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Quasisymmetric expansion of Hall-Littlewood symmetric functions

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Abstract. In our previous works we introduced a q-deformation of the generating functions for enriched P-partitions. We call the evaluation of this generating functions on labelled chains, the q-fundamental quasisymmetric functions. These functions interpolate between Gessel's fundamental ($q = 0$) and Stembridge's peak ($q = 1$) functions, the natural quasisymmetric expansions of Schur and Schur's Q-symmetric functions. In this paper, we show that our q-fundamental functions provide a quasisymmetric expansion of Hall-Littlewood S-symmetric functions with parameter $t = -q$.

Résumé. Dans nos travaux précédents, nous avons introduit une q-déformation des fonctions génératrices pour les P-partitions enrichies. Nous nommons l'évaluation de ces fonctions génératrices sur les chaînes étiquetées, les fonctions quasisymétriques q-fondamentales. Ces fonctions interpolent entre les fonctions fondamentales de Gessel ($q = 0$) et les fonctions de pics de Stembridge ($q = 1$) qui sont les expansions quasisymétriques naturelles des fonctions symétriques de Schur et Q Schur. Dans cet article, nous montrons que nos fonctions q-fondamentales fournissent une expansion quasisymétrique des fonctions symétriques Hall-Littlewood S avec paramètre $t = -q$.

Keywords: Hall-Littlewood, quasisymmetric functions, enriched P-partitions

1 Introduction

We define the q-fundamental quasisymmetric functions as the q-deformed generating functions for enriched P-partitions on labelled chains [5, 6]. These functions naturally interpolate between I. Gessel's fundamental ([1], $q = 0$) and J. Stembridge's peak ([12], $q = 1$) quasisymmetric functions and exhibit most of the nice properties of these two classical families. In particular, when q is not a complex root of unity they span the ring of quasisymmetric functions (QSym). When q is a root of unity, a subfamily of our q-fundamentals is the basis of the algebra of extended peaks [6], a proper subalgebra of QSym that coincides with Stembridge's algebra of peaks when $q = 1$. Fundamental and peak functions indexed by standard Young tableaux of shape λ are respectively the

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quasisymmetric expansions of Schur and Schur's Q-symmetric functions indexed by λ. Finding the analogous families of symmetric functions for general q appears as a natural question. We find out that q-fundamental functions provide a similar quasisymmetric expansion of the family $(S_\lambda(X;t))_\lambda$, the Hall-Littlewood S-symmetric functions with parameter $t = -q$. After recalling the required definitions, we state and prove our main result. Finally, we look at some important consequences regarding the quasisymmetric extension of the classical homorphism between Λ, the algebra of symmetric functions and the subalgebra of Λ spanned by Hall-Littlewood functions as well as some Cauchy like formulas for the $S_\lambda(X;t)$'s.

1.1 Integer partitions, Young tableaux and permutation statistics

Let $\mathbb{P}$ be the set of positive integers and $\mathbb{P}^\pm$ be the set of positive and negative integers ordered by $-1 < 1 < -2 < 2 < -3 < 3 < \ldots$. We embed $\mathbb{P}$ into $\mathbb{P}^\pm$ and let $-\mathbb{P} \subseteq \mathbb{P}^\pm$ be the set of all $-n$ for $n \in \mathbb{P}$. For $n \in \mathbb{P}$ write $[n] = \{1, \ldots, n\}$ and $\mathfrak{S}_n$ the symmetric group on $[n]$. A *partition* λ of an integer n, denoted $\lambda \vdash n$ is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ of $\ell(\lambda) = p$ parts sorted in decreasing order such that $|\lambda| = \sum_i \lambda_i = n$. We denote the one part partition (n) simply n. A partition λ is represented as a Young diagram of $n = |\lambda|$ boxes arranged in $\ell(\lambda)$ left justified rows so that the i-th row from the top contains λ_i boxes. Given a second partition μ with $\ell(\mu) \leq \ell(\lambda)$ such that $\mu_i \leq \lambda_i$, $(i \leq \ell(\mu))$ delete the μ_i leftmost boxes of the i-th row to get the diagram of *shape* λ/μ. A Young diagram whose boxes are filled with positive integers such that the entries are increasing along the rows and strictly increasing down the columns is called a *semistandard Young tableau*. If the entries are consecutive and strictly increasing along the rows, we call it a *standard Young tableau* and we denote $SYT(\lambda/\mu)$ (resp. $SSYT(\lambda/\mu)$) the set of standard (resp. semistandard) Young tableaux of shape λ/μ. A *marked semistandard Young tableau* is a Young diagram filled with integers in $\mathbb{P}^\pm$ such that the entries are increasing along rows and columns and such that each row contains at most once each negative integer and that each column contains at most once each positive integer. Denote $SSYT^\pm(\lambda/\mu)$ the

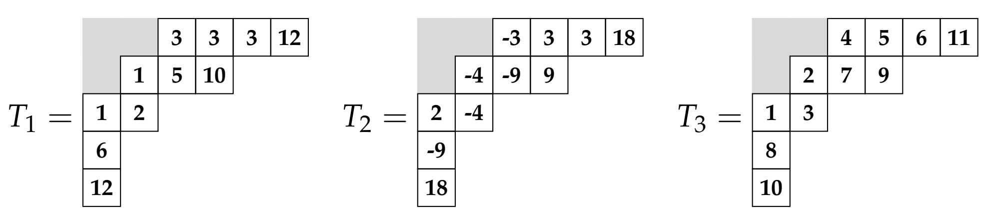

Figure 1: A semistandard, marked semistandard and standard tableau of shape $(6,4,2,1,1)/(2,1)$. The descent set of T_3 is $\{2,6,7,9\}$ while its peak set is $\{2,6,9\}$. T_2 has $neg(T_2) = 5$ negative entries.

set of marked semistandard Young tableaux of shape λ/μ. Define the *descent set of a standard Young tableau T* as $Des(T) = \{1 \leq i \leq n-1 \mid i \text{ is in a strictly higher row than } i+1\}$ and it *peak set* as $Peak(T) = \{2 \leq i \leq n-1 \mid i \in Des(T) \text{ and } i-1 \notin Des(T)\}$. Finally, denote the number of negative entries of a marked tableau T as $neg(T)$.

Example 1. *Figure 1 depicts a semistandard, a marked semistandard and a standard Young tableau with their shape and descent and peak set.*

Similarly, the descent set and peak set of a permutation π in $\mathfrak{S}_n$ are the subsets of $[n-1]$ defined as $Des(\pi) = \{1 \leq i \leq n-1 \mid \pi(i) > \pi(i+1)\}$ and $\text{Peak}(\pi) = \{2 \leq i \leq n-1 | \pi(i-1) < \pi(i) > \pi(i+1)\}$. Finally the *Robinson-Schensted (RS) correspondence* ([9, 10]) is a bijection between permutations π in $\mathfrak{S}_n$ and ordered pairs of standard Young tableaux (P, Q) of the same shape $\lambda \vdash n$. This bijection is descent preserving in the sense that $Des(\pi) = Des(Q)$, and $Des(\pi^{-1}) = Des(P)$.

1.2 Hall-Littlewood symmetric functions

Consider the set of indeterminates $X = \{x_1, x_2, x_3, \ldots\}$. Let Λ denote the ring of symmetric functions over $\mathbb{C}$. We use notations consistent with [7]. Namely, for $\lambda \vdash n$, denote $m_\lambda(X)$, $h_\lambda(X)$, $e_\lambda(X)$, $p_\lambda(X)$ and $s_\lambda(X)$ the *monomial*, *complete homogeneous*, *elementary*, *power sum* and *Schur* symmetric functions over X indexed by λ. Fix a parameter $t \in \mathbb{C}$ and define $q_n(X;t) \in \Lambda$ as $q_0(X;t) = 1$ and for any positive integer n as:

$$q_n(X;t) = (1-t)\sum_i x_i^n \prod_{j \neq i} \frac{x_i - tx_j}{x_i - x_j}. \tag{1.1}$$

The generating function for the q_n is given by

$$\sum_{n \geq 0} q_n(X;t)u^n = \prod_i \frac{1 - x_i tu}{1 - x_i u}. \tag{1.2}$$

The family $(q_n(X;t))_n$ generates a subalgebra of Λ that we denote Λ_t. In particular, Λ_t is a proper subalgebra of Λ when t is a root of unity.

Definition 1 (Hall-Littlewood S-symmetric functions)**.** *Let λ/μ be a skew shape, define the Hall-Littlewood S-symmetric functions indexed by λ/μ as*

$$S_{\lambda/\mu}(X;t) = det\left(q_{\lambda_i - \mu_j - i + j}(X;t)\right)_{i,j} \tag{1.3}$$

As a direct consequence of Definition 1, setting $t = 0$ yields $S_{\lambda/\mu}(X;0) = s_{\lambda/\mu}(X)$. When $t = -1$, $S_{\lambda/\mu}(X;-1)$ is a variant of *Schur's Q-function* indexed by λ/μ. We end this section with the definition of a classical ring homomorphism.

Definition 2 (Ring homomorphism)**.** *Define a ring homomorphism* θ_t: $\Lambda \longrightarrow \Lambda_t$ *by setting for any non-negative integer* n*,*

$$\theta_t(h_n)(X) = q_n(X;t).$$

In particular, one has $\theta_t(e_n)(X) = q_n(X;t)$*,* $\theta_t(p_n)(X) = (1-t^n)p_n(X)$ *and, as a consequence of Definition 1,*

$$\theta_t(s_{\lambda/\mu})(X) = S_{\lambda/\mu}(X;t).$$

1.3 Enriched P-partitions and q-deformed generating functions

We recall the main definitions regarding weighted posets, enriched P-partitions and their q-deformed generating functions. See [1, 4, 5, 11, 12] for more details.

Definition 3 (Labelled weighted poset, [4])**.** *A* labelled weighted poset *is a triple* $P = ([n], <_P, \epsilon)$ *where* $([n], <_P)$ *is a* labelled poset*, i.e., an arbitrary partial order* $<_P$ *on the set* $[n]$ *and* $\epsilon : [n] \longrightarrow \mathbb{P}$ *is a map (called the* weight function*). If* $\epsilon(i) = 1$ *for all* $i \in [n]$*, we may simply omit it.*

Each node of a labelled weighted poset is marked with its label and weight (Figure 2).

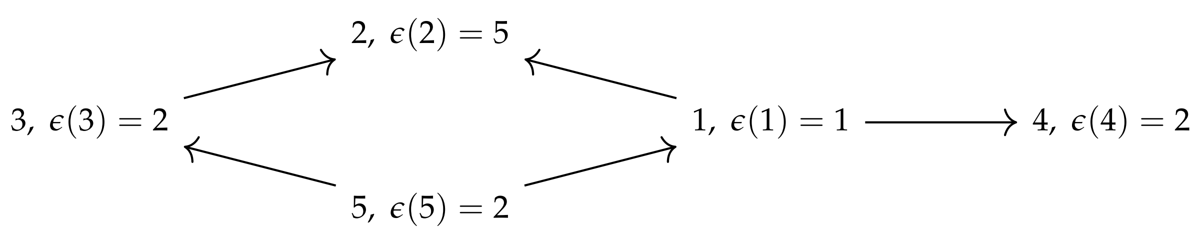

Figure 2: A 5-vertex labelled weighted poset. Arrows show the covering relations.

Definition 4 (Enriched P-partition, [12])**.** *Given a labelled weighted poset* $P = ([n], <_P, \epsilon)$*, an* enriched P-partition *is a map* $f : [n] \longrightarrow \mathbb{P}^{\pm}$ *that satisfies the two following conditions:*

(i) If $i <_P j$ *and* $i < j$*, then* $f(i) < f(j)$ *or* $f(i) = f(j) \in \mathbb{P}$*.*

(ii) If $i <_P j$ *and* $i > j$*, then* $f(i) < f(j)$ *or* $f(i) = f(j) \in -\mathbb{P}$*.*

We let $\mathcal{L}_{\mathbb{P}^{\pm}}(P)$ *denote the set of enriched* P*-partitions.*

Definition 5 (q-Deformed generating function, [5])**.** *Consider the ring* $\mathbb{C}[[X]]$ *of formal power series on* X *and let* $q \in \mathbb{C}$ *be an additional parameter. Given a labelled weighted poset* $([n], <_P, \epsilon)$*, define its generating function* $\Gamma^{(q)}([n], <_P, \epsilon) \in \mathbb{C}[[X]]$ *as*

$$\Gamma^{(q)}([n], <_P, \epsilon) = \sum_{f \in \mathcal{L}_{\mathbb{P}^{\pm}}([n], <_P, \epsilon)} \prod_{1 \le i \le n} q^{[f(i)<0]} x_{|f(i)|}^{\epsilon(i)},$$

where $[f(i) < 0] = 1$ *if* $f(i) < 0$ *and* 0 *otherwise.*

Finally, let $X^{\pm} = \{x_{-1}, x_1, x_{-2}, x_2, \ldots\}$. In the sequel we denote ϖ the homomorphism $\varpi : \mathbb{C}[[X^{\pm}]] \longrightarrow \mathbb{C}[[X]]$ defined by setting $\varpi(x_i) = q^{[i<0]}x_{|i|}$ for $x_i \in X^{\pm}$.

1.4 q-fundamental quasisymmetric functions

We recall results from [5] and [6].

Definition 6 (q-Fundamental quasisymmetric functions)**.** *Given a permutation $\pi = \pi_1 \ldots \pi_n$ of $\mathfrak{S}_n$, we let $P_\pi = ([n], <_\pi, 1^n)$ be the labelled weighted poset on the set $[n]$, where the order relation $<_\pi$ is such that $\pi_i <_\pi \pi_j$ if and only if $i < j$ and where all the weights are equal to 1 (see Figure 3). Define the q-*fundamental quasisymmetric function

$$L_\pi^{(q)} = \Gamma^{(q)}([n], <_\pi, 1^n).$$

$$\pi_1 \longrightarrow \pi_2 \longrightarrow \cdots\cdots\cdots \longrightarrow \pi_n$$

Figure 3: The labelled weighted poset P_π.

The q-fundamental quasisymmetric functions belong to the subalgebra of $\mathbb{C}[[X]]$ called the ring of *quasisymmetric functions (*QSym*)*, i.e. for any strictly increasing sequence of indices $i_1 < i_2 < \cdots < i_p$ the coefficient of $x_1^{k_1} x_2^{k_2} \cdots x_p^{k_p}$ is equal to the coefficient of $x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_p}^{k_p}$. The specialisations of $L_\pi^{(q)}$ to $L_\pi = L_\pi^{(0)}$ and $K_\pi = L_\pi^{(1)}$ are respectively the Gessel's fundamental [1] and Stembridge's peak [12] quasisymmetric functions indexed by permutation π. We have the following explicit expression.

$$L_\pi^{(q)} = \sum_{\substack{i_1 \le i_2 \le \cdots \le i_n; \\ j \in \mathrm{Peak}(\pi) \Rightarrow i_{j-1} < i_{j+1}}} q^{|\{j \in \mathrm{Des}(\pi) | i_j = i_{j+1}\}|} (q+1)^{|\{i_1, i_2, \ldots, i_n\}|} x_{i_1} x_{i_2} \ldots x_{i_n}. \tag{1.4}$$

Furthermore q-fundamental quasisymmetric functions admit a closed-form product and coproduct.

Proposition 1. *Let $q \in \mathbb{C}$, let π and σ be two permutations in $\mathfrak{S}_n$ and $\mathfrak{S}_m$. The product of $L_\pi^{(q)}$ and $L_\sigma^{(q)}$ is given by*

$$L_\pi^{(q)} L_\sigma^{(q)} = \sum_{\tau \in \pi \sqcup\!\sqcup \overline{\sigma}} L_\tau^{(q)}, \tag{1.5}$$

where $\overline{\sigma} = n + \sigma_1\, n + \sigma_2 \ldots n + \sigma_m$. Moreover, the coproduct $\Delta : \mathrm{QSym} \to \mathrm{QSym} \otimes \mathrm{QSym}$ of the Hopf algebra QSym *(see [3, §5.1]) acts on the q-fundamental quasisymmetric functions as follows:*

$$\Delta(L_\pi^{(q)}) = \sum_{i=0}^{n} L^{(q)}_{\mathrm{std}(\pi_1\pi_2\ldots\pi_i)} \otimes L^{(q)}_{\mathrm{std}(\pi_{i+1}\pi_{i+2}\ldots\pi_n)}.$$

Here, if γ is a sequence of non-repeating integers, $\operatorname{std}(\gamma)$ *is the permutation whose values are in the same relative order as the entries of γ.*

According to Equation (1.4), $L_\pi^{(q)}$ depends only on the descent set of π. We reindex our q-fundamentals by an integer n and a subset of $[n-1]$. We recall two significant results.

Proposition 2 ([5])**.** *$(L_{n,I}^{(q)})_{n\geq 0, I\subseteq[n-1]}$ is a basis of* QSym *if and only if $q \in \mathbb{C}$ is not a root of unity.*

Proposition 3 ([6])**.** *Let $p \in \mathbb{P}$ and $\rho_p \in \mathbb{C}$ such that $-\rho_p$ is a primitive $p+1$-th root of unity. For a subset I of $[n-1]$, write $I \subseteq_p [n-1]$ if $I \cup \{0\}$ does not contain more than p consecutive elements. Then $(L_{n,I}^{(\rho_p)})_{n\geq 0, I\subseteq_p[n-1]}$ is a basis of a proper subalgebra $\mathcal{P}^p$ of* QSym*.*

For general $q \in \mathbb{C}$ denote $\mathcal{P}^{(q)}$ the subalgebra of QSym spanned by the $(L_{n,I}^{(q)})_{n,I}$. If q is not a root of unity then $\mathcal{P}^{(q)} = \mathrm{QSym}$. If $q = \rho_p$ for some $p \in \mathbb{P}$ then $\mathcal{P}^{(q)} = \mathcal{P}^p$.

2 Relating Hall-Littlewood and q-fundamentals functions

The ring of symmetric functions Λ is a subalgebra of QSym and any symmetric function may be expanded in quasisymmetric bases. The relation between Schur functions (i.e Hall-Littlewood S-functions with parameter $t = 0$) and fundamental quasisymmetric functions is of particular interest. Let λ/μ be a skew shape, Gessel shows in [1]

$$S_{\lambda/\mu}(X;0) = s_{\lambda/\mu}(X) = \sum_{T \in SYT(\lambda/\mu)} L_{Des(T)}^{(0)}(X). \tag{2.1}$$

On the other hand, Stembridge shows in [12] that

$$S_{\lambda/\mu}(X;-1) = \sum_{T \in SYT(\lambda/\mu)} L_{Des(T)}^{(1)}(X). \tag{2.2}$$

As a result, understanding how these relations generalise for general q seems to be a very legitimate question. We state our result and some significant consequences.

2.1 Computing the q-deformed generating functions on skew diagrams

Let $n \in \mathbb{P}$ and λ and μ be two partitions such that λ/μ is a skew shape and $|\lambda| - |\mu| = n$. Label the skew Young diagram of shape λ/μ with the successive integers of $[n]$ from left to right and bottom to top. Define the partial order $<_{\lambda/\mu}$ on $[n]$ as $i <_{\lambda/\mu} j$ if and only if i lies northwest of j and denote the labelled poset $P_{\lambda/\mu} = ([n], <_{\lambda/\mu})$. As a direct consequence the set of enriched $P_{\lambda/\mu}$-partitions are precisely the marked semistandard Young tableaux of shape λ/μ, i.e. $\mathcal{L}_{\mathbb{P}^\pm}(P_{\lambda/\mu}) = SSYT^\pm(\lambda/\mu)$.

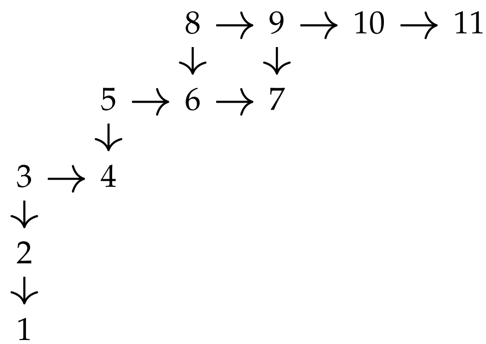

Figure 4: The labelled weighted poset $P_{(6,4,2,1,1)/(2,1)}$.

Theorem 1. *Let $n \in \mathbb{P}$ and λ/μ be a skew shape such that $|\lambda| - |\mu| = n$. The q-deformed generating function of $P_{\lambda/\mu}$ is exactly the Hall-Littlewood S-symmetric function with parameter $t = -q$.*

$$S_{\lambda/\mu}(X; -q) = \Gamma^{(q)}([n], <_{\lambda/\mu}). \tag{2.3}$$

The proof is postponed to Section 3. As a consequence to Theorem 1, we give an explicit quasisymmetric expansion of the Hall-Littlewood S-symmetric functions that is a natural generalisation of Equations (2.1) and (2.2).

Theorem 2. *Let λ/μ be a skew shape. The Hall-Littlewood S-symmetric function with parameter $t = -q$ is related to q-fundamental quasisymmetric functions through*

$$S_{\lambda/\mu}(X; -q) = \sum_{T \in SYT(\lambda/\mu)} L^{(q)}_{Des(T)}(X). \tag{2.4}$$

Proof. Let $n = |\lambda| - |\mu|$. Given a marked semistandard Young tableau $T \in SSYT^{\pm}(\lambda/\mu)$, define its standardisation as the standard tableau $T_0 \in SYT(\lambda/\mu)$ obtained by relabelling the boxes of T with the integers in $[n]$ such that:

- The entries of T and T_0 are in the same relative order
- Identical negative entries of T are relabelled from top to bottom
- Identical positive entries of T are relabelled from left to right.

Denote $T^{st} = T_0$. For instance, in Figure 1, $T_1^{st} = T_2^{st} = T_3$. Further denote $X^{|T|} = \prod_{i \in \mathbb{P}^{\pm}} x_{|i|}^{t_i}$ where t_i is the number of entries equal to i in T. Finally, use Theorem 1 to get

$$S_{\lambda/\mu}(X; -q) = \Gamma^{(q)}(P_{\lambda/\mu}) = \sum_{T \in SSYT^{\pm}(\lambda/\mu)} q^{neg(T)} X^{|T|}$$

$$= \sum_{T_0 \in SYT(\lambda/\mu)} \left(\sum_{T \in SSYT^{\pm}(\lambda/\mu),\ T^{st} = T_0} q^{neg(T)} X^{|T|} \right)$$

End the proof by noticing that the part between parentheses is exactly $L^{(q)}_{Des(T)}(X)$. □

Recall the ring homomorphism θ_t of Definition 2. The following result is a consequence of Theorem 2.

Theorem 3. *There is a ring homomorphism Θ_q: $\mathrm{QSym} \longrightarrow \mathcal{P}^{(q)}$ such that for any positive integer n and any subset $I \subseteq [n-1]$, $\Theta_q\left(L^{(0)}_{n,I}\right) = L^{(q)}_{n,I}$. Then the restriction of Θ_q to Λ is exactly θ_{-q} and the ring map diagram of Figure 5 is commutative.*

$$\begin{array}{ccc} \mathrm{QSym} & \xrightarrow{\Theta_q} & \mathcal{P}^{(q)} \\ \uparrow & & \uparrow \\ \Lambda & \xrightarrow{\theta_{-q}} & \Lambda_{-q} \end{array}$$

Figure 5: Map diagram relating QSym, $\mathcal{P}^{(q)}$, Λ and Λ_{-q}. Vertical maps are inclusion.

Proof. The existence and proper definition of Θ_q is a consequence of Equation (1.5). To end the proof, it suffices to show that for any n non-negative integer, $\Theta_q(h_n)(X) = q_n(X;-q)$. Indeed, one has

$$\begin{aligned} \Theta_q(h_n)(X) &= \Theta_q(s_n)(X) = \Theta_q(L^{(0)}_{n,\emptyset})(X) \\ &= L^{(q)}_{n,\emptyset}(X) = S_n(X;-q) \\ &= q_n(X;-q) \end{aligned}$$

This is the desired result. □

Remark 1. *Applying the morphism Θ_q to both the left-hand and right-hand sides of Equation (2.1) gives an alternative proof that $\theta_t\left(s_{\lambda/\mu}\right)(X) = S_{\lambda/\mu}(X;t)$. Indeed*

$$\Theta_q\left(s_{\lambda/\mu}\right)(X) = \sum_{T \in SYT(\lambda/\mu)} \Theta_q\left(L^{(0)}_{Des(T)}\right)(X) = \sum_{T \in SYT(\lambda/\mu)} L^{(q)}_{Des(T)}(X) = S_{\lambda/\mu}(X;-q)$$

2.2 Cauchy like formula for Hall-Littlewood symmetric functions

We use Theorem 2 to provide an alternative proof of a classical Cauchy like formula for Hall-Littlewood S-symmetric functions. Denote $Y = \{y_1, y_2, \ldots,\}$ an additional alphabet of commutating indeterminate independent of and commuting with X and denote the product alphabet $XY = \{x_i y_j\}_{i,j}$. We first show the following proposition.

Proposition 4. *Let $\pi \in \mathfrak{S}_n$ be a permutation. Extend the definition of $\Gamma^{(q)}$ to the alphabet XY by considering P_π-partitions $(f,g) : i \mapsto (f(i), g(i))$ with value in $\mathbb{P} \times \mathbb{P}^{\pm}$ that we equip with the lexicographic order. Assume also that for $(i,j) \in \mathbb{P} \times \mathbb{P}^{\pm}$, (i,j) is negative if and only if j is negative. We have*

$$\Gamma^{(q)}(P_\pi)(XY) = \sum_{(f,g) \in \mathcal{L}_{\mathbb{P}\times\mathbb{P}^{\pm}}([n],<_\pi)} \prod_{1 \le i \le n} q^{[g(i)<0]} x_{f(i)} y_{|g(i)|}$$

The q-fundamental indexed by π on the product of indeterminate XY satisfies

$$L_\pi^{(q)}(XY) = \Gamma^{(q)}(P_\pi)(XY) = \sum_{\sigma \circ \tau = \pi} L_\sigma^{(0)}(X) L_\tau^{(q)}(Y). \tag{2.5}$$

Proof. The proof is similar to the one in [8, thm 6.11] and not detailed here. □

In [7, III. 4. Eq. (4.7)], Macdonald provides a Cauchy like formula for Hall-Littlewood symmetric functions.

$$q_n(XY;t) = \sum_{\lambda \vdash n} s_\lambda(X) S_\lambda(Y;t). \tag{2.6}$$

Proposition 5. *Equation (2.6) is a direct consequence of Proposition 4 and Theorem 2.*

Proof. Fix $q \in \mathbb{C}$ and use Proposition 4 to write

$$q_n(XY;-q) = L_{id_n}^{(q)}(XY) = \sum_{\sigma \in \mathfrak{S}_n} L_{\sigma^{-1}}^{(0)}(X) L_\sigma^{(q)}(Y),$$

where $id_n \in \mathfrak{S}_n$ is the identity permutation. The RS correspondence allows to reindex the sum over standard Young tableaux.

$$\begin{aligned} q_n(XY;-q) &= \sum_{\lambda \vdash n} \sum_{T,U \in SYT(\lambda)} L_{Des(T)}^{(0)}(X) L_{Des(U)}^{(q)}(Y) \\ &= \sum_{\lambda \vdash n} \left(\sum_{T \in SYT(\lambda)} L_{Des(T)}^{(0)}(X) \right) \left(\sum_{U \in SYT(\lambda)} L_{Des(U)}^{(q)}(Y) \right) \end{aligned}$$

Applying Theorem 2 yields Equation (2.6). □

3 Proof of Theorem 1

Let $\prec$ be a total order on $\mathbb{P}^{\pm}$. Define the binary relation R as follows. For any two elements $i, j \in \mathbb{P}^{\pm}$, set

$$(i \; R \; j) \iff (i \preccurlyeq j \text{ but not } i = j \in -\mathbb{P}).$$

We define a formal power series on the alphabet $X^{\pm} = \{x_{-1}, x_1, x_{-2}, x_2, \dots\}$.

Definition 7. *For each non-negative integer n, define the formal power series*

$$H_n(X^{\pm}) = \sum_{\substack{(i_1,i_2,\ldots,i_n)\in(\mathbb{P}^{\pm})^n; \\ i_1 \ R \ i_2 \ R \ \cdots \ R \ i_n}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

Moreover, set $H_n = 0$ for all $n < 0$.

Define an alternative version of the generating function for enriched P-partitions $\Gamma^{\pm}([n], <_P) \in \mathbb{C}[[X^{\pm}]]$ as

$$\Gamma^{\pm}([n], <_P) = \sum_{f \in \mathcal{L}_{\mathbb{P}^{\pm}}([n], <_P)} \prod_{1 \le i \le n} x_{f(i)}.$$

Recall the homomorphism $\varpi : \mathbb{C}[[X^{\pm}]] \longrightarrow [[X]]$, such that $\varpi(x_i) = q^{[i<0]} x_{|i|}$ for $x_i \in X^{\pm}$. Clearly

$$\varpi(\Gamma^{\pm}([n], <_P)) = \Gamma^{(q)}([n], <_P).$$

Proposition 6. *Let λ and μ be two partitions such that λ/μ is a skew shape. We have*

$$\Gamma^{\pm}([n], <_{\lambda/\mu}) = \det\left(H_{\lambda_i - \mu_j - i + j}\right)_{i,j \in [k]} \tag{3.1}$$

Proof. We want to apply [2, §7]. To this end, we introduce a new relation. Let $\overline{R}$ be the complement of the binary relation R. (Thus, $\overline{R}$ is the binary relation on $\mathbb{P}^{\pm}$ defined by $\left(i \ \overline{R} \ j\right) \iff (\text{not } i \ R \ j)$.) It is easy to see that both relations R and $\overline{R}$ are transitive. Hence, the relation R is semitransitive (meaning that if $a, b, c, d \in \mathbb{P}^{\pm}$ satisfy $a \ R \ b \ R \ c$, then $a \ R \ d$ or $d \ R \ c$). Therefore, [2, Theorem 11] yields that the power series $s^R_{\lambda/\mu}$ (defined in [2, §7]) counts R-tableaux of shape λ/μ. But the R-tableaux of shape λ/μ are precisely the enriched $P_{\lambda/\mu}$-partitions. □

In order to prove Theorem 1 from Equation (3.1), we need to show that for any non-negative integer n, $\varpi(H_n(X^{\pm})) = q_n(X; -q)$. We proceed in three steps. First we have the following proposition.

Proposition 7. *Let $n \in \mathbb{N}$. Then,*

$$H_n = \sum_{k=0}^{n} \sum_{\substack{U \text{ is a size-}k \\ \text{subset of } -\mathbb{P}}} \sum_{\substack{V \text{ is a size-}(n-k) \\ \text{multisubset of } \mathbb{P}}} \left(\prod_{u \in U} x_u\right) \left(\prod_{v \in V} x_v\right)$$

(where the product over $v \in V$ takes each element with its multiplicity). In particular, H_n does not depend on the order $\prec$.

Secondly, we express $q_n(X; -q)$ in terms of elementary and complete homogeneous symmetric functions.

Lemma 1. *Let n be a non-negative integer and $q \in \mathbb{C}$.*

$$q_n(X;-q) = \sum_{k=0}^{n} q^k e_k h_{n-k}. \tag{3.2}$$

Proof. We have

$$\sum_n q_n(X;-q)u^n = \prod_{i\geq 1} \frac{1+qx_iu}{1-x_iu} = \underbrace{\left(\prod_{i\geq 1}(1+qx_iu)\right)}_{=\sum_n q^n e_n t^n} \underbrace{\left(\prod_{i\geq 1}\frac{1}{1-x_iu}\right)}_{=\sum_n h_n u^n}$$
$$= \left(\sum_n q^n e_n u^n\right)\left(\sum_n h_n u^n\right) = \sum_n \left(\sum_{k=0}^{n} q^k e_k h_{n-k}\right) u^n.$$

Extracting coefficients in u^n on both sides yields the desired result. □

Finally, use Proposition 7 and Lemma 1 to relate H_n and q_n.

Proposition 8. *Let $n \in \mathbb{Z}$. Then,*

$$\varpi\left(H_n(X^{\pm})\right) = q_n(X;-q)$$

Proof. From Proposition 7, we know that

$$H_n = \sum_{k=0}^{n} \sum_{\substack{U \text{ is a size-}k \\ \text{subset of } -\mathbb{P}}} \sum_{\substack{V \text{ is a size-}(n-k) \\ \text{multisubset of } \mathbb{P}}} \left(\prod_{u\in U} x_u\right)\left(\prod_{v\in V} x_v\right).$$

Applying the map ϖ to both sides of this equality, we obtain

$$\varpi(H_n) = \sum_{k=0}^{n} \sum_{\substack{U \text{ is a size-}k \\ \text{subset of } -\mathbb{P}}} \sum_{\substack{V \text{ is a size-}(n-k) \\ \text{multisubset of } \mathbb{P}}} \left(\prod_{u\in U} \underbrace{\varpi(x_u)}_{\substack{=qx_{-u} \\ (\text{since } u\in -\mathbb{P})}}\right)\left(\prod_{v\in V} \underbrace{\varpi(x_v)}_{\substack{=x_v \\ (\text{since } v\in \mathbb{P})}}\right)$$
$$(\text{since } \varpi \text{ is a continuous } \mathbf{k}\text{-algebra homomorphism})$$
$$= \sum_{k=0}^{n} \sum_{\substack{U \text{ is a size-}k \\ \text{subset of } -\mathbb{P}}} \sum_{\substack{V \text{ is a size-}(n-k) \\ \text{multisubset of } \mathbb{P}}} \underbrace{\left(\prod_{u\in U}(qx_{-u})\right)}_{\substack{=q^k(\prod_{u\in U} x_{-u}) \\ (\text{since } |U|=k)}} \left(\prod_{v\in V} x_v\right)$$
$$= \sum_{k=0}^{n} q^k \underbrace{\left(\sum_{\substack{U \text{ is a size-}k \\ \text{subset of } -\mathbb{P}}} \prod_{u\in U} x_{-u}\right)}_{=\underbrace{\sum_{\substack{U \text{ is a size-}k \\ \text{subset of } \mathbb{P}}} \prod_{u\in U} x_u}_{=e_k}} \underbrace{\left(\sum_{\substack{V \text{ is a size-}(n-k) \\ \text{multisubset of } \mathbb{P}}} \prod_{v\in V} x_v\right)}_{=h_{n-k}}$$

As a result $\varpi(H_n) = \sum_{k=0}^{n} q^k e_k h_{n-k} = q_n(X; -q)$ by Equation (3.2). $\square$

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