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Charge formulas for Macdonald polynomials at $t = 0$ from multiline queues and diagrams

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Abstract. Multiline queues are combinatorial objects coming from probability theory that give formulas for the q-Whittaker specialization $P_\lambda(X; q, 0)$ of the Macdonald polynomials. We define a charge statistic and an RSK-esque procedure on multiline queues that naturally recovers the Schur expansion of $P_\lambda(X; q, 0)$. We extend these results to *generalized multiline queues*, which are in bijection with binary matrices, and obtain a new family of formulas for $P_\lambda(X; q, 0)$ in terms of these objects. Multiline diagrams are the plethystic analogs of multiline queues that were recently found to give a formula for the modified Hall–Littlewood polynomials $\widetilde{H}_\lambda(X; q, 0)$. We obtain formulas for the latter through a cocharge statistic and an RSK-esque procedure on multiline diagrams.

Keywords: multiline queues, multiline diagrams, Macdonald polynomials, q-Whittaker, Hall–Littlewood, crystal operators, RSK, charge, cocharge.

1 Introduction

Macdonald polynomials $P_\lambda(X; q, t)$ [10] are symmetric functions in the variables $X = x_1, x_2, \ldots$ with coefficients in $\mathbb{Q}(q, t)$. They are indexed by partitions, and characterized as the unique basis satisfying certain triangularity and orthogonality axioms. They contain as specializations the *q-Whittaker polynomials* $P_\lambda(X; q, 0)$, the Hall–Littlewood polynomials $P_\lambda(X; 0, t)$, the Schur functions $s_\lambda = P_\lambda(X; 0, 0)$, and are connected to many other important families of symmetric functions. The *modified Macdonald polynomials* $\widetilde{H}_\lambda(X; q, t)$ were introduced by Garsia and Haiman [6] as a combinatorial version of $P_\lambda(X; q, t)$. They are obtained through plethysm from a scaled form J_λ of P_λ as $\widetilde{H}_\lambda(X; q, t) = t^{n(\lambda)} J_\lambda[X/(1 - t^{-1}); q, t^{-1}]$ (see [7] for details). The *modified Hall–Littlewood polynomial* is the specialization $\widetilde{H}_\lambda(X; q, 0)$ (which is equal to $\widetilde{H}_{\lambda'}(X; 0, q)$).

Expanding the q-Whittaker and the modified Hall–Littlewood polynomials in the Schur basis yields Kotska–Foulkes and modified Kotska–Foulkes coefficients:

$$P_\lambda(X; q, 0) = \sum_{\mu \le \lambda} K_{\lambda\mu}(q, 0) s_\mu(X) \quad \text{and} \quad \widetilde{H}_\lambda(X; q, 0) = \sum_{\mu \le \lambda} \widetilde{K}_{\lambda\mu}(q, 0) s_\mu(X). \tag{1.1}$$

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At $q = 0$, Lascoux and Schützenberger [8] gave a charge formula for $K_{\lambda\mu}(0,t)$ as a sum over semistandard tableaux of shape λ and content μ. The relation to the Kotska–Foulkes coefficients is given by the following set of formulas (see Definition 2.3):

$$K_{\lambda\mu}(0,t) = \sum_{T \in \mathrm{SSYT}(\lambda,\mu)} t^{\mathrm{charge}(T)}, \quad K_{\lambda\mu}(q,0) = K_{\lambda'\mu'}(0,q),$$
$$\widetilde{K}_{\lambda\mu}(0,t) = \widetilde{K}_{\lambda\mu'}(t,0) = t^{n(\mu)} K_{\lambda\mu}(0,1/t) = \sum_{T \in \mathrm{SSYT}(\lambda,\mu)} t^{\mathrm{cocharge}(T)}.$$

In this abstract, we study $P_\lambda(X;q,0)$ and $\widetilde{H}_\lambda(X;q,0)$ through multiline queues and multiline diagrams. Our constructions recover classical results, and provide variations and simplifications of formulas for these polynomials. This abstract is based on [11].

In Section 3, we introduce a weight-preserving RSK-esque procedure on multiline queues from which several classical results immediately follow, including (1.1) and the Cauchy identities. In Section 4, we obtain a new formula for the modified Hall–Littlewood polynomials via a cocharge statistic on multiline diagrams.

Theorem 1.1. *Let λ be a partition. The modified Hall–Littlewood polynomial is given by*

$$\widetilde{H}_\lambda(x_1,\ldots,x_n;q,0) = \sum_{D \in \mathrm{MLD}(\lambda,n)} q^{\widetilde{\mathrm{maj}}(D)} x^D = \sum_{D \in \mathrm{MLD}(\lambda,n)} q^{\mathrm{cocharge}(\widetilde{\mathrm{cw}}(D))} x^D. \tag{1.2}$$

We also get new formulas for $K_{\lambda\mu}(q,0)$ and $\widetilde{K}_{\lambda\mu}(q,0)$, bypassing the charge statistic.

Theorem 1.2. *For a partition ν, let $B_\nu \in \mathrm{MLQ}_0(\nu,\ell(\nu))$ be the multiline queue with all balls left-justified and let $\widetilde{B}_\nu \in \mathrm{MLD}_0(\nu,\ell(\nu))$ be the diagonal multiline diagram of type ν. Then*

$$K_{\lambda\mu}(q,0) = \sum_{\substack{M \in \mathrm{MLQ}(\mu',\lambda') \\ \rho_N(M) = B_{\lambda'}}} q^{\mathrm{maj}(M)} \quad \textit{and} \quad \widetilde{K}_{\lambda\mu}(q,0) = \sum_{\substack{D \in \mathrm{MLD}(\mu',\lambda) \\ \widetilde{\rho}_N(D) = \widetilde{B}_\lambda}} q^{\widetilde{\mathrm{maj}}(D)}.$$

Finally, in Section 5 we extend our results to generalized multiline queues, obtaining a new family of formulas, indexed by compositions, for the q-Whittaker polynomials.

Theorem 1.3. *Let λ be a partition, n an integer, and let α be a composition with $\alpha^+ = \lambda'$. Then*

$$P_\lambda(x_1,\ldots,x_n;q,0) = \sum_{M \in \mathrm{GMLQ}(\alpha,n)} q^{\mathrm{maj}_G(M)} x^M.$$

2 Preliminaries

Definition 2.1. *The* charge *of a permutation $\sigma \in \mathfrak{S}_n$ is* $\mathrm{charge}(\sigma) = \sum_{i \notin Des(\sigma)} (n-i)$, *where* $\mathrm{Des}(\sigma) = \{i : \sigma^{-1}(i) > \sigma^{-1}(i-1)\}$. *The* cocharge *is* $\mathrm{cocharge}(\sigma) = \binom{n}{2} - \mathrm{charge}(\sigma)$.

This definition generalizes to words with partition content by splitting the word into *charge subwords*. Let w be a word with content $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$. Extract the first subword $w^{(1)}$ by scanning w from left to right and finding the first occurrence of its largest letter $k := \mu_1'$, then $k-1, \ldots, 2, 1$, looping back around the word whenever needed. This subword $w^{(1)}$ is then extracted from w, and the remaining charge subwords are obtained recursively from the remaining letters, which now have content $(\mu_1 - 1, \mu_2 - 1, \ldots, \mu_k - 1)$. For each i, $w^{(i)}$ can be thought of as permutations in $\mathfrak{S}_{\mu_i'}$.

Definition 2.2. *For a word w with partition content $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$, its* charge *is given by* $\mathrm{charge}(w) = \mathrm{charge}(w^{(1)}) + \mathrm{charge}(w^{(2)}) + \ldots + \mathrm{charge}(w^{(k)})$.

Definition 2.3. *For a semistandard Young tableau T (in French notation), define its* row reading word, *denoted by* $\mathrm{rrw}(T)$, *to be the word obtained by recording the entries of the rows of T from top to bottom and from left to right within each row. If T has partition content, the* charge *of T is given by* $\mathrm{charge}(T) = \mathrm{charge}(\mathrm{rrw}(T))$.

We will make use of two types of related operations acting on words, described below. See [11] for details on how $\mathrm{charge}(w)$ can be restated in terms of these operations.

Definition 2.4 (Classical and cylindrical matching operators)**.** *Let n be a positive integer and let w be a word in the alphabet $\{1, \ldots, n\}$. For $1 \le i < n$, define $\pi_i(w)$ to be a word in open and closed parentheses $\{(,)\}$ that is obtained by reading w from left to right and recording a "(" for each $i+1$ and a ")" for each i. The signature rule (see, e.g. [3]) is the procedure of iteratively matching pairs of open and closed parentheses whenever they are adjacent or whenever there are only matched parentheses in between. Then $\pi_i(w)$ contains the data of which instances of i and $i+1$ in w are matched or unmatched following the signature rule applied to $\pi_i(w)$.*

Let $\pi_i^c(w)$ represent the word $\pi_i(w)$ on a circle, so that open and closed parentheses may match by wrapping around the word. Then the cylindrically unmatched *$i+1$'s and i's in w correspond respectively to the (cylindrically) unmatched open and closed parentheses in $\pi_i^c(w)$, according to the signature rule executed on a circle. The* wrapping *$i+1$'s and i's in w correspond respectively to the cylindrically matched open and closed parentheses in $\pi_i^c(w)$ that are unmatched in $\pi_i(w)$.*

Example 2.5. *For $w = 312214342131232$, the unmatched parentheses are show in red in $\pi_1(w) =)\,(\,(\,)\,(\,)\,)\,(\,($ and $\pi_1^c(w) =)\,(\,(\,)\,(\,)\,)\,(\,($. This corresponds to the unmatched 1's and 2's indicated by ˆ and the cylindrically unmatched 2 underlined: $w = 3\hat{1}2214342131\underline{\hat{2}}3\hat{2}$.*

3 Multiline queues and charge

The multiline queues we study are the $t = 0$ specialization of the multiline queues and their statistics defined by Corteel, Williams, and the first author in [4], and are in correspondence with the classical multiline queues introduced by Ferrari and Martin [5].

Definition 3.1. *Fix a partition λ, an integer $n \geq \ell(\lambda)$, and set $L := \lambda_1$. A* multiline queue *of* shape (λ, n) *is an arrangement of balls on an array with L rows numbered 1 through L from bottom to top and n columns numbered 1 through n from left to right, such that row j contains λ'_j balls. Denote the set of multiline queues of shape (λ, n) by $\mathrm{MLQ}(\lambda, n)$.*

A multiline queue can be viewed as a binary matrix by corresponding balls to 1's and vacancies to 0's. We represent a multiline queue as a tuple $M = (B_1, \ldots, B_L)$ of L subsets of $\{1, \ldots, n\}$ where $B_j = (b_1, \ldots, b_{\lambda'_j})$ is the set of labels of columns containing balls in row j of M. A site (r, j) of M refers to the cell in column j of row r of M; we say the site is empty if $j \notin B_r$, and contains a ball otherwise.

Definition 3.2. *The* column word *of a multiline queue M, denoted $\mathrm{cw}(M)$, is obtained by recording the row number of each ball by scanning the columns of M from* left to right *and from* top to bottom *within each column. For M in Example 3.14,* $\mathrm{cw}(M) = 421|3|41|521|32$.

Definition 3.3. *Let $n > 0$ and $S, T \subseteq [n]$, where we shall consider (S, T) as rows 1 and 2 of a multiline queue. Then $\pi(S, T) = \pi_1(\mathrm{cw}(S, T))$. Unmatched open parenthesis are referred to as* unmatched above *elements, and unmatched closed parenthesis are* unmatched below.

Example 3.4. *Let $S = \{2, 3, 5\}$ and $T = \{1, 4, 5, 6\}$ corresponding to rows 1 and 2 from B in Example 3.18. Then $\pi(S, T) = (\,)\,)\,(\,(\,)\,($ where the unmatched parentheses are in red, corresponding to $4, 6 \in T$ unmatched above and $3 \in S$ unmatched below.*

The Ferrari–Martin pairing process is an algorithm that deterministically assigns a label to each ball in a multiline queue M to obtain a labelled multiline queue $L(M)$.

Definition 3.5 (Ferrari–Martin algorithm)**.** *Let $M = (B_1, \ldots, B_L)$ be a multiline queue of shape (λ, n). Define the* labelled multiline queue $L(M)$ *by replicating M and sequentially labelling the balls, as follows. For each row r for $r = L, L-1, \ldots, 2$, each unlabelled ball in B_r is labelled r. Next, for $\ell = L, L-1, \ldots, r$, let $\mathrm{cw}(M)^{(r,\ell)}$ be the restriction of $\mathrm{cw}(M)$ to the balls labelled ℓ in B_r and the unlabelled balls in B_{r-1}. The balls in row $i-1$ that are cylindrically matched in $\pi^c_{r-1}(\mathrm{cw}(M)^{(r,\ell)})$ acquire the label ℓ. To complete the process, all unpaired balls in row 1 are labelled "1". Such a labelling is shown in Example 3.14.*

Definition 3.6. *Let $M \in \mathrm{MLQ}(\lambda, n)$ with labelling $L(M)$, and let $m_{r,\ell}$ be the number of wrapping balls labelled ℓ when cylindrically matched from row r to row $r-1$ in $L(M)$. Then*

$$\mathrm{maj}(M) = \sum_{2 \leq r \leq L} \sum_{r \leq \ell \leq L} m_{r,\ell}\,(\ell - r + 1).$$

When we restrict to multiline queues with major index equal to zero we obtain a set of objects that is in bijection with semistandard tableaux [11].

Definition 3.7. *If M satisfies* $\mathrm{maj}(M) = 0$*, we call it* non-wrapping. *We will denote the set of non-wrapping multiline queues of shape (λ, n) by $\mathrm{MLQ}_0(\lambda, n)$.*

We have an expression for Schur functions in terms of multiline queues [4].

$$s_\lambda(x_1, \ldots, x_n) = \sum_{M \in \mathrm{MLQ}_0(\lambda,n)} x^M. \tag{3.1}$$

Theorem 3.8. *Let M be a multiline queue. Then* $\mathrm{maj}(M) = \mathrm{charge}(\mathrm{cw}(M))$.

Notably, the theorem above eliminates the need for the Ferrari–Martin algorithm to determine $\mathrm{maj}(M)$. Thus we obtain the following formula for $P_\lambda(X; q, 0)$.

Theorem 3.9. *Let λ be a partition. The q-Whittaker polynomial is given by*

$$P_\lambda(x_1, \ldots, x_n; q, 0) = \sum_{M \in \mathrm{MLQ}(\lambda,n)} q^{\mathrm{maj}(M)} x^M = \sum_{M \in \mathrm{MLQ}(\lambda,n)} q^{\mathrm{charge}(\mathrm{cw}(M))} x^M \tag{3.2}$$

where the first equality is due to [4].

3.1 Collapsing on multiline queues via row operators

Let $\mathcal{M}_{(2)}$ be the set of binary matrices with finite support, and let $\mathcal{M}_{(2)}(L, n)$ be the set of such matrices with size $L \times n$. For $B \in \mathcal{M}_{(2)}(L, n)$ and every $1 \leq j \leq L$, let $B_j \subseteq [n]$ be the set of column labels of the balls (1's) of row j of B.

Definition 3.10. *Let $B \in \mathcal{M}_{(2)}$. The* dropping operator *e_i acts on B by dropping the ball corresponding to the leftmost unmatched above element in $\pi(B_i, B_{i+1})$ from B_{i+1} to B_i. Define $e_i^\star(B)$ to drop all balls that are unmatched above from B_{i+1} to B_i. By definition, $e_i(e_i^\star) = e_i^\star$.*

For $M \in \mathrm{MLQ}(\lambda, n)$, the operators e_i act on M as the classical crystal operators E_i (the standard lowering crystal operators in type A on words; see [3]) act on $\mathrm{cw}(M)$, so that $\mathrm{cw}(e_i(M)) = E_i(\mathrm{cw}(M))$. Moreover, the operators $e_i^\star$, which maximally apply e_i, satisfy the braid relations (i) $e_i^\star e_{i+1}^\star e_i^\star = e_{i+1}^\star e_i^\star e_{i+1}^\star$, and (ii) $e_i^\star e_j^\star = e_j^\star e_i^\star$ whenever $|i - j| \geq 2$. Applying the operators $e_i^\star$ from bottom to top defines a procedure that we call collapsing.

Definition 3.11. *For a pair of integers a and b with $a \leq b$, let $[a, b]$ be the interval of integers. Define $e_{[a,b]}^\star := e_a^\star e_{a+1}^\star \cdots e_b^\star$, where we use multiplicative notation for composition.*

Definition 3.12 (Collapsing). *Let $L, n > 0$. Define the* collapsing *map on $\mathcal{M}_{(2)}(L, n)$ as*

$$\rho \ : \ \mathcal{M}_{(2)}(L, n) \longrightarrow \bigcup_{\mu} \mathrm{MLQ}_0(\mu, n) \times \mathrm{SSYT}(\mu') \tag{3.3}$$

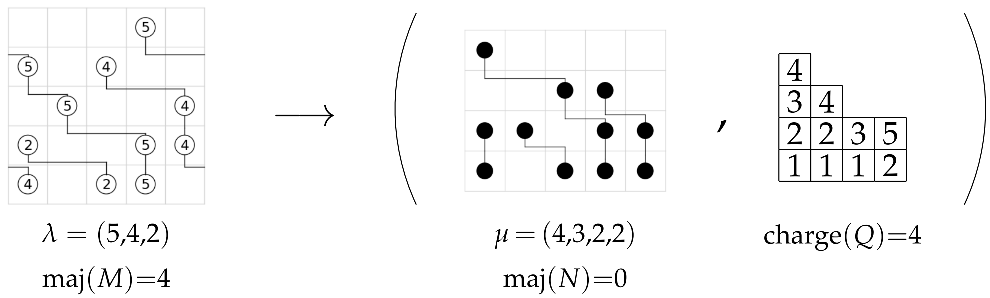
$$B \longmapsto (\rho_N(B), \rho_Q(B)) \tag{3.4}$$

where $\rho_N(B)$ is given by $\rho_N(B) = e_{[1,L-1]}^\star e_{[1,L-2]}^\star \cdots e_{[1,2]}^\star e_{[1,1]}^\star(B)$, and $\rho_Q(B)$ is the semistandard tableaux whose entries i record the difference in row content between $e_{[1,i]}^\star e_{[1,i-1]}^\star \cdots e_{[1,1]}^\star(B)$ and $e_{[1,i-1]}^\star e_{[1,i-2]}^\star \cdots e_{[1,1]}^\star(B)$ for $2 \leq i \leq L - 1$, and between $e_{[1,1]}^\star(B)$ and B for $i = 1$.

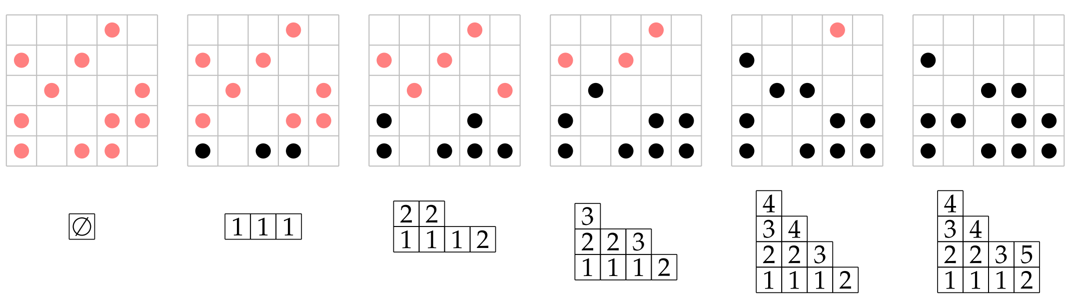
Restricting the previous map to the set of multiline queues $\mathrm{MLQ}(\lambda, n)$ yields a bijection to pairs of a non-wrapping multiline queue on n columns and a semistandard tableau of the conjugate shape with content λ'. By taking the preimage of $\{B_{\lambda'}\} \times \mathrm{SSYT}(\lambda, \mu)$ under this map we obtain Theorem 1.2 using the following result.

Theorem 3.13. *Let $M \in \mathrm{MLQ}(\lambda, n)$ be a multiline queue. Then $\mathrm{maj}(M) = \mathrm{charge}(\rho_Q(M))$.*

Example 3.14. *We show the collapsing $\rho(M) = (N, Q)$ of a multiline queue $M \in \mathrm{MLQ}(\lambda, 5)$ with $\lambda = (5, 4, 2)$.*



The step-by-step collapsing of the rows from bottom to top is shown, where the black balls are collapsed particles and the red/shaded balls are the remaining rows of the starting multiline queue, along with the recording tableaux corresponding to each step.



3.2 Multiline queue RSK

In [9], commuting crystal operators on rows and columns of integer matrices are introduced to recover some classical tableaux operations such as the RSK correspondence and jeu de taquin. These operators correspond to bi-directional collapsing in the setting of multiline queues (and in Section 4.2, multiline diagrams).

Definition 3.15. *For a matrix $B \in \mathcal{M}_{(2)}$, define $\mathrm{rot}(B)$ to be the rotation of B by $90°$ counterclockwise. We use the same notation to describe the rotation of a multiline queue M by identifying it with its associated binary matrix. We define $e_i^{\downarrow} = e_i$ from Definition 3.10 and $e_i^{\leftarrow} = \mathrm{rot}^{-1} \circ e_i \circ \mathrm{rot}$ as the operator that drops unmatched balls to the left. We also define $\rho^{\downarrow}(B) := \rho_N(B)$, and $\rho^{\leftarrow}(B) := \mathrm{rot}^{-1}(\rho_N(\mathrm{rot}(B)))$. See Example 3.18.*

Theorem 3.16. *Let $P(L, n)$ be the set of partitions λ with $\ell(\lambda) \leq n$ and $\ell(\lambda') \leq L$. The map*

$$\mathrm{mRSK} \; : \; \mathcal{M}_{(2)}(L, n) \longrightarrow \bigcup_{\lambda \in P(L,n)} \mathrm{MLQ}_0(\lambda, n) \times \mathrm{MLQ}_0(\lambda', L)$$

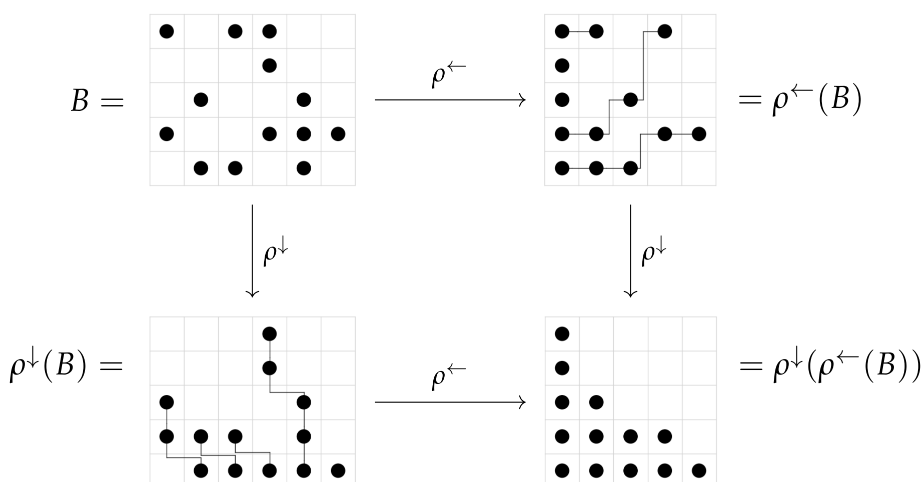
given by $\mathrm{mRSK}(B) = (\rho^{\downarrow}(B), \rho^{\leftarrow}(B))$ *is a bijection.*

The following fact can be obtained from [9, Lemma 1.3.7].

Lemma 3.17. *Let $B \in \mathcal{M}_{(2)}$. Then $e_i^{\downarrow}(e_j^{\leftarrow}(B)) = e_j^{\leftarrow}(e_i^{\downarrow}(B))$ for all i and j. Moreover, if B is a multiline queue,* $\mathrm{maj}(e_i^{\leftarrow}(B)) = \mathrm{maj}(B)$.

The previous lemma implies that $\rho^{\downarrow}(\rho^{\leftarrow}(B)) = \rho^{\leftarrow}(\rho^{\downarrow}(B))$. Since the major index is preserved while collapsing to the left when $M \in \mathrm{MLQ}(\lambda)$, examining the construction of the recording tableau $\rho_Q(\rho^{\leftarrow}(M))$ leads to a simple proof of Theorem 3.13. Furthermore, Theorem 3.16 gives a bijective proof of the dual Cauchy identity in view of Equation (3.1): $\sum_{\lambda} s_{\lambda}(X) s_{\lambda'}(Y) = \prod_{i,j}(1 + x_i y_j)$.

Example 3.18. *For the matrix $B \in \mathcal{M}_{(2)}(5, 6)$ in the upper left, we show $\rho^{\leftarrow}(B)$ in the upper right, $\rho^{\downarrow}(B)$ in the bottom left, and the double collapsing in the bottom right.*



4 Multiline diagrams and cocharge

A *multiline diagram* is a configuration of balls on a rectangular grid with no restriction on the number of balls occupying each cell, and such that the number of balls in each row is weakly decreasing from bottom to top. Multiline diagrams have appeared in the context of a family of statistical mechanics processes called the *totally asymmetric zero range process* (see [2]). They are also in bijection with inversion-free Haglund–Haiman–Loehr tableaux [7] and in (weight preserving) bijection with queue-inversion-free tableaux [2], which give formulas for the modified Hall–Littlewood polynomials. Thus, as a reference to the plethystic correspondence between the q-Whittaker polynomials $P_\lambda(X;q,0)$ and the modified Hall–Littlewood polynomials $\widetilde{H}_\lambda(X;q,0)$, we think of multiline diagrams as the *plethystic analog* of multiline queues.

Definition 4.1. *Let λ be a partition and $n > 0$. A* multiline diagram *of shape (λ, n) is a configuration of particles on a $\lambda_1 \times n$ grid, such that each site can contain any number of particles, and row j contains λ'_j particles (labelled from bottom to top). Denote the set of multiline diagrams of type (λ, n) by* $\mathrm{MLD}(\lambda, n)$.

We represent a multiline diagram by the tuple $D = (D_1, \ldots, D_{\lambda_1})$, where each D_i is a *multiset* of $[n]$ of size λ'_i

Definition 4.2. *For a word $w = w_1 \ldots w_n$, define $\mathrm{rev}(w) = w_n \ldots w_1$. Define the multiline diagram column reading word as $\widetilde{\mathrm{cw}}(D) := \mathrm{rev}(\mathrm{cw}(D))$, where $\mathrm{cw}(D)$ is given by the multiline queue reading order. See Example 4.11 for reference.*

Definition 4.3. *Let $n > 0$ and let S, T be multisets in $[n]$; we shall consider (S, T) as rows 1 and 2 of a multiline diagram. Then $\widetilde{\pi}(S, T) = \pi(\widetilde{\mathrm{cw}}(S, T))$.*

Example 4.4. *Let $S = \{2, 3, 3, 3\}$ and $T = \{1, 3, 4, 4\}$, corresponding to the second and third rows of Example 4.11. Then $\widetilde{\pi}(S, T) = (\,(\,)\,)\,)\,(\,)\,($ where the unmatched parentheses are in red, corresponding to $1 \in T$ unmatched above and $3 \in S$ unmatched below.*

There is a pairing process on multiline diagrams, where particles are paired *strictly to the left*, that is analogous to the Ferrari–Martin algorithm and produces a major index statistic. See Example 4.11.

Definition 4.5 (Major index for multiline diagrams)**.** *The* major index *of a multiline diagram D, denoted by $\widetilde{\mathrm{maj}}(D)$, is determined by the non-wrapping pairings. Let $m_{r,\ell}(D)$ be the number of balls labelled ℓ that wrap when matched from row r to row $r-1$. Then*

$$\widetilde{\mathrm{maj}}(D) = \sum_{r,\ell} (\lambda'_r - m_{r,\ell})(r - \ell + 1).$$

The following lemma implies one of our main results, Theorem 1.1.

Lemma 4.6. *Let $D = (D_1, \ldots, D_L)$ be a multiline diagram. Then $\widetilde{\mathrm{maj}}(D) = \mathrm{cocharge}(\widetilde{\mathrm{cw}}(D))$.*

The lemma follows from the same argument as the proof of Theorem 3.8 with an appropriate modification to the parentheses matching algorithm.

4.1 Collapsing on multiline diagrams via row operators

Definition 4.7. *A multiline diagram $D \in \mathrm{MLD}(\lambda, n)$ is called* non-wrapping *if there are no wrapping pairings between any pair of rows. Denote the set of non-wrapping multiline diagrams by $\mathrm{MLD}_0(\lambda, n)$. Note that these multiline diagrams satisfy $\widetilde{\mathrm{maj}}(D) = n(\lambda')$.*

For each non-wrapping multiline diagram $D \in \mathrm{MLD}_0(\lambda, n)$, there is a unique semistandard tableau in $\mathrm{SSYT}(\lambda', n)$ whose row contents match those of D, implying that

$$s_\lambda(x_1, \dots, x_n) = \sum_{D \in \mathrm{MLD}_0(\lambda', n)} x^D. \tag{4.1}$$

Definition 4.8. *Let $D = (D_1, \dots, D_L)$ be a multiline diagram. The* dropping operator *$\widetilde{e}_i$ acts on D by moving the rightmost element unmatched above in $\widetilde{\pi}(D_i, D_{i+1})$ from D_{i+1} to D_i. The operator $\widetilde{e}_i^\star$ is defined as the operator that maximally applies $\widetilde{e}_i$, as an analog to Definition 3.10.*

Let $\mathcal{M}$ denote the set of nonnegative matrices with finite support, and let $\mathcal{M}(L, n)$ be the subset of such matrices of on L rows and n columns.

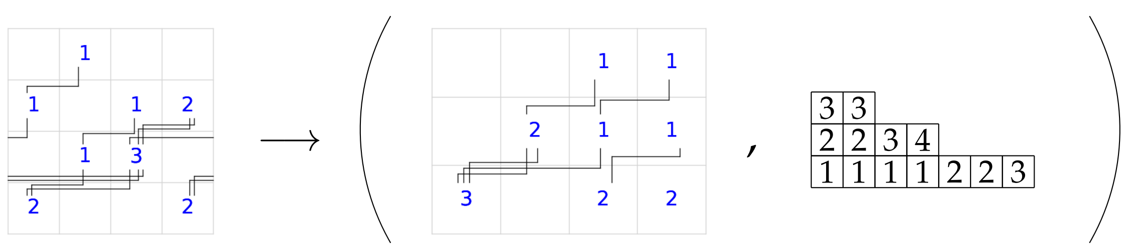
Definition 4.9 (Collapsing)**.** *Let L, n be positive integers. In analogy to Definition 3.12, define* collapsing *for nonnegative integer matrices $\mathcal{M}(L, n)$ by $\widetilde{\rho}(D) = (\widetilde{\rho}_N(D), \widetilde{\rho}_Q(D))$:*

$$\widetilde{\rho} \;:\; \mathcal{M}(L, n) \longrightarrow \bigcup_{\mu} \mathrm{MLD}_0(\mu, n) \times \mathrm{SSYT}(\mu'). \tag{4.2}$$

Restricting the collapsing map to the set of multiline diagrams $\mathrm{MLD}(\lambda, n)$, $\widetilde{\rho}$ yields a bijection to pairs of non-wrapping multiline diagrams and semistandard tableau of the conjugate shape with content λ. By taking the preimage of $\{\widetilde{B}_\lambda\} \times \mathrm{SSYT}(\lambda, \mu)$, we obtain the formula for $\widetilde{K}_{\lambda\mu}(q, 0)$ from Theorem 1.2 using the following theorem.

Theorem 4.10. *Let $D \in \mathrm{MLD}(\lambda, n)$ be a multiline diagram. Then $\widetilde{\mathrm{maj}}(D) = \mathrm{cocharge}(\widetilde{\rho}_Q(D))$.*

Example 4.11. *The collapsing of $D = (\{1,1,4,4\}, \{2,3,3,3\}, \{1,3,4,4\}, \{2\}) \in \mathrm{MLD}(\lambda, 4)$ with $\lambda = (4,3,3,3)$ is shown, with integers representing the number of particles at each site.*



$\widetilde{\mathrm{maj}}(D)=9$ $\quad$ $\mu = (3,3,2,2,1,1,1)$ $\quad$ $\mathrm{cocharge}(\widetilde{\rho}_Q(D)) = 9$

$\widetilde{\mathrm{cw}}(D)$=1133.2223.24.113

4.2 Multiline diagram RSK

Recall that multiline diagram pairing is done strictly to the left. With the appropriate modification on the construction of the parenthesis word $\widetilde{\pi}(S,T)$ we can set the pairing direction to be strictly to the right to get an equivalent set of objects.

Definition 4.12. *Let $P \in \{L, R\}$ (left or right) be a direction of pairing. The set of multiline diagrams of shape (λ, n) with pairing direction P is denoted by $\mathrm{MLD}_P(\lambda, n)$. Similarly, the set of non-wrapping multiline diagrams with pairing in direction P is denoted by $\mathrm{MLD}_{0,P}(\lambda, n)$.*

Definition 4.13. *For a pairing direction $P \in \{L, R\}$, let $\tilde{e}^{\downarrow}_{P,i}$ be the operator acting on matrices $D \in \mathcal{M}$ that drops the ball that is furthest in the opposite direction of P among balls in row $i+1$ that are unmatched above. Extending the definition of the rotation operators* rot *on matrices, we similarly define leftward operators $\tilde{e}^{\leftarrow}_{P,i} = \mathrm{rot}^{-1} \circ \tilde{e}^{\downarrow}_{P,i} \circ \mathrm{rot}$.*

The interplay between the pairing and collapsing directions plays an important role when defining the RSK analog for multiline diagrams. In particular, opposite pairing directions are required for the following crucial lemma to hold.

Lemma 4.14. *Let $D \in \mathcal{M}$. Then $\tilde{e}^{\downarrow}_{L,i}(\tilde{e}^{\leftarrow}_{R,j}(D)) = \tilde{e}^{\leftarrow}_{R,j}(\tilde{e}^{\downarrow}_{L,i}(D))$ for all i and j.*

From the definition of $\tilde{e}^{\downarrow}_{P,i}$ and in analogy to the presented collapsing procedures, collapsing downwards and leftwards with pairing direction left and right can be defined from these operators and from the rotation operator.

Theorem 4.15. *Let L, n be positive integers, and let $\mathcal{M}(L, n)$ represent the set of $L \times n$ nonnegative integer matrices. The following map, given by $\mathrm{dRSK}(B) = (\widetilde{\rho}^{\downarrow}_L(B), \widetilde{\rho}^{\leftarrow}_R(B))$, is a bijection:*

$$\mathrm{dRSK} \; : \; \mathcal{M}(L, n) \longrightarrow \bigcup_{\lambda \,:\, \lambda_1 \leq \min\{L,n\}} \mathrm{MLD}_{0,L}(\lambda, n) \times \mathrm{MLD}_{0,R}(\lambda, L).$$

This theorem, together with Equation (4.1), gives a bijective proof, using multiline diagrams, of the Cauchy identity $\sum_\lambda s_\lambda(X)s_\lambda(Y) = \prod_{i,j}(1 - x_i y_j)^{-1}$. Moreover, by Theorem 4.15 and the fact that multiline queues and diagrams are in bijection [11], we have the following formulations of the (q,t)-Kostka polynomials at $t = 0$ in terms of multiline queues and multiline diagrams.

Corollary 4.16. *The (modified) Kostka polynomial at $t = 0$ is given by*

$$K_{\lambda\mu}(q, 0) = \sum_{M \in \mathrm{MLQ}_0(\lambda,\mu)} q^{\mathrm{maj}(\mathrm{rot}(M))} \quad \textit{and} \quad \widetilde{K}_{\lambda\mu}(q, 0) = \sum_{D \in \mathrm{MLD}_0(\lambda',\mu)} q^{\widetilde{\mathrm{maj}}(\mathrm{rot}(D))}.$$

5 Generalized multiline queues

A generalized multiline queue is a multiline queue in which we relax the condition that the number of balls in each row must be weakly decreasing from bottom to top.

Denote by α^+ the partition obtained by rearranging the parts of the composition α.

Definition 5.1. *Let λ be a partition, α a composition such that $\alpha^+ = \lambda'$, and $n > \ell(\lambda)$ a positive integer. A* generalized multiline queue *of type (α, n) is a tuple of subsets $(B_1, \ldots, B_L)$ such that $B_j \subseteq [n]$ and $|B_j| = \alpha_j$ for $1 \leq j \leq L$. Denote the set of generalized multiline queues corresponding to a composition α by $\mathrm{GMLQ}(\alpha, n)$. Then $\mathrm{MLQ}(\lambda, n) = \mathrm{GMLQ}(\lambda', n)$.*

In generalized multiline queues we consider the vacancies to be "anti-particles". There is a pairing algorithm that generalizes the Ferrari–Martin procedure by sequentially assigning labels to both the particles and the anti-particles in M, by pairing sites between adjacent rows from top to bottom such that *particles are paired weakly to the right*, while *anti-particles are paired weakly to the left*, and propagating the labels upon pairing. This is done in a certain priority order: see [1, Section 2] for the details of the procedure. When applied to a (regular) multiline queue, the labelling of the particles coincides with that in Definition 3.5.

Definition 5.2. *Let $M \in \mathrm{GMLQ}(\alpha, n)$ with an associated labelling. For $1 \leq r, \ell \leq L$, let $m_{r,\ell}$ (*resp. *$a_{r,\ell}$) be the number of particles (*resp. *anti-particles) of type ℓ that wrap when pairing to the right (*resp. *left) from row r to row $r-1$, as shown in Example 5.5. Define*

$$\mathrm{maj}_G(M) = \sum_{1 \leq r, \ell \leq L} m_{r,\ell}(\ell - r + 1) - a_{r,\ell}(\ell - r + 1).$$

When $M \in \mathrm{MLQ}(\lambda, n)$, every anti-particle at row r is labelled $r-1$, so $\mathrm{maj}_G(M) = \mathrm{maj}(M)$.

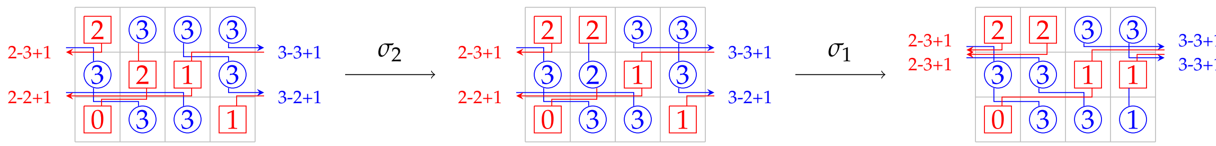
In [1], a row-swapping involution acting on GMLQ is defined to show that certain statistics and distributions are preserved between the set $\mathrm{GMLQ}(\alpha)$ and the set $\mathrm{MLQ}(\lambda)$, where $\alpha^+ = \lambda$. We generalize the result of [1] by showing that the distribution of the maj_G statistic is also preserved, thus recovering Theorem 1.3, which is a formula for $P_\lambda(x_1, \ldots, x_n; q, 0)$ as a sum over $\mathrm{GMLQ}(\alpha, n)$ where $\alpha^+ = \lambda'$.

Definition 5.3. *For $(B_1, \ldots, B_L) \in \mathrm{GMLQ}(\alpha, n)$ and $1 \leq i \leq L-1$, define the involution σ_i by* exchanging *cylindrically unmatched particles in $\pi_i^c(\mathrm{cw}(B_i, B_{i+1}))$ between B_i and B_{i+1}.*

Proposition 5.4. *Let α be a composition with $\alpha^+ = \lambda'$, $L := \ell(\alpha)$, $M \in \mathrm{GMLQ}(\alpha)$, and let $1 \leq i \leq L-1$. Then $\rho^{\downarrow}(M) = \rho^{\downarrow}(\sigma_i(M))$ and $\mathrm{maj}_G(M) = \mathrm{maj}_G(\sigma_i(M))$.*

Since the σ_i's satisfy the Moore–Coxeter relations and $\mathrm{maj}_G(M) = \mathrm{maj}(M)$ when $M \in \mathrm{MLQ}(\lambda)$, we obtain Theorem 1.3.

Example 5.5. *We show the labelled anti-particles (squares) and particles (circles) corresponding to* $M = (\{2,3\},\{1,4\},\{2,3,4\}) \in \mathrm{GMLQ}((2,2,3),4)$*,* $\sigma_2(M) = (\{2,3\},\{1,2,4\},\{3,4\}) \in \mathrm{GMLQ}((2,3,2),4)$ *and* $\sigma_1(\sigma_2(M)) = (\{2,3,4\},\{1,2\},\{3,4\}) \in \mathrm{GMLQ}((3,2,2),4)$*. We show the positive and negative contributions to* maj_G *for each, totalling* $\mathrm{maj}_G = 2$ *in each case.*



If M is a multiline queue, $\mathrm{maj}(M)$ can be computed directly from $\mathrm{charge}(\mathrm{cw}(M))$, bypassing the Ferrari–Martin procedure. There is a natural question of whether one could compute charge directly from a GMLQ without the anti-particles, and without the operators σ_i. This would allow us to define charge on generalized MLDs to get analogous results for $\widetilde{H}_\lambda(X;q,0)$.

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