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Locally Invariant Vectors in Representations of Symmetric Groups

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Abstract. For each permutation w in S_n and each irreducible representation $(\rho_\lambda, V_\lambda)$ of S_n, we determine when $\rho_\lambda(w)$ admits a non-zero invariant vector in V_λ. We find that non-zero invariant vectors exist in most cases, with very few exceptions.

Keywords: symmetric group, locally invariant vector, cyclic permutation representation, global conjugacy class

1 Introduction

This extended abstract is based on the results of [8]. The main results of this article are motivated by various problems in representation theory which we survey in the first four subsections of this introduction. The main results are stated in Section 1.5. Section 2 contains an outline of the proof of our main theorem. Details can be found in [8]. In Section 3, we list some interesting questions for further study.

1.1 Locally Invariant Vectors

Let G be a finite group and let V be a complex representation of G. A G-invariant vector in V is a vector $v \in V$ such that $g \cdot v = v$ for all $g \in G$. The representation V admits a G-invariant vector if and only if it contains the trivial representation of G as a subrepresentation.

In this article, we will be concerned with *locally* G-invariant vectors. Fixing an element $g \in G$, we ask if there exists a non-zero vector $v \in V$ such that $g \cdot v = v$. It is easy to see that the existence of such a vector depends on $g \in G$ only through its conjugacy class.

Let $C(G)$ denote the set of conjugacy classes of G. Let $\mathrm{Irr}(G)$ denote the set of irreducible complex representations of G up to isomorphism.

Question 1. Given a finite group G, for which pairs $(C, V) \in C(G) \times \mathrm{Irr}(G)$ does there exist a non-zero vector $v \in V$ such that $g \cdot v = v$ for some element $g \in C$?

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1.2 Cyclic Permutation Representations

A *cyclic representation* of G is a representation that is induced from a multiplicative character of a cyclic subgroup of G. Artin [1] proved that every complex representation of any finite group can be expressed as a virtual rational linear combination of cyclic representations. This allowed him to show that some integer power of the Artin L-function associated to any representation of G extends to a meromorphic function on the complex plane.

Brauer [3] showed that every representation of G is a virtual *integer* linear combination of representations induced from linear characters of (not necessarily Abelian) subgroups of G. Brauer showed that the subgroups of G can all be taken to be *elementary* (product of a p-group with a cyclic group of order coprime to p for some prime p). This variation on Artin's theorem was used to improve Artin's result on L-functions, concluding that Artin L-functions extend to meromorphic functions on the complex plane.

A *permutation representation* is a representation induced from the trivial representation of a subgroup of G.

Definition 2 (Cyclic permutation representation)**.** A *cyclic permutation representation* of a finite group G is a representation that is induced from the trivial representation of a cyclic subgroup of G.

In general (even for symmetric groups), it is not true that every representation of G is a virtual rational linear combination of cyclic permutation representations.

Given $g \in G$, let $V_g = \mathrm{Ind}_{\langle g \rangle}^{S_n} 1$ denote the representation of G induced from the trivial representation of the cyclic group $\langle g \rangle$ generated by g. The isomorphism class of V_g depends only on the conjugacy class of g in G. By Frobenius reciprocity, given $V \in \mathrm{Irr}(G)$, g admits a non-zero invariant vector in V if and only if V occurs in the decomposition of V_g into irreducibles. Thus Question 1 can be reformulated in terms of cyclic permutation representations as follows:

Question 3. Given a finite group G, for which pairs $(C, V) \in C(G) \times \mathrm{Irr}(G)$ does V occur in V_g?

Let $Z_G(g)$ denote the centralizer of g in G. The following definition is due to Heide and Zalessky [4].

Definition 4 (Global conjugacy class)**.** Let G be a finite group. The conjugacy class of an element $g \in G$ is said to be a *global conjugacy class* if every irreducible representation of G occurs in the permutation representation $\mathrm{Ind}_{Z_G(g)}^{G} 1$.

The group algebra $\mathbf{C}[G]$ of G can be thought of as a representation of G via the action of G on itself by conjugation, called the *adjoint representation* of G. Heide, Saxl, Tiep, and Zalessky [5] showed that the adjoint representation of G contains every irreducible representation of G for every finite simple group G except $G = SU(n, q^2)$ when n is odd and coprime to $q+1$. Since $\mathbf{C}[G]$ is a direct sum of the representations $\mathrm{Ind}_{Z_G(g)}^G 1$ as g runs over the conjugacy classes of G, if G admits a global conjugacy class then the adjoint representation of G contains every irreducible representation of G. Heide and Zalessky [4, Conjecture 1.5] conjectured that the converse is true: if every irreducible representation of a finite simple group G occurs in its adjoint representation then G admits a global conjugacy class. They proved this conjecture for alternating groups A_n, $n > 4$, and for all sporadic simple groups.

Sheila Sundaram [10, Theorem 5.1] characterized global conjugacy classes for all symmetric groups (see Theorem 7). She showed [10, Theorem 1.1] that a symmetric group S_n admits a global conjugacy class if and only if $n = 6$ or $n \geq 8$.

For every element $g \in G$, the cyclic group $\langle g \rangle$ generated by g is a subgroup of the centralizer group $Z_G(g)$. It follows that $\mathrm{Ind}_{Z_G(g)}^G 1$ is a subrepresentation of $\mathrm{Ind}_{\langle g \rangle}^G 1$. Thus, if the conjugacy class of G is a global class, then every irreducible representation V of G admits a non-zero vector $v \in V$ such that $g \cdot v = v$.

Definition 5. Let G be a finite group, and let C be a conjugacy class in G. We say that C is a *cyclically global* class if $\mathrm{Ind}_{\langle g \rangle} 1$ contains every irreducible representation of G.

A complete answer to the equivalent Questions 1 or 3 will result in the characterization of cyclically global conjugacy classes in G.

1.3 Immersion of Representations

Prasad and Raghunathan [9] proposed a partial order on automorphic representations called immersion. Adapted to finite groups, it may be defined as follows.

Definition 6. Given representations (ρ, V) and (σ, W) of G, say that V *is immersed in* W, denoted $V \preccurlyeq W$, if for every $g \in G$ and every $\lambda \in \mathbf{C}$, the multiplicity of λ as an eigenvalue of $\rho(g)$ does not exceed the multiplicity of λ as an eigenvalue of $\sigma(g)$.

In particular, if V is a subrepresentation of W, then $V \preccurlyeq W$.

Let 1 denote the trivial representation of G. Then $1 \preccurlyeq V$ if and only if, for every $g \in G$, there exists a non-zero vector $v \in V$ such that $g \cdot v = v$.

1.4 Results for Symmetric Groups

In this section, we follow standard notation from the theory of symmetric functions. See e.g., Macdonald [7].

Let S_n denote the nth symmetric group. The conjugacy class of $w \in S_n$ is completely determined by the cycle type of w which is a partition $\mu \vdash n$. For each $\mu \vdash n$, let w_μ denote a permutation with cycle type μ.

Following Schur, irreducible representations of S_n are elegantly characterized by associated symmetric functions. If the representation V of S_n has character $\chi : S_n \to \mathbf{C}$, its Frobenius characteristic is defined as the symmetric function

$$\mathrm{ch}_n\, \chi = \sum_{\mu \vdash n} \frac{\chi(w_\mu)}{z_\mu} p_\mu,$$

where p_μ denotes the power sum symmetric function associated to the partition μ and z_μ denotes the number of permutations in S_n that commute with w_μ.

For every partition $\lambda \vdash n$, there is a unique irreducible representation V_λ of S_n whose character χ^λ satisfies

$$\mathrm{ch}_n\, \chi^\lambda = s_\lambda,$$

where s_λ is the Schur function associated to $\lambda \vdash n$. The representations $\{V_\lambda \mid \lambda \vdash n\}$ are the irreducible representations of S_n.

Sundaram's characterization of global conjugacy classes for symmetric groups is the following.

Theorem 7 (Sundaram [10, Theorem 5.1])**.** *Let $n \neq 4, 8$. A partition of n is the cycle type of a global conjugacy class in S_n if and only if it has at least two parts, and all its parts are odd and distinct.*

When $\mu = (n)$, w_μ is an n-cycle in S_n and $Z_{S_n}(w_{(n)}) = \langle w_{(n)} \rangle$. The decomposition of the cyclic permutation representation of S_n induced from $\langle w_{(n)} \rangle$ into irreducible representations has a nice combinatorial interpretation.

Theorem 8 (Kraśkiewicz and Weyman [6])**.** *Let χ_r denote the character of $\langle w_{(n)} \rangle$ which takes $w_{(n)}$ to $e^{2\pi i r/n}$. For any $\lambda \vdash n$, the multiplicity of V_λ in $\mathrm{Ind}_{\langle w_{(n)} \rangle}^{S_n} \chi_r$ is given by the number $a_{\lambda,r}$ of standard tableaux of shape λ whose major index is congruent to r modulo n.*

However, it is not easy to say when $a_{\lambda,r}$ is positive. This question was resolved by Swanson [12, Theorem 1.5]. When $r = 0$, his results prove a conjecture of Sundaram [11, Remark 4.8].

Theorem 9. *For $\lambda \vdash n$, V_λ occurs in $\mathrm{Ind}_{\langle w_{(n)} \rangle}^{S_n} 1$ unless λ is one of*

1. *$(n-1, 1)$,*
2. *$(2, 1^{n-2})$ with n odd,*
3. *(1^n) with n even.*

1.5 Our Main Results

Let A_n denote the alternating group, a subgroup of index 2 in S_n.

Main Theorem. *The only pairs of partitions (λ, μ) of a given integer n such that w_μ does not admit a nonzero invariant vector in V_λ are the following:*

1. $\lambda = (1^n)$, *μ is any partition of n for which $w_\mu \notin A_n$,*
2. $\lambda = (n-1, 1)$, $\mu = (n)$, $n \geq 2$,
3. $\lambda = (2, 1^{n-2})$, $\mu = (n)$, $n \geq 3$ *is odd,*
4. $\lambda = (2^2, 1^{n-4})$, $\mu = (n-2, 2)$, $n \geq 5$ *is odd,*
5. $\lambda = (2, 2)$, $\mu = (3, 1)$,
6. $\lambda = (2^3)$, $\mu = (3, 2, 1)$,
7. $\lambda = (2^4)$, $\mu = (5, 3)$,
8. $\lambda = (4, 4)$, $\mu = (5, 3)$,
9. $\lambda = (2^5)$, $\mu = (5, 3, 2)$.

It follows that most irreducible representations of S_n admit w-invariant vectors for every permutation w. In terms of the notion of immersion (Definition 6), we have

Theorem 10. *Given a partition $\lambda \vdash n$, $V_{(n)} \preccurlyeq V_\lambda$ if and only if λ is not one of*

1. (1^n),
2. $(n-1, 1)$ *for* $n \geq 2$,
3. $(2, 1^{n-2})$ *when* $n \geq 3$ *is odd,*
4. $(2^2, 1^{n-4})$, *when* $n \geq 5$ *is odd,*
5. $(2, 2)$, (2^3), (2^4), (4^2) *and* (2^5).

Because the sign representation does not admit any non-zero invariant vector for a permutation that does not lie in A_n, conjugacy classes of S_n that are not contained in A_n cannot be cyclically global. We find that most conjugacy classes of S_n which are contained in A_n are cyclically global.

Theorem 11. *Given a partition $\mu \vdash n$ the conjugacy class in S_n consisting of permutations with cycle type μ is cyclically global if and only if it is contained in A_n and μ is not one of*

1. (n) *for* $n \geq 2$*,*
2. $(n-2, 2)$ *for* $n \geq 5$ *odd,*
3. $(3, 1)$*,* $(5, 3)$*.*

While conjugacy classes that are not contained in A_n cannot be cyclically global, for most of them, the only obstruction to being cyclically global is the sign representation.

Definition 12 (Persistent class)**.** A permutation $w \in S_n$ is said to be persistent if $\mathrm{Ind}_{\langle w \rangle}^{S_n} 1$ contains V_λ for every $\lambda \vdash n$ with the possible exception of $\lambda = (1^n)$. If w is persistent then every permutation in its conjugacy class is persistent.

By Frobenius reciprocity, w is persistent if there exists a non-zero $v \in V_\lambda$ such that $w \cdot v = v$ for all $\lambda \vdash n$ such that $\lambda \neq (1^n)$. It turns out that for most partitions μ, w_μ is persistent.

Theorem 13. *Given* $\mu \vdash n$*,* w_μ *is persistent unless* μ *is one of the following:*

1. (n) *when* $n \geq 2$*,*
2. $(n-2, 2)$*, when* $n \geq 5$ *is odd,*
3. $(3, 1)$*,* $(3, 2, 1)$*,* $(5, 3)$*,* $(5, 3, 2)$*.*

2 Proof of the Main Theorem

The main theorem is proved using Swanson's theorem (Theorem 9) and the Littlewood-Richardson rule. We outline the main steps in the proof in this section.

2.1 Reformulation in terms of Symmetric Functions

Definition 14. Given symmetric functions f and g with integer coefficients, say that $f \geq g$ if $f - g$ is a non-negative integer combination of Schur functions.

Define

$$f_\mu = \mathrm{ch}_n \mathrm{Ind}_{\langle w_\mu \rangle}^{S_n} 1.$$

Then V_λ occurs in $\mathrm{Ind}_{\langle w_\mu \rangle}^{S_n} 1$ if and only if

$$f_\mu \geq s_\lambda \tag{2.1}$$

If $\mu = (\mu_1, \ldots, \mu_k)$ let $S_\mu = S_{\mu_1} \times \cdots \times S_{\mu_k}$ be the Young subgroup corresponding to the cycles of w_μ. Let D_μ be the subgroup of S_μ generated by the cycles of w_μ. Thus D_μ is a product of cyclic groups of orders $\mu_1, \mu_2, \ldots, \mu_k$. Using induction in stages,

$$f_\mu = \operatorname{ch} \operatorname{Ind}_{S_\mu}^{S_n} \operatorname{Ind}_{D_\mu}^{S_\mu} \operatorname{Ind}_{C_\mu}^{D_\mu} 1.$$

Therefore

$$f_\mu \geq \operatorname{ch} \operatorname{Ind}_{S_\mu}^{S_n} \operatorname{Ind}_{D_\mu}^{S_\mu} 1 = \prod_{i=1}^{k} f_{(\mu_i)}. \tag{2.2}$$

Swanson's theorem (Theorem 9) tells us that $f_{(n)} \geq s_\lambda$ for most partitions λ of n. We will use this fact, together with the inequality (2.2), to establish (2.1) in most cases using the Littlewood-Richardson rule. Recall that the Littlewood-Richardson coefficients $c^\lambda_{\alpha\beta}$ are defined by

$$s_\alpha s_\beta = \sum_\lambda c^\lambda_{\alpha\beta} s_\lambda.$$

The Littlewood-Richardson rule [7, Section I.9] asserts that $c^\lambda_{\alpha\beta}$ is the number of LR-tableaux of shape λ/α and weight β. Recall that an LR-tableau is a semistandard skew-tableau whose reverse row reading word is a lattice permutation.

2.2 The Basic Lemmas

Our proof of the main will make frequent use of the following lemmas.

Lemma 15. *For every partition λ of $p+q$, and every partition α of p that is contained in λ, there exists a partition β of q such that $s_\alpha s_\beta \geq s_\lambda$.*

Proof. Let $T_{\lambda\alpha}$ denote the skew-tableau obtained by putting i in the ith cell of each column of λ/α. Let β be the weight of $T_{\lambda\alpha}$. For example, if $\lambda = (5,4,4,1)$ and $\alpha = (3,2,1)$ then

$$T_{\lambda\alpha} = \begin{array}{|c|c|c|c|c|} \hline & & & 1 & 1 \\ \hline & & 1 & 2 \\ \cline{1-4} & 1 & 2 & 3 \\ \cline{1-4} 1 \\ \cline{1-1} \end{array},$$

and β is $(5,2,1)$. Since every $i+1$ occurs below an i, $T_{\lambda\alpha}$ is an LR-tableau. The Littlewood-Richardson rule implies that $s_\alpha s_\beta \geq s_\lambda$. □

Lemma 15 is nothing more than the well-known statement that the skew-Schur function $s_{\lambda/\alpha}$ is non-zero whenever $\lambda \supset \alpha$. However, the method of constructing β in the proof is also used in our proof of the main theorem.

Lemma 16. *Given integers $p \geq 2$, $q \geq 1$, and a partition $\lambda \vdash (p+q)$ different from $(1^{(p+q)})$, there exists a partition $\beta \vdash q$ such that $f_{(q)} \geq s_\beta$ and $\beta \subset \lambda$.*

Proof. We consider the following cases:

Case 1: $\lambda \supset (q-1,1)$

Since $p \geq 2$, the skew shape $\lambda/(q-1,1)$ has at least two cells. If at least one of these cells lies in the first row of λ, then choose $\beta = (q)$. If at least one of these cells lies in the first column of λ, then choose $\beta = (q-2,1,1)$ If neither of the above happens, then $\lambda/(q-1,1)$ has at least two cells in its second row. In this case $q-1 \geq 3$. Choose $\beta = (q-2,2)$. The possible placements of the cells of $\lambda/(q-1,1)$ are shown in Figure 1.

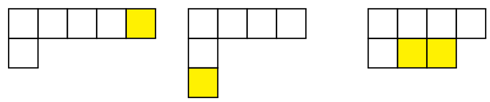

Figure 1: Possible placements of two cells of $\lambda/(q-1,1)$

In all these cases, Theorem 9 implies that $f_{(q)} \geq s_\beta$.

Case 2: $\lambda \supset (1^q)$ **and** q **is even**

Since $\lambda \neq (1^{p+q})$, the skew-shape $\lambda/(1^q)$ must contain at least one cell in the first row. Take $\beta = (2,1^{q-2})$. By Theorem 9, $f_{(q)} \geq s_\beta$, since q is even.

Case 3: $\lambda \supset (2,1^{q-2})$ **and** q **is odd**

If $\lambda/(2,1^{q-2})$ has a cell in its first row, take $\beta = (3,1^{q-3})$. If $\lambda/(2,1^{q-2})$ has a cell in its first column, take $\beta = (1^q)$. By Theorem 9, $f_{(q)} \geq s_\beta$, since q is odd. Otherwise the second column of $\lambda/(2,1^{q-2})$ must have at least two cells in its second column. In this case $q \geq 4$. Take $\beta = (2,2,1^{q-4})$. The possible placements of the cells of $\lambda/(2,1^{q-2})$ are

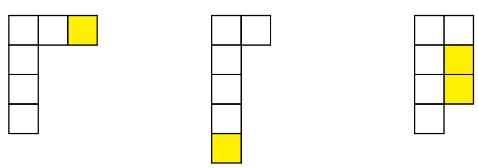

Figure 2: Possible placements of cells of $\lambda/(2,1^{q-2})$.

shown in Figure 2.

All remaining λ**:**

Take β to be any partition of q that is contained in λ. Since λ does not contain any of the exceptions of Theorem 9, $f_{(q)} \geq s_\beta$. □

2.3 The Main Steps of the Proof

For convenience, we will say that a partition μ is persistent if w_μ is persistent in the sense of Definition 12. The goal is to prove that, for $n \geq 11$, every partition $\mu \vdash n$ is persistent, except for the partitions $\mu = (n)$, and $\mu = (n-2, 2)$ when n is odd. The cases $n \leq 10$ are easily solved by computer calculation, using, for example, the Sage Mathematical Software [13]. The following lemma deals with most partitions that have two parts.

Lemma 17. *If $\mu = (p, q)$ where $p \geq q \geq 4$, then μ is persistent.*

For the details of the proof, we refer the reader to [8]. We only outline the main idea here.

We wish to show that $f_{(p,q)} \geq s_\lambda$ for every $\lambda \vdash (p+q)$ different from (1^{p+q}). By (2.2), $f_{(p,q)} \geq f_{(p)} f_{(q)}$. Hence, in order to show that $f_{(p,q)} \geq s_\lambda$, it suffices to find $\alpha \vdash p$ and $\beta \vdash q$ such that

$$f_{(p)} \geq s_\alpha,\ f_{(q)} \geq s_\beta, \text{ and } s_\alpha s_\beta \geq s_\lambda. \tag{2.3}$$

Lemma 16 allows us to choose $\beta \vdash q$ such that $\beta \subset \lambda$ and $f_{(q)} \geq s_\beta$. Using Lemma 15 with the roles of α and β reversed, we may choose $\alpha \vdash p$ such that $s_\alpha s_\beta \geq s_\lambda$. If $f_{(p)} \geq s_\alpha$ we are done. Otherwise, α must be one of the partitions occurring in Swanson's theorem (Theorem 9). Most of these cases are dealt with by prescribing a replacement for α and β so that (2.3) holds.

Lemma 17 can be leveraged to deal with most partitions with more than two parts using the following lemma.

Lemma 18. *A partition $\mu = (\mu_1, \ldots, \mu_k) \vdash n$ with $k \geq 2$ is persistent if the partition $\tilde{\mu}$ obtained by removing a part μ_i from μ is persistent and $n - \mu_i \geq 4$.*

In order to prove the lemma, we wish to show that $f_\mu \geq s_\lambda$ for every $\lambda \vdash n$ except $\lambda = (1^n)$. Suppose $\mu = (\mu_1, \ldots, \mu_k)$. Noting that $C_{\tilde{\mu}} \times C_{(\mu_i)} \subset C_\mu$ and $D_{\tilde{\mu}} \times D_{(\mu_i)} = D_\mu$, we have

$$\mathrm{Ind}_{C_\mu}^{D_\mu} 1 \geq \mathrm{Ind}_{C_{\tilde{\mu}}}^{D_{\tilde{\mu}}} 1 \otimes \mathrm{Ind}_{C_{(\mu_i)}}^{D_{(\mu_i)}} 1$$

Inducing to $S_{p_1} \times \cdots \times S_{p_k}$, and then to $S_{p_1 + \cdots + p_k}$ gives

$$f_\mu \geq f_{\tilde{\mu}} f_{(\mu_i)}.$$

Hence it suffices to show that $f_{\tilde{\mu}} f_{(\mu_i)} \geq s_\lambda$ for all $\lambda \vdash n$ except $\lambda = (1^n)$. As before, it suffince to find $\alpha \vdash n - \mu_i$ and $\beta \vdash \mu_i$ such that $\alpha \neq (1^n)$, $f_{(\mu_i)} \geq \beta$ and $s_\alpha s_\beta \geq s_\lambda$. Again, using Lemma 16, we may choose β such that $\beta \subset \lambda$ and $f_{(\mu_i)} \geq s_\beta$. Again, using Lemma 15 with the roles of α and β reversed, we may choose $\alpha \vdash n - \mu_i$ such that $s_\alpha s_\beta \geq s_\lambda$. In this case, there is a way to replace α and β with another pair which have the required properties. The details of the proof are found in [8].

Lemmas 17 and 18 take care of most cases of partitions. To complete the proof, they need to be carefully put together with a few more cases, for which we refer the reader to [8].

3 Futher Questions

We conclude this extended abstract by enumerating a few interesting open questions.

Question 19. Classify the global conjugacy classes of the alternating group A_n.

Heide and Zalesski [4] proved the existence of at least one such class for each n and gave an algorithm to find it.

Question 20. Find "effective" versions of Artin and Brauer induction theorem (discussed in Section 3) for symmetric groups.

For the Artin induction theorem, this would mean finding, for each positive integer n, a set of pairs (μ, χ) where $\mu \vdash n$ and $\chi : \langle w_\mu \rangle \to \mathbf{C}$ is a multiplicative character such that the representations $\mathrm{Ind}_{\langle w_\mu \rangle}^{S_n} \chi$ form a basis for the space of class functions on S_n. For the Brauer induction theorem, this would mean finding, for each positive integer n and each prime p, a set of pairs (A, χ) where A is an elementary subgroup of S_n and $\chi : A \to \mathbf{C}$ is a multiplicative character such that the characters of the representations $\mathrm{Ind}_A^{S_n} \chi$ form a basis for the space of class functions on S_n. Techniques developed by Boltje, Snaith and Symonds [2] may be useful in this context.

Consider the representation $U_\mu^\chi = \mathrm{Ind}_{\langle w_\mu \rangle}^{S_n} \chi$, where $\chi : \langle w_\mu \rangle \to \mathbf{C}$ is a primitive multiplicative character.

Question 21. Determine the set of triples (λ, μ, χ) such that U_μ^χ contains V_λ.

Since Swanson [12] solves this problem for $\mu = (n)$, this problem could also be amenable to the methods of [8].

Question 22. Find subgroups H of S_n that are maximal among subgroups for which V_λ occurs in $\mathrm{Ind}_H^{S_n} 1$ for every $\lambda \vdash n$.

Theorem 11 shows that there are many cyclic subgroups with this property, and hence there should be a large class of such maximal subgroups.

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