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Vines and MAT-labeled graphs

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Abstract. The present note explores a connection between two concepts arising from different fields of mathematics. The first concept, called vine, is a graphical model for dependent random variables. This concept first appeared in a work of Joe (1994), and the formal definition was given later by Cooke (1997). Vines have nowadays become an active research area whose applications can be found in probability theory and uncertainty analysis. The second concept, called MAT-freeness, is a combinatorial property in the theory of freeness of logarithmic derivation module of hyperplane arrangements. This concept was first studied by Abe-Barakat-Cuntz-Hoge-Terao (2016), and soon afterwards investigated further by Cuntz-Mücksch (2020).

In the particular case of graphic arrangements, the last two authors (2023) recently proved that the MAT-freeness is completely characterized by the existence of certain edge-labeled graphs, called MAT-labeled graphs. In this paper, we first introduce a poset characterization of a vine. Then we show that, interestingly, there exists an explicit equivalence between the categories of locally regular vines and MAT-labeled graphs. In particular, we obtain an equivalence between the categories of regular vines and MAT-labeled complete graphs.

Several applications will be mentioned to illustrate the interaction between the two concepts. Notably, we give an affirmative answer to a question of Cuntz-Mücksch that MAT-freeness can be characterized by a generalization of the root poset in the case of graphic arrangements.

Keywords: MAT-labeling, graph, poset, vine copula, hyperplane arrangement, MAT-freeness

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1 Motivation

The starting point of our note is a question of Cuntz-Mücksch [8] (Question 1.3) in the theory of *free hyperplane arrangements*.

Let V be a finite dimensional vector space. A **hyperplane** in V is a 1-codimensional linear subspace of V. Let $\{x_1, \ldots, x_\ell\}$ be a basis for the dual space V^*. Any hyperplane in V can be described by a linear equation of the form $a_1x_1 + \cdots + a_\ell x_\ell = 0$ where at least one of the a_i's is non-zero.

A **hyperplane arrangement** $\mathcal{A}$ is a finite set of hyperplanes in V. The **intersection lattice** of $\mathcal{A}$ is the set of all intersections of hyperplanes in $\mathcal{A}$, which is often referred to as the **combinatorics** of $\mathcal{A}$. An arrangement is said to be **free** if its *module of logarithmic derivations* is a free module. For basic definitions and properties of free arrangements, we refer the interested reader to [17, 14]. Freeness is an algebraic property of hyperplane arrangements which has been a major topic of research since the 1970s. A central question in the theory is to study the freeness of an arrangement by combinatorial structures, especially by the intersection lattice of the arrangement.

Among others, *MAT-freeness* is an important concept which was first used by Abe-Barakat-Cuntz-Hoge-Terao [1] to settle the conjecture of Sommers-Tymoczko [15] on the freeness of *ideal subarrangements of Weyl arrangements*. This concept is formally defined later by Cuntz-Mücksch [8] and we will use their definition throughout. For a hyperplane $H \in \mathcal{A}$, define the **restriction** $\mathcal{A}^H$ of $\mathcal{A}$ to H by

$$\mathcal{A}^H := \{K \cap H \mid K \in \mathcal{A} \setminus \{H\}\}.$$

Definition 1.1 (MAT-partition and MAT-free arrangement [8])**.** Let $\mathcal{A}$ be a nonempty arrangement. A partition (disjoint union of nonempty subsets) $\pi = (\pi_1, \ldots, \pi_n)$ of $\mathcal{A}$ is called an **MAT-partition** if the following three conditions hold for every $1 \le k \le n$.

1. The hyperplanes in π_k are linearly independent.
2. There does not exist $H' \in \mathcal{A}_{k-1}$ such that $\bigcap_{H \in \pi_k} H \subseteq H'$, where $\mathcal{A}_{k-1} := \pi_1 \sqcup \cdots \sqcup \pi_{k-1}$ (disjoint union) and $\mathcal{A}_0 := \emptyset$ is the empty arrangement.
3. For each $H \in \pi_k$, $|\mathcal{A}_{k-1}| - |(\mathcal{A}_{k-1} \cup \{H\})^H| = k - 1$.

An arrangement is called **MAT-free** if it is empty or admits an MAT-partition.

As the name suggests, any MAT-free arrangement is a free arrangement. This follows from the remarkable Multiple Addition Theorem by Abe-Barakat-Cuntz-Hoge-Terao [1, Theorem 3.1] (justifying the abbreviation MAT). MAT-freeness is a helpful combinatorial tool (as it depends only on the intersection lattice) to examine the freeness of arrangements. One of its most famous applications we mentioned earlier is a proof that the ideal subarrangements of Weyl arrangements are free. The MAT-freeness has received increasing attention in recent years, see [2, 3, 13, 7] for some other applications.

Let $V = \mathbb{R}^\ell$ with the standard inner product $(\cdot, \cdot)$. Let Φ be an irreducible (crystallographic) root system in V, with a fixed positive system $\Phi^+ \subseteq \Phi$ and the associated set of simple roots $\Delta := \{\alpha_1, \ldots, \alpha_\ell\}$. For $\alpha \in \Phi$, define $H_\alpha := \{x \in V \mid (\alpha, x) = 0\}$. For $\Sigma \subseteq \Phi^+$, the **Weyl subarrangement** $\mathcal{A}_\Sigma$ is defined by $\mathcal{A}_\Sigma := \{H_\alpha \mid \alpha \in \Sigma\}$. In particular, $\mathcal{A}_{\Phi^+}$ is called the **Weyl arrangement**.

We can make Φ^+ into a *poset* (partially ordered set) by defining a partial order $\leq$ on Φ^+ as follows: $\beta_1 \leq \beta_2$ if $\beta_2 - \beta_1 \in \sum_{i=1}^{\ell} \mathbb{Z}_{\geq 0}\alpha_i$. The poset $(\Phi^+, \leq)$ is called the **root poset** of Φ. For an *ideal* I (Definition 2.7) of the root poset Φ^+, the corresponding Weyl subarrangement $\mathcal{A}_I$ is called the **ideal subarrangement**.

Theorem 1.2 ([1, Theorem 1.1])**.** *Any ideal subarrangement $\mathcal{A}_I$ is MAT-free, hence free.*

The ideal subarrangements form a significant subclass of MAT-free arrangements. However, there are many MAT-free arrangements (or MAT-partitions of a given MAT-free arrangement) that do not arise from ideal subarrangements (Example 3.7). One may wonder if the hyperplanes in an arbitrary MAT-free arrangement satisfy some poset structure similar to the root poset? This question was asked by Cuntz-Mücksch [8] and is the main motivation of our work.

Question 1.3 ([8, Problem 47])**.** *Given an MAT-free arrangement $\mathcal{A}$, can we characterize all possible MAT-partitions of $\mathcal{A}$ by a poset structure generalizing the classical root poset?*

Cuntz-Mücksch's question is difficult in general as the number of different MAT-partitions of a given MAT-free arrangement might be very large. Also, the definition of an MAT-partition itself does not reveal a natural choice of the desirable partial order. In the present note, we pursue this question along *graphic arrangements*, a well-behaved class of arrangements in which both freeness and MAT-freeness are completely characterized by combinatorial properties of graphs.

Let G be a simple graph (i.e. no loops and no multiple edges) with vertex (or node) set $N_G = \{v_1, \ldots, v_\ell\}$ and edge set E_G. The **graphic arrangement** $\mathcal{A}_G$ is an arrangement in an ℓ-dimensional vector space V defined by

$$\mathcal{A}_G := \{x_i - x_j = 0 \mid \{v_i, v_j\} \in E_G\}.$$

A graph is **chordal** if it does not contain an induced *cycle* of length greater than three. A chordal graph is **strongly chordal** if it does not contain a *sun graph* as an induced subgraph. Here an ***n*-sun** S_n ($n \geq 3$) is a graph with vertex set $N_{S_n} = \{u_1, \ldots, u_n\} \cup \{v_1, \ldots, v_n\}$ and edge set

$$E_{S_n} = \{\{u_i, u_j\} \mid 1 \leq i < j \leq n\} \cup \{\{v_i, u_j\} \mid 1 \leq i \leq n, j \in \{i, i+1\}\},$$

where we let $u_{n+1} = u_1$.

Theorem 1.4 ([16], [9, Theorem 3.3])**.** *The graphic arrangement $\mathcal{A}_G$ is free if and only if G is chordal.*

Theorem 1.5 ([18, Theorem 2.10])**.** *The graphic arrangement $\mathcal{A}_G$ is MAT-free if and only if G is strongly chordal.*

While the definition of an MAT-free arrangement may seem technical at first glance, Theorem 1.5 enables us to view MAT-freeness as a rather natural property. Furthermore, the correspondence between MAT-freeness and strong chordality establishes a nice analog[1] of the classical correspondence between freeness and chordality.

The good thing about graphs is that MAT-partition of a graphic arrangement can be rephrased in terms of a special edge-labeling of graphs, the so-called *MAT-labeling* (Definition 2.1). A graph together with such a labeling is called an **MAT-labeled graph**. To approach Question 1.3 for graphic arrangements, the first question would be how many non-isomorphic MAT-labelings can a (strongly chordal) graph have? A computation aided by computer for complete graphs on up to 8 vertices gives us the sequence $1, 1, 1, 2, 6, 40, 560, 17024$. Surprisingly, we found out that this sequence coincides with the number of equivalence classes of *(graphical) regular vines* (or *R-vines*) in dimension up to 8 given in [12, §10.3]. This observation is indeed compelling as it leads us to the notion of the *node poset* of a *graphical vine* (Definitions 2.9 and 2.10), which is a perfect candidate for the poset structure we are looking for.

2 Definitions

2.1 MAT-labeled graphs

All graphs in this paper are undirected, finite and simple. Let $G = (N_G, E_G)$ be a graph with the set N_G of vertices (or nodes) and the set E_G of edges (unordered pairs of vertices). In this paper, a vertex and a node in a graph are synonyms. The former will be used more often for graphs, while the latter will be used for an element in a poset.

An **edge-labeled graph** is pair (G, λ) where G is a simple graph and $\lambda\colon E_G \longrightarrow \mathbb{Z}_{>0}$ is a map, called **(edge-)labeling**. The following definition of an MAT-labeling is equivalent to the original one in [18, Definition 4.2].

Definition 2.1 (MAT-labeling)**.** Let (G, λ) be an edge-labeled graph. For $k \in \mathbb{Z}_{>0}$, let $\pi_k := \lambda^{-1}(k) \subseteq E_G$ denote the set of edges of label k. Define $\pi_{\le k} := \pi_1 \sqcup \cdots \sqcup \pi_k$ and $\pi_{<1} := \varnothing$. The labeling λ is an **MAT-labeling** if the following two conditions hold for every $k \in \mathbb{Z}_{>0}$.

[1] Many important concepts in the classical theory such as *simplicial vertex* and *perfect elimination ordering* of chordal graphs have their analogs in MAT-labeled graphs (see [18] for more details).

1. Any edge $e \in \pi_{\le k}$ does not form a cycle with edges in π_k.
2. Every edge $e \in \pi_k$ forms exactly $k-1$ triangles with edges in $\pi_{<k}$.

Given an edge $e \in \pi_k$, a **conditioning vertex** of e is a vertex that together with the endvertices of e forms two edges both of label $< k$. Condition (2) above can be rephrased as every edge e of label k has exactly $k-1$ conditioning vertices.

Definition 2.2 (MAT-labeled (complete) graph)**.** An edge-labeled graph (G, λ) is an **MAT-labeled graph** if λ is an MAT-labeling of G. In particular, an MAT-labeled graph (G, λ) is an **MAT-labeled complete graph** if G is a complete graph.

MAT-partition of a graphic arrangement is nothing but MAT-labeling of the underlying graph [18, Proposition 4.3]. Thus, MAT-free graphic arrangement and MAT-labeled graph are essentially the same object.

Recall that a **clique** of a graph is a subset of vertices such that every two distinct vertices in the clique are adjacent.

Lemma 2.3 (Principal clique)**.** *Let (G, λ) be an MAT-labeled graph. Let $e = \{i, j\} \in \pi_k$ be an edge in G of label k and $h_1, \dots, h_{k-1}$ be the conditioning vertices of e. Then the set $K_e := \{i, j, h_1, \dots, h_{k-1}\}$ is a clique of G. We call K_e the* ***principal clique generated by*** *e.*

Definition 2.4 (Label-preserving isomomorphism)**.** Let (G, λ) and (G', λ') be two edge-labeled graphs. A **label-preserving homomorphism** from (G, λ) to (G', λ'), written $\sigma\colon (G, \lambda) \longrightarrow (G', \lambda')$ is a map $\sigma\colon N_G \longrightarrow N_{G'}$ such that for all $u, v \in N_G$, $\{u, v\} \in E_G$ implies $\{\sigma(u), \sigma(v)\} \in E_{G'}$ and $\lambda(u, v) = \lambda'(\sigma(u), \sigma(v))$.

We call σ an **isomorphism** if σ is bijective and its inverse is a label-preserving homomorphism. The edge-labeled graphs (G, λ) and (G', λ') are said to be **isomorphic**, written $(G, \lambda) \simeq (G', \lambda')$ if there exists an isomorphism $\sigma\colon (G, \lambda) \longrightarrow (G', \lambda')$. If $(G, \lambda) \simeq (G, \lambda')$, we say that two labelings λ and λ' are the same (or isomorphic).

If $(G, \lambda) \simeq (G', \lambda')$ and (G, λ) is an MAT-labeled graph, then (G', λ') is also an MAT-labeled graph.

Definition 2.5 (Category of MAT-labeled (complete) graphs)**.** The **category** MG **of MAT-labeled graphs** is the category whose objects are the MAT-labeled graphs and whose morphisms are the label-preserving homomorphisms. The **category** MCG **of MAT-labeled complete graphs** is a full subcategory of MG whose objects are the MAT-labeled complete graphs.

2.2 Vines: graphical and poset definitions

All posets $\mathcal{P} = (\mathcal{P}, \le_{\mathcal{P}})$ in this note are finite. Denote by $\max(\mathcal{P})$ (resp. $\min(\mathcal{P})$) the set of all maximal (resp. minimal) elements in a poset $\mathcal{P}$.

Definition 2.6 (Graded poset)**.** A finite poset $\mathcal{P}$ is **graded** if there exists a **rank function** $\mathrm{rk} = \mathrm{rk}_{\mathcal{P}} : \mathcal{P} \longrightarrow \mathbb{Z}_{\geq 0}$ satisfying the following three properties:

1. For any $x, y \in \mathcal{P}$, if $x < y$ then $\mathrm{rk}(x) < \mathrm{rk}(y)$.
2. If y covers x, then $\mathrm{rk}(x) = \mathrm{rk}(y) - 1$.
3. All minimal elements of $\mathcal{P}$ have the same rank. In this note, we assume[2] $\mathrm{rk}(x) = 1$ for all $x \in \min(\mathcal{P})$.

Equivalently, for every $x \in \mathcal{P}$, all maximal chains among those with x as greatest element have the same length.

The **dimension**[3] $\dim(\mathcal{P})$ of $\mathcal{P}$ is defined as $\dim(\mathcal{P}) := |\min(\mathcal{P})|$. The **rank** $\mathrm{rk}(\mathcal{P})$ of a graded poset $\mathcal{P}$ with rank function rk is defined as

$$\mathrm{rk}(\mathcal{P}) := \max\{\mathrm{rk}(x) \mid x \in \mathcal{P}\}.$$

Definition 2.7 (Ideal, principal ideal)**.** Let $\mathcal{P}$ be a poset. An (order) **ideal** $\mathcal{I}$ of $\mathcal{P}$ is a downward-closed subset, i.e. for every $x \in \mathcal{P}$ and $y \in \mathcal{I}$, $x \leq y$ implies that $x \in \mathcal{I}$. For $a \in \mathcal{P}$, the ideal

$$\mathcal{P}_{\leq a} := \{x \in \mathcal{P} \mid x \leq a\}$$

is called the **principal** ideal of $\mathcal{P}$ generated by a.

Definition 2.8 (Poset homomorphism)**.** Let $\mathcal{P}$ and $\mathcal{P}'$ be posets. A **(poset) homomorphism** $\varphi : \mathcal{P} \longrightarrow \mathcal{P}'$ is an order-preserving map, i.e. $x \leq y$ implies $\varphi(x) \leq \varphi(y)$ for all $x, y \in \mathcal{P}$. We call φ a **join-preserving** homomorphism if for any $x, y \in \mathcal{P}$ such that the join $x \vee y$ exists, then $\varphi(x) \vee \varphi(y)$ exists and $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$. We call φ an **isomorphism** if φ is bijective and its inverse is a homomorphism. The posets $\mathcal{P}$ and $\mathcal{P}'$ are said to be **isomorphic**, written $\mathcal{P} \simeq \mathcal{P}'$ if there exists an isomorphism $\varphi : \mathcal{P} \longrightarrow \mathcal{P}'$. When $\mathcal{P} = (\mathcal{P}, \mathrm{rk})$ and $\mathcal{P}' = (\mathcal{P}', \mathrm{rk}')$ are graded posets, a homomorphism $\varphi : \mathcal{P} \longrightarrow \mathcal{P}'$ is called **rank-preserving** if $\mathrm{rk}'(\varphi(x)) = \mathrm{rk}(x)$ for all $x \in \mathcal{P}$.

Now we recall the graphical definition of a *vine* following [4, Definition 4.1].

Definition 2.9 (Graphical definition of vine)**.** Let $1 \leq n \leq \ell$ be positive integers. A **(graphical) vine** $\mathcal{V}$ on ℓ elements $[\ell] = \{1, \dots, \ell\}$ (or more generally, on an ℓ-element set called N_1) is an ordered n-tuple $\mathcal{V} = (F_1, F_2, \dots, F_n)$ such that

[2] A motivation for this assumption is the equivalence between D-vine and root poset of type A (Remark 3.4). The latter is graded by heights of positive roots, and all the minimal elements (simple roots) have rank (height) 1.

[3] The term "dimension" of a poset may have a different meaning in the other context. The present definition is to make a compatibility for dimensions of a vine (Remark 2.11) and the ambient space of graphic arrangements.

1. F_1 is a forest with nodes $N_1 = [\ell]$ and a set of edges denoted E_1,
2. for $2 \leq i \leq n$, F_i is a forest with nodes $N_i = E_{i-1}$ and edge set E_i.

We call F_i the **i-th associated forest** of $\mathcal{V}$. A graphical vine is uniquely determined by its associated forests. Denote by $N(\mathcal{V}) = N_1 \cup \cdots \cup N_n$ the set of nodes (of the associated forests) of $\mathcal{V}$. We call the numbers n and ℓ the **rank** and **dimension** of $\mathcal{V}$, respectively.

If node u is an element of node v, i.e. $u \in v$, we say that u is a **child** of v. If v is reachable from u via the membership relation: $u \in u_1 \in \cdots \in v$, we say that u is a **descendant** of v.

Definition 2.10 (Node poset)**.** Let $\mathcal{V}$ be a graphical vine with node set $N(\mathcal{V})$. The **node poset** $\mathcal{P} = \mathcal{P}(\mathcal{V})$ of $\mathcal{V}$ is the poset $(N(\mathcal{V}), \leq)$ defined as follows: For any $u, v \in N(\mathcal{V})$,

$$u \leq v \quad \text{if} \quad u \text{ is a descendant of } v.$$

Remark 2.11. We emphasize that a graphical vine is uniquely determined by its node poset. The terminology "rank" of a vine has motivation from poset theory. If a vine $\mathcal{V}$ is an ordered n-tuple, then $\mathcal{P} = \mathcal{P}(\mathcal{V})$ is a graded poset with rank function $\mathrm{rk}(v) = i$ for $v \in N_i$ $(1 \leq i \leq n)$. Thus this number n equals the rank of $\mathcal{P}$. In addition, the dimension of $\mathcal{V}$ equals the number of minimal elements in $\mathcal{P}$, or the dimension of $\mathcal{P}$.

Assumption & Notation 2.12. From now on, unless otherwise stated we assume that $\mathcal{P}$ is a finite graded poset with a rank function $\mathrm{rk} : \mathcal{P} \longrightarrow \mathbb{Z}_{>0}$. Denote $n := \mathrm{rk}(\mathcal{P})$ and $\ell := \dim(\mathcal{P})$. For $v \in \mathcal{P}$, denote by $\mathcal{E}(v)$ the set of elements covered by v. For $i \geq 0$, define $\mathcal{P}_i := \{v \in \mathcal{P} \mid \mathrm{rk}(v) = i\}$ and $\mathcal{E}(\mathcal{P}_i) := \{\mathcal{E}(v) \mid v \in \mathcal{P}_i\}$. If $\mathcal{P}$ is an ℓ-dimensional poset, we assume $\mathcal{P}_1 = \min(\mathcal{P}) = [\ell]$.

As noted earlier in Remark 2.11, we may think of a graphical vine and its node poset essentially as the same object. It is thus natural to look for a characterization of the node poset of a vine. We give below such a characterization obtained immediately from Definition 2.9.

Definition & Proposition 2.13 (Poset definition of vine)**.** A finite graded poset $\mathcal{P}$ is the node poset of a graphical vine if and only if $\mathcal{P}$ satisfies the following conditions:

1. Every non-minimal node covers exactly two other nodes, and any two distinct nodes of the same rank are covered by at most one node.
2. For each $1 \leq i \leq n = \mathrm{rk}(\mathcal{P})$, the graph $F_i = (N_i, E_i)$ with node set $N_i := \mathcal{P}_i$ and edge set $E_i := \mathcal{E}(\mathcal{P}_{i+1})$ is a forest.

Assumption & Notation 2.14. From now on, unless otherwise stated, by a vine $\mathcal{P}$ we mean a finite graded poset satisfying the two conditions in 2.13. We will also retain the notion i-th associated forest $F_i = (\mathcal{P}_i, \mathcal{E}(\mathcal{P}_{i+1}))$ $(1 \leq i \leq n)$ of $\mathcal{P}$. If v is a node in a vine $\mathcal{P}$ and $\mathcal{E}(v) = \{a, b\}$, we will often abuse notation and write $v = \{a, b\}$. This notation is compatible with the notation of node/edge in the graphical definition of a vine.

The main reason why we choose the poset definition of a vine is because many terms and properties of a (graphical) vine have natural meanings in the language of posets. Under this consideration, the following poset definition of a *regular vine* is equivalent to the well-known graphical definition of it in the literature, e.g. [4, Definition 4.1].

Definition 2.15 (R-vine)**.** A vine $\mathcal{P}$ is a **regular vine**, or an *R-vine* for short, if $\mathcal{P}$ satisfies the following conditions:

1. $\text{rk}(\mathcal{P}) = \dim(\mathcal{P})$, i.e. $n = \ell$.
2. Each associated forest $F_i = (\mathcal{P}_i, \mathcal{E}(\mathcal{P}_{i+1}))$ is a tree $(1 \le i \le n)$.
3. **Proximity**: For any distinct nodes $a, b \in \mathcal{P}_i$ for $i \ge 2$, if a and b are covered by a common node, then a and b cover a common node.

Next we introduce the notion of a *locally* regular vine.

Definition 2.16 (LR-vine)**.** A vine $\mathcal{P}$ is a **locally regular vine**, or an *LR-vine* for short, if every principal ideal of $\mathcal{P}$ is an R-vine.

Remark 2.17. Intuitively, an LR-vine is a vine that "locally" looks like an R-vine. In particular, any R-vine is an LR-vine. Any ideal of a vine (resp. an LR-vine) is itself a vine (resp. an LR-vine).

The following theorem indicates the equivalence between the ideals of an R-vine and LR-vines.

Theorem 2.18. *Let $\mathcal{P}$ be a vine. The following are equivalent:*

1. *$\mathcal{P}$ is an ideal of an R-vine.*
2. *$\mathcal{P}$ satisfies the proximity condition.*
3. *$\mathcal{P}$ is an LR-vine.*

Definition 2.19 (Category of (L)R-vines)**.** The **category** LRV **of LR-vines** is the category whose objects are the LR-vines and whose morphisms are the homomorphisms preserving rank and join. The **category** RV **of R-vines** is a full subcategory of LRV whose objects are the R-vines.

3 The main result

Having introduced the concepts, we are ready to state our main result.

Theorem 3.1. *The categories* MG *and* LRV *are equivalent. In particular, the categories* MCG *and* RV *are equivalent.*

To prove the equivalence between MG and LRV, we construct two functors $\Psi\colon \mathsf{MG} \longrightarrow \mathsf{LRV}$ and $\Omega\colon \mathsf{LRV} \longrightarrow \mathsf{MG}$. The former amounts to constructing an LR-vine from a given MAT-labeled graph which is presented in Theorem 3.2 below. The proof is direct and largely dependent upon the notion of *MAT-perfect elimination ordering* developed in an earlier work of the last two authors [18]. The argument on the functor Ω is however more complicated, and the details are omitted.

Theorem 3.2. *Let (G, λ) be an MAT-labeled graph with $N_G = [\ell]$. Define a finite graded poset $\mathcal{P} = (\mathcal{P}, \leq_{\mathcal{P}}, \mathrm{rk}_{\mathcal{P}})$ from (G, λ) as follows:*

1. *$\mathcal{P}$ consists of the sets $\{i\}$ for $1 \leq i \leq \ell$ and all the principal cliques in (G, λ) (Lemma 2.3).*
2. *For $u, v \in \mathcal{P}$, $u \leq_{\mathcal{P}} v$ if u is a subset of v.*
3. *$\mathrm{rk}_{\mathcal{P}}(v) = |v|$ for all $v \in \mathcal{P}$.*

Then the poset $\mathcal{P}$ is an LR-vine. In particular, if (G, λ) is an MAT-labeled complete graph, then $\mathcal{P}$ is an R-vine.

We give two examples to illustrate the construction in Theorem 3.2.

Definition 3.3 (D-vine)**.** An R-vine is called a **D-Vine** if each associated tree has the smallest possible number of vertices of degree 1. Equivalently, each associated tree is a *path graph*.

Remark 3.4. Let Φ be an irreducible root system in $\mathbb{R}^\ell$ with a fixed positive system $\Phi^+ \subseteq \Phi$ and the associated set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$. Suppose that Φ is of type A_ℓ and the Dynkin diagram of Φ is the path graph $\alpha_1 - \alpha_2 - \cdots - \alpha_\ell$. Then the positive roots of Φ are given by

$$\Phi^+ = \left\{ \sum_{i \leq k \leq j} \alpha_k \,\middle|\, 1 \leq i \leq j \leq m \right\}.$$

It is not hard to show that the D-vine $\mathcal{P}$ with the first associated tree $1 - 2 - \cdots - \ell$ is isomorphic to the root poset of type A_ℓ.

Example 3.5. Figure 1 depicts a 4-dimensional D-vine (middle) that can be constructed in three ways. First, it is the node poset of a graphical vine on [4] (left). Second, it is the poset defined an MAT-labeled complete graph (right) via Theorem 3.2. Third, it is the root poset of type A_4 by Remark 3.4. The elements in the poset are written without set symbol for simplicity. The conditioned set of a non-minimal node is given to the left of the "|" sign, while the conditioning set appears on the right. For example, the top node $\{1, 2, 3, 4\}$ (or the largest clique generated by $\{v_1, v_4\}$) is written by $14|23$.

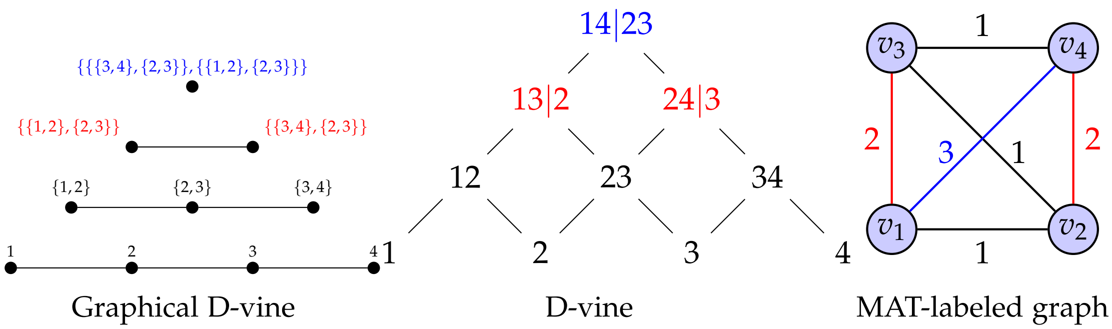

Figure 1: An MAT-labeled complete graph on 4 vertices (right), the D-vine (middle) (= type A root poset) defined by the graph via Theorem 3.2, and the corresponding graphical vine (left).

Definition 3.6 (C-vine)**.** An R-vine is called a **C-Vine** if each associated tree has the largest possible number of vertices of degree 1. Equivalently, each associated tree is a *star graph*.

D-vine and C-vine can be regarded as the "extreme" cases of R-vines.

Example 3.7. In dimension 4, there are exactly two non-isomorphic R-vine structures: D-vine and C-vine. Likewise, there are exactly two non-isomorphic MAT-labeled complete graphs on 4 vertices. Figure 2 depicts a graphical C-vine on $[4]$ (left), the corresponding node poset (middle), and the corresponding MAT-labeled complete graph (right) via Theorem 3.2. The C-vine in dimension ≥ 4 is not an ideal of any D-vine hence the corresponding MAT-partition is not obtained from an ideal of the type A root poset.

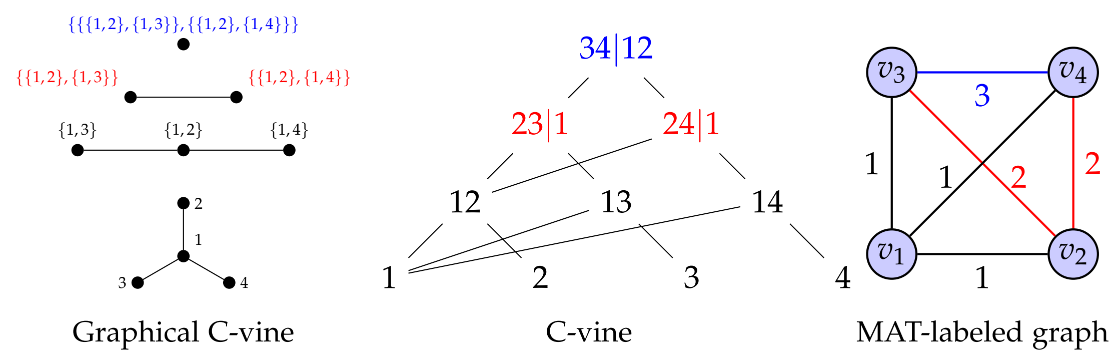

Figure 2: C-vine on 4 elements and the corresponding graphical version, MAT-labeled complete graph from Example 3.7.

4 Applications

From the view point of category theory, the equivalence establishes a strong similarity between the categories and allows many properties and structures to be translated from one to the other. We obtain two main applications from LR-vines to MAT-labeled graphs. First, LR-vine is an answer for Question 1.3 in the case of graphic arrangements. We find it interesting that although the class of MAT-free arrangements is strictly larger than that of ideal subarrangements in general, any MAT-free *graphic* arrangement is characterized by being an ideal of a poset structure (Theorem 2.18). Second, an explicit formula for the number of non-isomorphic MAT-labelings of complete graphs is obtained. This equals the number of equivalence classes of regular vines whose explicit formula is known [12, §10.3].

A vine is a graphical tool for representing the joint distribution of random variables. The first construction of a vine was given by Joe [10], and the formal definition was given and refined further by Cooke, Bedford and Kurowicka [5, 4, 11]. Vines have been studied extensively and proved to have various applications in probability theory and related areas. We refer the reader to [12] for a comprehensive handbook of vines. Our main result gives a new appearance and applications of vines in the arrangement theory. In the present note, we do not pursue the probabilistic or applied aspects of vines (neither does the proof of the main result) but emphasize and develop more on the theoretical aspects. In the full version of this extended abstract, we give several new combinatorial properties of vines, hoping that they will be useful for the future research on vines. For instance, we give an alternative way to associate an *m-vine* to a strongly chordal graph compared with the work of Zhu-Kurowicka [19], and an extension of the notion of *sampling order* [6] from R-vine to LR-vine.

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