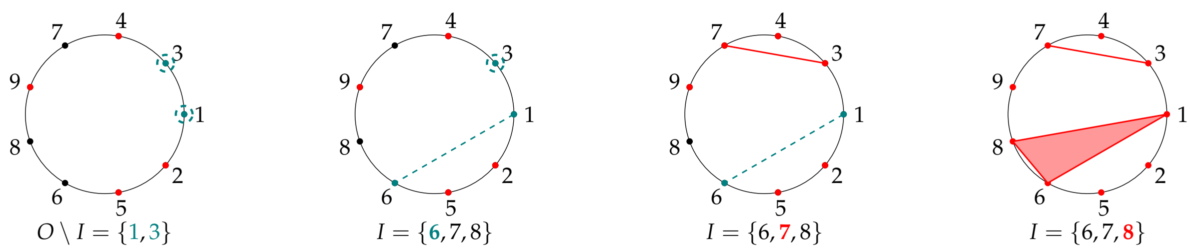

Figure 3: The construction of a noncrossing partition in $\mathrm{NC}(S_9, c)$ for $c = s_2 s_1 s_3 s_6 s_5 s_4 s_8 s_7 = (1\,3\,4\,7\,9\,8\,6\,5\,2)$ starting with the outgoing set $O = \{1, 3, 6\}$ and incoming set $I = \{6, 7, 8\}$. We depict open blocks in dashed teal, and closed blocks in solid red; we circle open and closed singletons using the same color code.

3.2 Intimate families

We draw the root poset for S_n by placing the root (i, j) in the plane with coordinates $((i + j - 1)/2, (j - i)/2)$ and—since the label (i, j) is implied by the position—we may omit the labels on the roots. For $1 \leq i \leq n$, we draw n additional points labeled by i at coordinates $(i - 1/2, 0)$ and call these extra points *integral vertices*. For $\heartsuit \subseteq \Phi^+$, we call a root (i, j) *charmed* if $(i, j) \in \heartsuit$ and *ordinary* otherwise. We depict charmed roots using

[1] To be precise, both O and I depend on c, however since c will always be clear from the context we omit it in the notation.

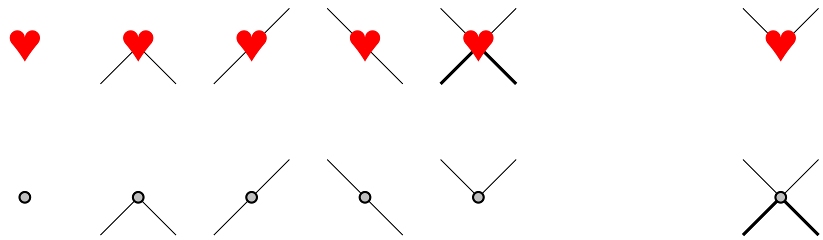

Figure 4: *Left:* the five allowed local configurations for charmed families. *Right:* the two local configurations are forbidden for charmed families.

hearts ♥ and ordinary roots using circles—an example of c-charmed roots is illustrated in Figure 1.

A *path* is a lattice path with step set $\{(1/2, 1/2), (1/2, -1/2)\}$ that starts and ends at an integral vertex and stays strictly above the x-axis. We call $(1/2, 1/2)$-steps *up*, and $(1/2, -1/2)$-steps *down*; a *peak* (resp. *valley*) of a path is a root contained in an up step to its left (resp. right) and a down step to its right (resp. left). Two paths are *kissing* if they do not cross or share edges—they may *meet* at a vertex, where they are said to *kiss*. (Note, though, that kisses are not required for two paths to be kissing.) A family of paths is *kissing* if they are pairwise kissing. A path *feints* at a root (i, j) if (i, j) is a valley, but the path does not kiss any path at (i, j). These definitions are illustrated on the right of Figure 4: the top configuration is a feint at a charmed root, while the bottom one is a kiss at an ordinary root.

For $♥ \subseteq \Phi^+$, a family $\mathcal{L}$ of kissing paths is called *♥-charmed* if:

- paths only kiss at charmed roots and
- paths only feint at ordinary roots.

In other words, a family of paths is charmed if it avoids the two local configurations shown on the right of Figure 4.

Definition 3. *A family $\mathcal{L}$ of ♥-charmed kissing paths is called* ♥-intimate *if:*

- *every ordinary root either lies above all paths in $\mathcal{L}$ or is contained in some path in $\mathcal{L}$ and*
- *no path contains a root above a charmed peak of a path in $\mathcal{L}$, unless that charmed peak is the location of a kiss.*

3.3 Balanced pairs and intimate families

We now relate balanced pairs and intimate families of paths. For a family $\mathcal{L}$ of paths, we call an integral vertex on the x-axis *outgoing* if it is incident to an up step and *incoming* if it is incident to a down step (an integral vertex can be both outgoing and incoming). Denote by $\mathrm{Out}(\mathcal{L})$ (resp. $\mathrm{In}(\mathcal{L})$) the set of labels of outgoing (resp. incoming) vertices of $\mathcal{L}$. It is clear that $(\mathrm{Out}(\mathcal{L}), \mathrm{In}(\mathcal{L}))$ is balanced.

Lemma 1. *Let* $\heartsuit \subseteq \Phi^+$ *and let* $(O, I) \in \mathrm{Bal}(n)$*. Then there is a unique* $\heartsuit$*-inimate family* $\mathcal{L}_{(O,I)}$ *with* $\mathrm{Out}(\mathcal{L}_{(O,I)}) = O$ *and* $\mathrm{In}(\mathcal{L}_{(O,I)}) = I$*.*

Proof. We first construct a well-formed word of parentheses from the subsets O and I. For i from 1 to n, write:

- a $(_i$ parenthesis if $i \in O \setminus I$,
- a $)_i$ parenthesis if $i \in I \setminus O$, and
- $)_i(_i$ parentheses if $i \in O \cap I$.

We now construct an $\heartsuit$-intimate family $\mathcal{L}$ recursively, starting with the empty family of paths. At each step, we pick neighbouring parentheses of the form $(_i)_j$, delete them, and add the path P that starts at i and ends at j that takes a down step whenever possible without violating the condition that the family is charmed, and an up step otherwise. Then $\mathcal{L} \cup \{P\}$ is intimate:

- If there were an ordinary root below $\mathcal{L} \cup \{P\}$ that wasn't part of a path, then that root would lie between p and $\mathcal{L}$, since $\mathcal{L}$ was intimate. But then P took an up step instead of a possible down step, contradicting the definition of P.
- If a previously constructed path P' in $\mathcal{L}$ started at an integral vertex after i, ended before j, and had a charmed peak which is not the location of a kiss, then our new path P will kiss P' at that charmed peak.

The order of choosing two neighbouring parentheses is irrelevant. The family produced is unique, since if at any point a path uses a step different from those prescribed by the algorithm above, then the resulting family of paths will be non-intimate. This non-intimacy will persist, regardless of how the family is extended. □

An example of the algorithm used in the proof of Lemma 1 is given in Figure 5.

Let $\mathcal{L}$ be an $\heartsuit$-intimate family of paths. We define the order ideal $J(\mathcal{L})$ of $\mathcal{L}$ to be the set of all roots (i, j) which lie on or below a path in $\mathcal{L}$. It is clear that $J(\mathcal{L})$ is an order ideal and hence is in $\mathrm{NN}(S_n)$.

Lemma 2. *Let* $\heartsuit \subseteq \Phi^+$ *and* $J \in \mathrm{NN}(S_n)$*. Then there exists a unique* $\heartsuit$*-intimate family* $\mathcal{L}_J$ *with* $J(\mathcal{L}_J) = J$*.*

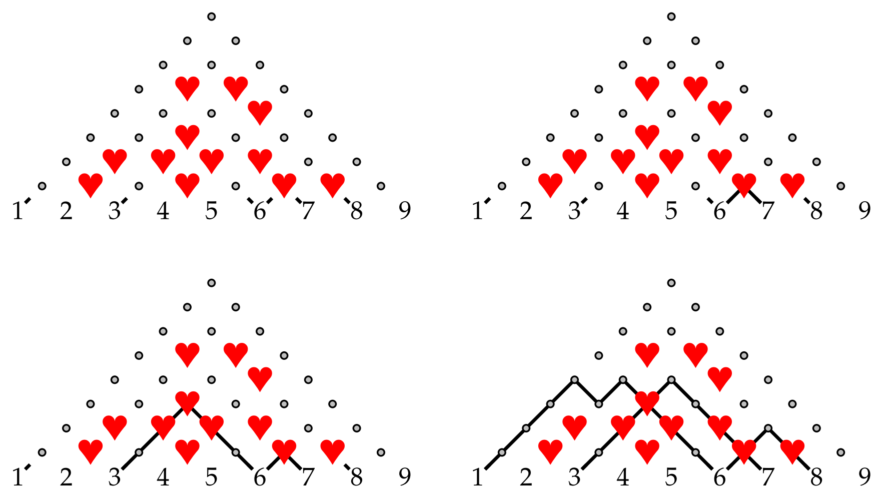

Figure 5: The construction of an intimate family of paths with outgoing set $\{1,3,6\}$ and incoming set $\{6,7,8\}$. The corresponding word of parentheses is $(_1(_3)_6(_6)_7)_8$.

Proof. We construct an ♥-intimate family $\mathcal{L}$ recursively, starting with the empty family of paths. At each step, we add a maximal path p to $\mathcal{L}$ such that all roots contained in p lie in J. We then replace J by the order ideal generated by all ordinary roots in J not contained in a path of $\mathcal{L}$, all charmed feints of paths of $\mathcal{L}$, and roots in J not lying below a path of $\mathcal{L}$. The recursion stops when J is empty. It is clear that the resulting family $\mathcal{L}$ is the unique ♥-intimate family of paths with order ideal J. □

An example of the algorithm used in the proof of Lemma 2 is given in Figure 6.

3.4 Charmed bijections between balanced pairs and nonnesting partitions

As a direct consequence of Lemmas 1 and 2, we obtain the following family of bijections between balanced pairs and nonnesting partitions.

Proposition 2. *Fix a collection of charmed roots* $♥ \subseteq \Phi^+$. *Then the map* $J_♥\colon \mathrm{Bal}(n) \to \mathrm{NN}(S_n)$ *defined by* $J_♥(O,I) = J(\mathcal{L}_{(O,I)})$ *is a bijection.*

Charmed roots along the upper boundary of Φ^+ do not affect the bijection of Proposition 2. On the other hand, each of the $2^{\binom{n-2}{2}}$ charming choices for the the roots (i,j) with $1 < i < j < n$ gives rise to a distinct bijection between $\mathrm{NN}(S_n)$ and $\mathrm{Bal}(n)$.

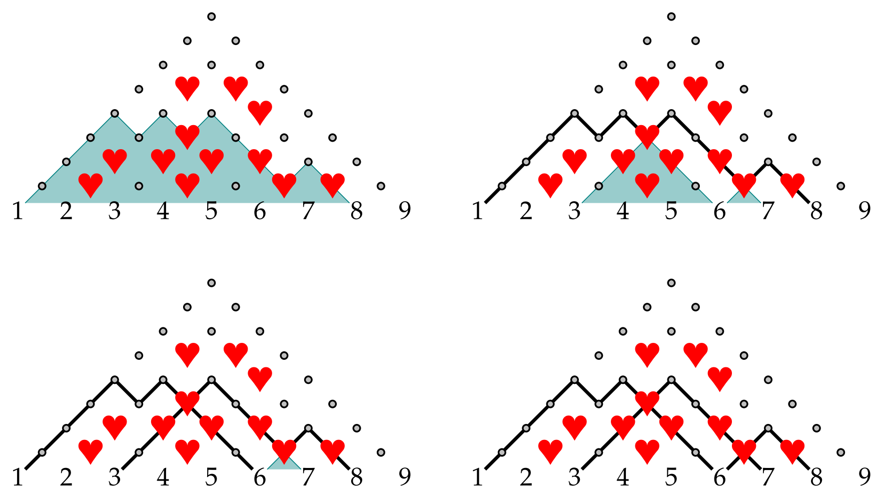

Figure 6: The construction of the intimate family of paths with order ideal J, where J contains the roots in the grey-shaded region of the top left picture. At each step, the order ideal under consideration consists of the roots contained in the teal-shaded region.

4 Charmed bijections between noncrossing and nonnesting partitions

Let $c \in S_n$ be a Coxeter element. Using Proposition 1 and Proposition 2, we now construct a bijection between c-noncrossing partitions and nonnesting partitions *that depends on the choice of Coxeter element c*, resolving Incongruity (2).

Definition 4. *The c-charmed bijection between c-noncrossing partitions and nonnesting partitions is given by*

$$\begin{aligned} \mathrm{Charm}_c \colon \mathrm{NC}(S_n, c) &\to \mathrm{NN}(S_n) \\ \pi &\mapsto J_{\heartsuit_c}(O(\pi), I(\pi)), \end{aligned}$$

where the set $\heartsuit_c$ of c-charmed roots is defined in Definition 1 and the map $J_{\heartsuit_c}$ is defined in Proposition 2.

Theorem 2. *For all Coxeter elements c, the bijection Charm_c is the unique support-preserving bijection between $\mathrm{NC}(S_n, c)$ and $\mathrm{NN}(S_n)$ satisfying $\mathrm{Krow}_c \circ \mathrm{Charm}_c = \mathrm{Charm}_c \circ \mathrm{Krew}_c$.*

For reasons of space, we only sketch the idea of the proof and refer the reader to the full version of this extended abstract for the details [7]. We say that a simple reflection s is *initial* in c if $\ell_S(sc) \leq \ell_S(c)$. If $s = s_k = (k, k+1)$ is initial in c, then $c' = scs$ is also a Coxeter element of S_n, and we will denote this by writing $c \xrightarrow{k} c'$. Theorem 2 (and hence Theorem 1) are proven using *Cambrian induction*—that is, we show that the theorem holds for a particular Coxeter element c_1 (the *base case*), and then we show that if $c \xrightarrow{k} c'$ and the theorem holds for c, then the theorem also holds for c' (the *inductive step*). Since all Coxeter elements in S_n are conjugate by a sequence of conjugations by initial simple reflections [15, Lemma 1.7], the theorem holds for all Coxeter elements.

As a consequence of our proof, we obtain a simple description for reading the blocks of the noncrossing partition from the corresponding intimate family.

Corollary 1. *Let $\mathcal{L}$ be an $\heartsuit_c$-intimate family and $\pi \in \mathrm{NC}(S_n, c)$ the corresponding noncrossing partition with $(\mathrm{Out}(\mathcal{L}), \mathrm{In}(\mathcal{L})) = (O(\pi), I(\pi))$. The blocks of π consist of the integers which are connected by paths in $\mathcal{L}$ after reinterpreting each kiss between a pair of paths in $\mathcal{L}$ at a charmed root as a crossing.*

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