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Crystals for variations of decomposition tableaux

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Abstract. Our previous work introduced a category of extended queer crystals, whose connected normal objects have unique highest weight elements and characters that are Schur Q-polynomials. Our initial models for such crystals were based on semistandard shifted tableaux. Here, we introduce a simpler construction using certain "primed" decomposition tableaux, which slightly generalize the decomposition tableaux used in work of Grantcharov et al. This leads to a new, much shorter proof of the highest weight properties of the normal subcategory of extended queer crystals. We also describe a natural crystal structure on set-valued decomposition tableaux. Our results give the first crystal constructions for shifted set-valued tableaux, and lead to partial progress on a conjectural formula of Cho and Ikeda for K-theoretic Schur P-functions.

Keywords: Crystals, K-theoretic Schur P-functions, queer Lie superalgebras, decomposition tableaux, set-valued tableaux

1 Introduction

Crystals are an abstraction for the crystal bases of quantum group representations, and can be viewed as acyclic directed graphs with labeled edges and weighted vertices, satisfying certain axioms. Crystals for $\mathfrak{gl}_n$ and other classical Lie algebras were first studied by Kashiwara [9, 10] and Lusztig [12, 13] in the 1990s. More recent work by Grantcharov et al. [3, 4] introduced crystals for the queer Lie superalgebra $\mathfrak{q}_n$.

Our previous work [14] defined a slightly modified category of $\mathfrak{q}_n^+$-crystals, which share many nice features with $\mathfrak{gl}_n$-crystals and $\mathfrak{q}_n$-crystals. For example, $\mathfrak{q}_n^+$-crystals have a natural tensor product and a standard crystal corresponding to the vector representation of the quantum group $U_q(\mathfrak{q}_n)$. This lets one define a subcategory of *normal crystals*, consisting of crystals whose connected components can each be embedded in some tensor power of the standard crystal.

In [14], our primary models for normal $\mathfrak{q}_n^+$-crystals were derived from *semistandard shifted tableaux*, using crystal operators with very technical formulas. One of the main results of this note is to introduce a much simpler model for normal $\mathfrak{q}_n^+$-crystal based on a "primed" generalization of *decomposition tableaux*. The latter tableaux served as the original model for normal (non-extended) $\mathfrak{q}_n$-crystals in [3].

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After formally defining primed decomposition tableaux, we equip them with a natural family of $\mathfrak{q}_n^+$-crystal operators, identify their highest weight elements, and construct a primed generalization of a useful "insertion scheme" from [3]. As an application, we give a short, alternate proof that normal $\mathfrak{q}_n^+$-crystals are determined up to isomorphism by their characters (which range over all Schur Q-positive symmetric polynomials in n variables), and also by by their multisets of highest weights (which range over all strict partitions with at most n parts).

Our other main results concern a new crystal structure on a "set-valued" generalization of decomposition tableaux. Several authors (for example, [5, 18, 20]) have recently studied $\mathfrak{gl}_n$-crystal structures on unshifted set-valued tableaux. The characters of these crystals give K-theoretic symmetric functions of independent interest. It has been an open problem to extend such constructions to shifted tableaux.

Addressing this open problem, we show that a certain natural family of *set-valued decomposition tableaux* has a normal $\mathfrak{gl}_n$-crystal structure. This structure is formally similar to the one in [18] for unshifted set-valued tableaux, though somewhat more technical. Cho and Ikeda [6] has conjectured that the weight generating function for set-valued decomposition tableaux recovers the K-theoretic Schur P-function GP_λ. As partial progress on this conjecture, our results imply that this generating function is at least symmetric and equal to GP_λ plus a (possibly infinite) $\mathbb{Z}$-linear combination of GP_μ's with $|\mu| > |\lambda|$.

2 Abstract crystals

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{P} = \{1, 2, 3, \dots\}$. Fix $n \in \mathbb{N}$ and let $[n] = \{1, 2, \dots, n\}$. Let $\mathcal{B}$ be a set with maps $\mathrm{wt}\colon \mathcal{B} \to \mathbb{N}^n$ and $e_i, f_i\colon \mathcal{B} \to \mathcal{B} \sqcup \{0\}$ for $i \in [n-1]$, where $0 \notin \mathcal{B}$. We assume that if $b, c \in \mathcal{B}$ then $f_i(b) = c$ if and only if $e_i(c) = b$. This means that the maps e_i and f_i encode a directed graph with vertex set $\mathcal{B}$, to be called the *crystal graph*, with an edge $b \xrightarrow{i} c$ if $f_i(b) = c$. The *string lengths* $\varepsilon_i, \varphi_i\colon \mathcal{B} \to \{0, 1, 2, \dots\} \sqcup \{\infty\}$ are

$$\varepsilon_i(b) := \sup\left\{k \geq 0 \mid e_i^k(b) \neq 0\right\} \text{ and } \varphi_i(b) := \sup\left\{k \geq 0 : f_i^k(b) \neq 0\right\}. \tag{2.1}$$

We assume that $\varepsilon_i(b)$ and $\varphi_i(b)$ are always finite. If the set $\mathcal{B}$ is finite then its *character* is the polynomial $\mathrm{ch}(\mathcal{B}) := \sum_{b \in \mathcal{B}} \prod_{i \in [n]} x_i^{\mathrm{wt}(b)_i} \in \mathbb{N}[x_1, x_2, \dots, x_n]$. Finally, let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n \in \mathbb{Z}^n$ be the standard basis.

Definition 2.1. The set $\mathcal{B}$ is a *$\mathfrak{gl}_n$-crystal* if for all $i \in [n-1]$ and $b \in \mathcal{B}$ we have (a) $\mathrm{wt}(e_i(b)) = \mathrm{wt}(b) + \mathbf{e}_i - \mathbf{e}_{i+1}$ if $e_i(b) \neq 0$, and (b) $\varphi_i(b) - \varepsilon_i(b) = \mathrm{wt}(b)_i - \mathrm{wt}(b)_{i+1}$.

We refer to wt as the *weight map* and to each e_i as a *raising operator*. Each connected component of the crystal graph of $\mathcal{B}$ may be viewed as a $\mathfrak{gl}_n$-crystal by restricting the weight map and crystal operators; these objects are called *full subcrystals*. A *crystal isomorphism* is a weight-preserving bijection that induces an isomorphism of crystal graphs.

Example 2.2. The *standard* $\mathfrak{gl}_n$*-crystal* $\mathbb{B}_n = \left\{\boxed{i} : i \in [n]\right\}$ has crystal graph

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \xrightarrow{3} \cdots \xrightarrow{n-1} \boxed{n} \quad \text{with } \mathrm{wt}(\boxed{i}) := \mathbf{e}_i.$$

The set of formal tensors $\mathcal{B} \otimes \mathcal{C} := \{b \otimes c : b \in \mathcal{B},\ c \in \mathcal{C}\}$ has a unique $\mathfrak{gl}_n$-crystal structure with $\mathrm{wt}(b \otimes c) := \mathrm{wt}(b) + \mathrm{wt}(c)$ and with

$$e_i(b \otimes c) := \begin{cases} b \otimes e_i(c) & \text{if } \varepsilon_i(b) \leq \varphi_i(c) \\ e_i(b) \otimes c & \text{if } \varepsilon_i(b) > \varphi_i(c) \end{cases} \tag{2.2}$$

for $i \in [n-1]$, where it is understood that $b \otimes 0 = 0 \otimes c = 0$ [1, §2.3]. This follows the "anti-Kashiwara convention," which reverses the tensor product order in [3, 4].

3 Queer crystals

The general linear Lie algebra $\mathfrak{gl}_n$ has two super-analogues, one of which is the *queer Lie superalgebra* $\mathfrak{q}_n$. Grantcharov et al. developed a theory of crystals for $\mathfrak{q}_n$ in [3, 4], which we review here. Assume $n \geq 2$. Let $\mathcal{B}$ be a $\mathfrak{gl}_n$-crystal with maps $e_{\overline{1}}, f_{\overline{1}} : \mathcal{B} \to \mathcal{B} \sqcup \{0\}$ satisfying $f_{\overline{1}}(b) = c$ if and only if $b = e_{\overline{1}}(c)$ when $b, c \in \mathcal{B}$. Define $\varepsilon_{\overline{1}}, \varphi_{\overline{1}} : \mathcal{B} \to \mathbb{N} \sqcup \{\infty\}$ as in (2.1) but with $i = \overline{1}$. Below, we say that one map $\phi : \mathcal{B} \to \mathcal{B} \sqcup \{0\}$ *preserves* another map $\eta : \mathcal{B} \to \mathcal{X}$ if $\eta(\phi(b)) = \eta(b)$ whenever $\phi(b) \neq 0$.

Definition 3.1. The $\mathfrak{gl}_n$-crystal $\mathcal{B}$ is a $\mathfrak{q}_n$*-crystal* if for all $b \in \mathcal{B}$:

(a) $\mathrm{wt}(e_{\overline{1}}(b)) = \mathrm{wt}(b) + \mathbf{e}_1 - \mathbf{e}_2$ whenever $e_{\overline{1}}(b) \neq 0$,

(b) $\varphi_{\overline{1}}(b) + \varepsilon_{\overline{1}}(b)$ is 0 if $\mathrm{wt}(b)_1 = \mathrm{wt}(b)_2 = 0$ and 1 otherwise, and

(c) $e_{\overline{1}}$ and $f_{\overline{1}}$ commute with e_i, f_i while preserving ε_i, φ_i for all $3 \leq i \leq n-1$.

Assume $\mathcal{B}$ is a $\mathfrak{q}_n$-crystal. The corresponding $\mathfrak{q}_n$*-crystal graph* has vertex set $\mathcal{B}$ and edges $b \xrightarrow{i} c$ whenever $f_i(b) = c$ for any $i \in \{\overline{1}, 1, 2, \ldots, n-1\}$.

Example 3.2. The *standard* $\mathfrak{q}_n$*-crystal* $\mathbb{B}_n = \left\{\boxed{i} : i \in [n]\right\}$ has crystal graph

$$\boxed{1} \underset{1}{\overset{\overline{1}}{\rightrightarrows}} \boxed{2} \xrightarrow[2]{} \boxed{3} \xrightarrow[3]{} \cdots \xrightarrow[n-1]{} \boxed{n} \quad \text{with } \mathrm{wt}(\boxed{i}) := \mathbf{e}_i.$$

Suppose $\mathcal{B}$ and $\mathcal{C}$ are $\mathfrak{q}_n$-crystals. The set $\mathcal{B} \otimes \mathcal{C}$ already has a $\mathfrak{gl}_n$-crystal structure. There is a unique way of viewing this object as a $\mathfrak{q}_n$-crystal [3, Thm. 1.8] with

$$e_{\overline{1}}(b \otimes c) := \begin{cases} b \otimes e_{\overline{1}}(c) & \text{if } \mathrm{wt}(b)_1 = \mathrm{wt}(b)_2 = 0 \\ e_{\overline{1}}(b) \otimes c & \text{otherwise.} \end{cases} \tag{3.1}$$

4 Extended crystals

We continue to assume $n \geq 2$. The following theory of *extended* $\mathfrak{q}_n$*-crystals* (abbreviated as $\mathfrak{q}_n^+$*-crystals* from now on) was introduced in our previous work [14]. Suppose $\mathcal{B}$ is a $\mathfrak{q}_n$-crystal with additional maps $e_0, f_0 : \mathcal{B} \to \mathcal{B} \sqcup \{0\}$ satisfying $f_0(b) = c$ if and only if $b = e_0(c)$ when $b, c \in \mathcal{B}$. Define $\varepsilon_0, \varphi_0 : \mathcal{B} \to \mathbb{N} \sqcup \{\infty\}$ by the formula (2.1) with $i = 0$.

Definition 4.1. The $\mathfrak{q}_n$-crystal $\mathcal{B}$ is a $\mathfrak{q}_n^+$*-crystal* if for all $b \in \mathcal{B}$:

(a) $\mathrm{wt}(e_0(b)) = \mathrm{wt}(b)$ if $e_0(b) \neq 0$,

(b) $\varphi_0(b) + \varepsilon_0(b)$ is 0 if $\mathrm{wt}(b)_1 = 0$ and 1 otherwise, and

(c) e_0 and f_0 commute with e_i, f_i while preserving ε_i, φ_i for all $2 \leq i \leq n-1$.

Assume $\mathcal{B}$ is a $\mathfrak{q}_n^+$-crystal. The corresponding $\mathfrak{q}_n^+$*-crystal graph* has vertex set $\mathcal{B}$ and edges $b \xrightarrow{i} c$ whenever $f_i(b) = c$ for any $i \in \{\overline{1}, 0, 1, 2, \dots, n-1\}$.

Example 4.2. The *standard* $\mathfrak{q}_n^+$*-crystal* $\mathbb{B}_n^+$ has crystal graph

$$\begin{tikzcd}
\boxed{1} \arrow[r, dashed, "\overline{1}"] \arrow[r, swap, "1"] \arrow[d, dotted, swap, "0"] & \boxed{2} \arrow[r, "2"] & \boxed{3} \arrow[r, "3"] & \cdots \arrow[r, "n-1"] & \boxed{n} \\
\boxed{1'} \arrow[r, dashed, "\overline{1}"] \arrow[r, swap, "1"] & \boxed{2'} \arrow[r, "2"] & \boxed{3'} \arrow[r, "3"] & \cdots \arrow[r, "n-1"] & \boxed{n'}
\end{tikzcd} \qquad \text{with } \mathrm{wt}(\boxed{i}) = \mathrm{wt}(\boxed{i'}) := \mathbf{e}_i.$$

If $\mathcal{B}$ and $\mathcal{C}$ are $\mathfrak{q}_n^+$-crystals then the $\mathfrak{gl}_n$-crystal $\mathcal{B} \otimes \mathcal{C}$ has a $\mathfrak{q}_n^+$-crystal structure with

$$e_0(b \otimes c) := \begin{cases} e_0(b) \otimes c & \text{if } \mathrm{wt}(b)_1 \neq 0 \\ b \otimes e_0(c) & \text{if } \mathrm{wt}(b)_1 = 0 \end{cases} \tag{4.1}$$

and

$$e_{\overline{1}}(b \otimes c) := \begin{cases} b \otimes e_{\overline{1}}(c) & \text{if } \mathrm{wt}(b)_1 = \mathrm{wt}(b)_2 = 0 \\ f_0 e_{\overline{1}}(b) \otimes e_0(c) & \text{if } \mathrm{wt}(b)_1 = 0,\ f_0 e_{\overline{1}}(b) \neq 0, \text{ and } e_0(c) \neq 0 \\ e_0 e_{\overline{1}}(b) \otimes f_0(c) & \text{if } \mathrm{wt}(b)_1 = 0,\ e_0 e_{\overline{1}}(b) \neq 0, \text{ and } f_0(c) \neq 0 \\ e_{\overline{1}}(b) \otimes c & \text{otherwise} \end{cases} \tag{4.2}$$

where it is again understood that $b \otimes 0 = 0 \otimes c = 0$ [14, Thm. 3.14].

Remark 4.3. For $i \in \mathbb{Z}$ define $i' := i - \frac{1}{2} \in \mathbb{Z}' := \mathbb{Z} - \frac{1}{2}$. We refer to elements of $\mathbb{Z} \sqcup \mathbb{Z}'$ as *primed numbers*. A *primed word* is a finite sequence of primed numbers. We identify each primed word $w = w_1 w_2 \cdots w_m$ with $w_i \in \{1' < 1 < \cdots < n' < n\}$ with the formal tensor $w_1 \otimes w_2 \otimes \cdots \otimes w_m \in (\mathbb{B}_n^+)^{\otimes m}$. This allows us to evaluate $\mathrm{wt}(w)$, $e_i(w)$, and $f_i(w)$ for $i \in [n-1]$ using the definition of the $\mathfrak{q}_n^+$-crystal $(\mathbb{B}_n^+)^{\otimes m}$. For example, the weight of w becomes the vector whose ith component is the number of letters equal to i or i'.

There are well-known, explicit signature rules to compute the crystals operators on tensor powers of the standard $\mathfrak{gl}_n$-, $\mathfrak{q}_n$-, and $\mathfrak{q}_n^+$-crystals (and hence on primed words). We omit this background material in this extended abstract; see [2, 14] for the full details.

5 Decomposition tableaux

Assume $\lambda = (\lambda_1 > \lambda_2 > \cdots > 0)$ is a *strict* integer partition. Let $\ell(\lambda)$ be the number of nonzero parts of λ. The *shifted diagram* of λ is the $\mathsf{SD}_\lambda := \{(i, i+j-1) : i \in [\ell(\lambda)] \text{ and } j \in [\lambda_i]\}$. We often refer to the pairs $(i,j) \in \mathsf{SD}_\lambda$ as boxes. A box $(i,j) \in \mathsf{SD}_\lambda$ is on the *diagonal* if $i = j$. A *shifted tableau* of shape λ is an assignment of numbers to the boxes in SD_λ.

A *hook word* is a sequence of positive integers $w = w_1 w_2 \cdots w_n$ such that $w_1 \geq w_2 \geq \cdots \geq w_m < w_{m+1} < w_{m+2} < \cdots < w_n$ for some $m \in [n]$. The *weakly decreasing part* of such a hook word w is the (always nonempty) subword $w_1 w_2 \cdots w_m$, while the *increasing part* of w is the (possibly empty) subword $w_{m+1} w_{m+2} \cdots w_n$.

Following [3], we define a *(semistandard) decomposition tableau* of shape λ to be a shifted tableau T of shape λ such that if ρ_i denotes row i of T, then (1) each ρ_i is a hook word and (2) ρ_i is a hook subword of maximal length in $\rho_{i+1}\rho_i$ for each $i \in [\ell(\lambda) - 1]$. Note that this definition is different from Serrano's definition in [19], which uses the opposite weak/strict inequality convention for hook words. Let $\mathsf{DecTab}_n(\lambda)$ be the set of decomposition tableaux of shape λ with all entries in $[n]$.

Example 5.1. We draw tableaux in French notation, so that row indices increase from bottom to top and column indices increase from left to right. Then $\begin{array}{|c|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & 1 & \multicolumn{1}{c}{} \\ \hline 2 & 1 & 1 \\ \hline \end{array} \in \mathsf{DecTab}_2(\lambda)$ for $\lambda = (3,2)$, but $T = \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 2 & 1 \\ \hline 2 & 2 & 3 \\ \hline \end{array}$ is a not a decomposition tableau even though its rows are hook words, as $\rho_2\rho_1 = 21223$ contains the hook subword 2223, which is longer than 223.

Remark 5.2. The maximal hook subword condition in the definition of a decomposition tableau is equivalent to a set of inequalities that must hold for certain triples of entries. Concretely, a shifted tableau is a decomposition tableau if and only none of the following patterns with $a \leq b \leq c$ and $x < y < z$ occur in consecutive rows [3, Prop. 2.3]:

$$\begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & \cdots & b \\ \hline a & \cdots & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|c|} \cline{2-5} \multicolumn{1}{c|}{} & \cdots & c & \cdots & b \\ \hline & \cdots & a & \cdots & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & \cdots & x \\ \hline y & \cdots & z \\ \hline \end{array}, \quad \text{or} \quad \begin{array}{|c|c|c|c|c|} \cline{2-5} \multicolumn{1}{c|}{} & \cdots & & \cdots & x \\ \hline & \cdots & y & \cdots & z \\ \hline \end{array}.$$

Here, the leftmost boxes are on the main diagonal and the ellipses "$\cdots$" indicate sequences of zero or more columns.

Define the *middle element* of a hook word w to be the last letter in the weakly decreasing subword $w\downarrow$. Suppose T is a decomposition tableau of strict partition shape λ. We call any tableau formed by adding primes to the middle elements in a subset of rows in T a *primed decomposition tableau* of shape λ. Let $\mathsf{DecTab}_n^+(\lambda)$ denote the set of such tableaux with all entries in $\{1' < 1 < \cdots < n' < n\}$.

Example 5.3. $\begin{array}{ccc} & \boxed{1} & \\ \boxed{2} & \boxed{1} & \boxed{2} \end{array}$, $\begin{array}{ccc} & \boxed{1'} & \\ \boxed{2} & \boxed{1} & \boxed{2} \end{array}$, $\begin{array}{ccc} & \boxed{1} & \\ \boxed{2} & \boxed{1'} & \boxed{2} \end{array}$, and $\begin{array}{ccc} & \boxed{1'} & \\ \boxed{2} & \boxed{1'} & \boxed{2} \end{array}$ are all in $\mathsf{DecTab}_2^+(\lambda)$ for $\lambda = (3,1)$.

The *row reading word* of a shifted tableau T is the word $\mathsf{row}(T)$ formed by reading the rows from left to right, but starting with last row. The *reverse reading word* of T is the reversal of $\mathsf{row}(T)$; we denote this by $\mathsf{revrow}(T)$. For example, $\mathsf{revrow}\left(\begin{array}{ccc} & \boxed{1} & \\ \boxed{2} & \boxed{1} & \boxed{1'} \end{array}\right) = 1'121$.

A *crystal embedding* is a weight-preserving injective map $\phi : \mathcal{B} \to \mathcal{C}$ between crystals that commutes with all crystal operators, in the sense that $\phi(e_i(b)) = e_i(\phi(b))$ and $\phi(f_i(b)) = f_i(\phi(b))$ for all $b \in \mathcal{B}$ when we set $\phi(0) = 0$. Our first new result is the following theorem, which extends [3, Thm. 2.5(a)] from $\mathfrak{q}_n$-crystals to $\mathfrak{q}_n^+$-crystals.

Theorem 5.4. There is a unique $\mathfrak{q}_n^+$-crystal structure on $\mathsf{DecTab}_n^+(\lambda)$ that makes $\mathsf{revrow} : \mathsf{DecTab}_n^+(\lambda) \to (\mathbb{B}_n^+)^{\otimes|\lambda|}$ into a $\mathfrak{q}_n^+$-crystal embedding. This structure restricts to a $\mathfrak{q}_n$-crystal on $\mathsf{DecTab}_n(\lambda)$, for which $\mathsf{revrow} : \mathsf{DecTab}_n(\lambda) \to \mathbb{B}_n^{\otimes|\lambda|}$ is a $\mathfrak{q}_n$-crystal embedding. Finally, the characters of these crystals are the symmetric polynomials

$$\mathrm{ch}(\mathsf{DecTab}_n(\lambda)) = P_\lambda(x_1, x_2, \dots, x_n) \quad \text{and} \quad \mathrm{ch}(\mathsf{DecTab}_n^+(\lambda)) = Q_\lambda(x_1, x_2, \dots, x_n)$$

where P_λ and Q_λ are the *Schur P- and Q-functions* of λ.

An important property of many crystals is the existence of unique *highest weight elements*. For $\mathfrak{gl}_n$-crystals, such elements are exactly the sources in the crystal graph. The precise definitions of highest weight elements for $\mathfrak{q}_n$ and $\mathfrak{q}_n^+$-crystals from [3, 14] are more technical, and given as follows.

Assume $\mathcal{B}$ is a $\mathfrak{gl}_n$-crystal. An *i-string* in $\mathcal{B}$ is a connected component in the subgraph of the crystal graph retaining only the $\xrightarrow{i}$ arrows. Let $\sigma_i : \mathcal{B} \to \mathcal{B}$ be the involution that reverses each i-string, so that the first and last elements are swapped, the second and second-to-last elements are swapped, and so on.

Definition 5.5. An element b in a $\mathfrak{q}_n$-crystal $\mathcal{B}$ is *$\mathfrak{q}_n$-highest weight* if $e_i(b) = e_{\bar{i}}(b) = 0$ for $i \in [n-1]$, where $e_{\bar{i}} := (\sigma_{i-1}\sigma_i)\cdots(\sigma_2\sigma_3)(\sigma_1\sigma_2)e_{\bar{1}}(\sigma_2\sigma_1)(\sigma_3\sigma_2)\cdots(\sigma_i\sigma_{i-1})$ for $2 \le i < n$.

Definition 5.6. An element b in a $\mathfrak{q}_n^+$-crystal $\mathcal{B}$ is *$\mathfrak{q}_n^+$-highest weight* if it is $\mathfrak{q}_n$-highest weight with $\sigma_{i-1}\cdots\sigma_2\sigma_1 e_0 \sigma_1\sigma_2\cdots\sigma_{i-1}(b) = 0$ for all $i \in [n]$.

Let λ be a strict partition with $\ell(\lambda) = k$. The *first border strip* of a shifted shape SD_λ is the minimal subset S containing $(1, \lambda_1)$ such that if $(i,j) \in S$ and $i \neq j$, then either $(i+1, j) \in S$, or $(i, j-1) \in S$ when $(i+1, j) \notin \mathsf{SD}_\lambda$.

Let $\mathsf{SD}_\lambda^{(1)}$ be the first border strip of SD_λ. The set difference $\mathsf{SD}_\lambda - \mathsf{SD}_\lambda^{(1)}$ is either empty when $k = 1$ or equal to SD_μ for a strict partition μ with $\ell(\mu) = k - 1$. For

$i \in [k-1]$ let $\mathsf{SD}_\lambda^{(i+1)}$ be the first border strip of $\mathsf{SD}_\lambda - (\mathsf{SD}_\lambda^{(1)} \sqcup \cdots \sqcup \mathsf{SD}_\lambda^{(i)})$. Finally, let $T_\lambda^{\mathsf{highest}}$ be the shifted tableau of shape λ with all i entries in $\mathsf{SD}_\lambda^{(i)}$.

Example 5.7. If $\lambda = (6, 4, 2, 1)$ then the boxes with $\bullet$ below make up the first border strip

$$\begin{array}{|c|c|c|c|c|c|} \cline{4-4} \multicolumn{3}{c|}{} & \bullet & \multicolumn{2}{c}{} \\ \cline{3-4} \multicolumn{2}{c|}{} & & \bullet & \multicolumn{2}{c}{} \\ \cline{2-5} \multicolumn{1}{c|}{} & & & \bullet & \bullet & \multicolumn{1}{c}{} \\ \hline & & & & \bullet & \bullet \\ \hline \end{array} \quad \text{and we have} \quad T_\lambda^{\mathsf{highest}} = \begin{array}{|c|c|c|c|c|c|} \cline{4-4} \multicolumn{3}{c|}{} & 1 & \multicolumn{2}{c}{} \\ \cline{3-4} \multicolumn{2}{c|}{} & 2 & 1 & \multicolumn{2}{c}{} \\ \cline{2-5} \multicolumn{1}{c|}{} & 3 & 2 & 1 & 1 & \multicolumn{1}{c}{} \\ \hline 4 & 3 & 2 & 2 & 1 & 1 \\ \hline \end{array}\ .$$

The $\mathfrak{q}_n$-part of the following is [3, Thm. 2.5(b)], while the $\mathfrak{q}_n^+$-extension is new:

Theorem 5.8. The shifted tableau $T_\lambda^{\mathsf{highest}}$ is the unique $\mathfrak{q}_n$-highest weight element of $\mathsf{DecTab}_n(\lambda)$ and also the unique $\mathfrak{q}_n^+$-highest weight element of $\mathsf{DecTab}_n^+(\lambda)$.

6 Decomposition insertion

This section introduces a "primed" extension of Grantcharov et al.'s insertion scheme from [3, §3]. Suppose T is a primed decomposition tableau and $x \in \mathbb{Z} \sqcup \mathbb{Z}'$. We will form another primed decomposition tableau $x \xrightarrow{\mathsf{dec}} T$ by the following insertion procedure. On step i of this algorithm, a number a_i is inserted into row i of T, starting with $a_1 := x$.

To compute the insertion on step i, set $a = \lceil a_i \rceil$ and remove any prime from middle element m_i of row i (if the row is nonempty). The (unprimed) number a is added to the end of the (now unprimed) row if this creates a hook word; otherwise, a replaces the leftmost entry b from the increasing part of the row with $b \geq a$, then b replaces the leftmost entry c from the weakly decreasing part of the row with $c < b$.

Now we must decide the value a_{i+1} and whether to add back a prime to the middle element of the row. There are two cases:

(A) Suppose the row was initially empty, or the location of the middle element has moved (necessarily to the right). If $a_i \in \mathbb{Z}'$ then we add a prime to the new middle element. If no entries were bumped from the row, then the algorithm halts at this step and we say the insertion is *even* if $m_i \in \mathbb{Z}$ and *odd* if $m_i \in \mathbb{Z}'$. Otherwise, we set $a_{i+1} = c$ when $m_i \in \mathbb{Z}$ and $a_{i+1} = c'$ when $m_i \in \mathbb{Z}'$. For example:

$$\boxed{4}\boxed{2}\boxed{2}\boxed{1^\circ}\boxed{3} \leftarrow 1^\bullet = a_i \quad \rightsquigarrow \quad a_{i+1} = 2^\circ \leftarrow \boxed{4}\boxed{3}\boxed{2}\boxed{1}\boxed{1^\bullet}.$$

Here $\circ$ and $\bullet$ indicate arbitrary, unspecified choice of primes.

(B) Suppose instead that the location of the row's middle element has not changed. If $m_i \in \mathbb{Z}'$ then we add back a prime to the middle element. If no entries were bumped from the row, then the algorithm halts at this step and we say the insertion is *even* if $a_i \in \mathbb{Z}$ and *odd* if $a_i \in \mathbb{Z}'$. Otherwise, we set $a_{i+1} = c$ when $a_i \in \mathbb{Z}$ and $a_{i+1} = c'$ when $a_i \in \mathbb{Z}'$. For example:

$$\boxed{4}\boxed{2}\boxed{2}\boxed{1^\circ}\boxed{3} \leftarrow 3^\bullet = a_i \quad \rightsquigarrow \quad a_{i+1} = 2^\bullet \leftarrow \boxed{4}\boxed{3}\boxed{2}\boxed{1^\circ}\boxed{3}.$$

Definition 6.1. Given any primed word $w = w_m \cdots w_2 w_1$, form

$$P_{\mathsf{dec}}(w) := w_m \xrightarrow{\mathsf{dec}} (\cdots \xrightarrow{\mathsf{dec}} (w_2 \xrightarrow{\mathsf{dec}} (w_1 \xrightarrow{\mathsf{dec}} \varnothing)) \cdots)$$

by inserting the letters of w into the empty tableau $\varnothing$. Let $Q_{\mathsf{dec}}(w)$ be the tableau with the same shape as $P_{\mathsf{dec}}(w)$ that has i (respectively, i') in the box added by $w_i \xrightarrow{\mathsf{dec}}$ if this insertion is even (respectively, odd).

Example 6.2. For $w = 4'4332'3'32'1'$, $P_{\mathsf{dec}}(w) = \begin{array}{|c|c|c|c|c|} \cline{3-3} \multicolumn{2}{c|}{} & 1' & \multicolumn{2}{c}{} \\ \cline{2-4} \multicolumn{1}{c|}{} & 2 & 2' & 3 & \multicolumn{1}{c}{} \\ \hline 4 & 3 & 3 & 3 & 4 \\ \hline \end{array}$ and $Q_{\mathsf{dec}}(w) = \begin{array}{|c|c|c|c|c|} \cline{3-3} \multicolumn{2}{c|}{} & 7 & \multicolumn{2}{c}{} \\ \cline{2-4} \multicolumn{1}{c|}{} & 4 & 5' & 9' & \multicolumn{1}{c}{} \\ \hline 1 & 2' & 3 & 6 & 8 \\ \hline \end{array}$.

A shifted tableau with n boxes is *standard* if its rows and columns are increasing and it has exactly one entry equal to i' or i for each $i \in [n]$.

Theorem 6.3. The map $w \mapsto (P_{\mathsf{dec}}(w), Q_{\mathsf{dec}}(w))$ is a bijection from the set of all words with letters in $\{1' < 1 < 2' < 2 < \dots\}$ to the set of pairs (P, Q) of shifted tableaux with the same shape such that P is a primed decomposition tableau and Q is a standard shifted tableau with no primed diagonal entries.

Let $w^{\mathbf{r}}$ be the reverse of w. On unprimed words, the map $w \mapsto (P_{\mathsf{dec}}(w^{\mathbf{r}}), Q_{\mathsf{dec}}(w^{\mathbf{r}}))$ is [3, Def. 4.1] and gives a bijection to pairs (P, Q) where P is an (unprimed) decomposition tableau and Q is a standard shifted tableau of the same shape with no primed entries.

A map $\phi : \mathcal{B} \to \mathcal{C}$ between ($\mathfrak{gl}_n$, $\mathfrak{q}_n$, or $\mathfrak{q}_n^+$) crystals is a *quasi-isomorphism* if for each full subcrystal $\mathcal{B}' \subseteq \mathcal{B}$ there is a full subcrystal $\mathcal{C}' \subseteq \mathcal{C}$ such that $\phi|_{\mathcal{B}'}$ is an isomorphism $\mathcal{B}' \to \mathcal{C}'$. The $\mathfrak{q}_n$ part of the following more substantial result is [3, Thm. 4.5].

Theorem 6.4. The map P_{dec} defines $\mathfrak{q}_n$ and $\mathfrak{q}_n^+$ crystal quasi-isomorphisms

$$\mathbb{B}_n^{\otimes m} \to \bigsqcup_{\substack{\text{strict } \lambda \vdash m \\ \ell(\lambda) \le n}} \mathsf{DecTab}_n(\lambda) \quad \text{and} \quad (\mathbb{B}_n^+)^{\otimes m} \to \bigsqcup_{\substack{\text{strict } \lambda \vdash m \\ \ell(\lambda) \le n}} \mathsf{DecTab}_n^+(\lambda).$$

Moreover, and the full $\mathfrak{q}_n$-subcrystals of $\mathbb{B}_n^{\otimes m}$ and the full $\mathfrak{q}_n^+$-subcrystals of $(\mathbb{B}_n^+)^{\otimes m}$ are the subsets on which Q_{dec} is constant.

7 Applications to normal crystals

A ($\mathfrak{gl}_n$-, $\mathfrak{q}_n$-, or $\mathfrak{q}_n^+$-) crystal is *normal* if each of its full subcrystals is isomorphic to a full subcrystal of a tensor power of the relevant standard crystal. Normal crystals are automatically preserved by disjoint unions and tensor products.

One motivation for the new results in this article was to provide a simpler and more intuitive proof of the following theorem, which was our main result in [14]. One application of this theorem is a new Littlewood-Richardson rule for multiplying Schur Q-functions [14, Cor. 1.7].

Theorem 7.1. The following properties hold for normal $\mathfrak{q}_n^+$-crystals:

(a) Suppose $\mathcal{B}$ is a connected normal $\mathfrak{q}_n^+$-crystal. Then $\mathcal{B}$ has a unique $\mathfrak{q}_n^+$-highest weight element, whose weight λ is a strict partition with at most n parts, and it holds that $\mathcal{B} \cong \mathsf{DecTab}_n^+(\lambda)$ and $\mathrm{ch}(\mathcal{B}) = Q_\lambda(x_1, x_2, \dots, x_n)$.

(b) For each strict partition λ with at most n parts, there is a connected normal $\mathfrak{q}_n^+$-crystal with highest weight λ.

(c) Finite normal $\mathfrak{q}_n^+$-crystals are isomorphic if and only if they have the same characters, which range over all Schur Q-positive symmetric polynomials in $x_1, x_2, \dots, x_n$.

Proof. If $\mathcal{B}$ is a connected normal $\mathfrak{q}_n^+$-crystal then $\mathcal{B} \cong \mathsf{DecTab}_n^+(\lambda)$ for some strict partition λ with $\ell(\lambda) \leq n$ by Theorem 6.4. Theorem 5.8 implies that $\mathcal{B}$ has a unique $\mathfrak{q}_n^+$-highest weight element of weight λ. This proves part (a). Part (b) follows from Theorems 5.4 and 5.8. Part (c) holds since Schur Q-polynomials are linearly independent. □

The crux of this proof is Theorem 6.4 regarding decomposition insertion. Proving Theorem 6.4 is not a trivial exercise, but this is significantly easier than for the analogous result used in [14], which involves a more technical insertion algorithm defined in [15].

By essentially the same proof, one can derive a $\mathfrak{q}_n$-version of this theorem (involving Schur P-polynomials in place of Schur Q-polynomials); this proof strategy is similar to what appears in [3]. There is also a classical version of Theorem 7.1 for normal $\mathfrak{gl}_n$-crystals (see [1, Thms. 3.2 and 8.6] or [14, Thm. 1.1]) which implies that the character of every finite normal $\mathfrak{gl}_n$-crystal is Schur positive.

8 Set-valued tableaux

Let $\mathbb{M} = \{1' < 1 < 2' < 2 < \dots\}$ and define $\mathsf{Set}(\mathbb{M})$ to be the set of finite, nonempty subsets of $\mathbb{M}$. For $S, T \in \mathsf{Set}(\mathbb{M})$ write $S \prec T$ if $\max(S) < \min(T)$ and $S \preceq T$ is $\max(S) \leq \min(T)$. Finally, for $S \in \mathsf{Set}(\mathbb{M})$ let $x^S = \prod_{i \in S} x_{\mathsf{unprime}(i)}$.

Fix a strict partition λ. A *set-valued shifted tableau* of shape λ is a filling T of the shifted diagram SD_λ by elements of $\mathsf{Set}(\mathbb{M})$. For such tableau T define $x^T = \prod_{(i,j) \in T} x^{T_{ij}}$ where T_{ij} is the entry of T in box (i,j). A set-valued shifted tableau T is *semistandard* if it has all of the following properties: (1) no unprimed number appears twice in the same row, (2) no primed number appears twice in the same columns, and (3) rows and columns are weakly increasing in the sense that $T_{ij} \preceq T_{i,j+1}$ and $T_{ij} \preceq T_{i+1,j}$ for all relevant positions.

Let $\mathsf{SetShTab}^+(\lambda)$ be the set of all semistandard set-valued shifted tableaux of shape λ, and let $\mathsf{SetShTab}_n^+(\lambda)$ be the subset with all entries at most n. Let $\mathsf{SetShTab}(\lambda) \subseteq \mathsf{SetShTab}^+(\lambda)$ and $\mathsf{SetShTab}_n(\lambda) \subseteq \mathsf{SetShTab}_n^+(\lambda)$ be the subsets of tableaux with no primed numbers in any diagonal boxes. The *K-theoretic Schur P- and Q-functions* of λ, as introduced by Ikeda and Naruse [7], are the power series

$$GP_\lambda = \sum_{T \in \mathsf{SetShTab}(\lambda)} x^T \in \mathbb{N}[\![x_1, x_2, \dots]\!] \quad \text{and} \quad GQ_\lambda = \sum_{T \in \mathsf{SetShTab}^+(\lambda)} x^T \in \mathbb{N}[\![x_1, x_2, \dots]\!].$$

Often the definitions of these power series involve a bookkeeping parameter β. Here, for simplicity, we have set $\beta = 1$.

Remark 8.1. It turns out that GP_λ and GQ_λ are both Schur positive symmetric functions, though of unbounded degree [17, Thms. 3.27 and 3.40]. Specializations of GP_λ and GQ_λ give equivariant K-theory representatives for Schubert varities in the maximal isotropic Grassmannians of orthogonal and symplectic types [7, Cor. 8.1]. These symmetric functions have a number of remarkable positivity properties; see [8, 11, 16].

A *distribution* of a tableau with set-valued entries is a tableau of the same shape formed by replacing every set-valued entry by one of its elements. A semistandard set-valued shifted tableau is just a set-valued tableau whose distributions are all semistandard shifted tableaux. Analogously, define a *(semistandard) set-valued decomposition tableau* of strict partition shape λ to be a set-valued shifted tableau whose distributions are each (semistandard) decomposition tableaux of shape λ. Let $\mathsf{SetDecTab}(\lambda)$ be the set of all such tableaux and let $\mathsf{SetDecTab}_n(\lambda)$ be the subset with all entries at most n.

Conjecture 8.2 (Cho–Ikeda [6])**.** It holds that $GP_\lambda = \sum_{T \in \mathsf{SetDecTab}(\lambda)} x^T$.

Remark 8.3. It would be natural to define $\mathsf{SetDecTab}^+(\lambda)$ as the set of set-valued shifted tableaux with entries from $\mathsf{Set}(\mathbb{M})$ whose distributions are each primed decomposition tableaux of shape λ. But in general $GQ_\lambda \neq \sum_{T \in \mathsf{SetDecTab}^+(\lambda)} x^T$ and it remains an open problem to find even a conjectural decomposition tableau formula for GQ-functions.

Crystals for $\mathfrak{gl}_n$ have been identified on unshifted (semistandard) set-valued tableaux (see, e.g., [5, 18, 20]), and it is a natural open problem to find similar structures on shifted tableaux. We have identified one such crystal structure on set-valued decomposition tableaux, which implies a weaker form of Conjecture 8.2.

Fix a strict partition λ and $T \in \mathsf{SetDecTab}(\lambda)$. The *reverse reading word* of T is the word $\mathsf{revrow}(T)$ formed by iterating over the boxes of T in the reverse reading word order (starting with the last box of the first row and proceeding row by row, reading each row right to left), and listing the entries of each box in decreasing order. Define $\mathrm{wt}(T) = \mathrm{wt}(\mathsf{revrow}(T))$. For example, if $T = \begin{array}{ccc} \cline{2-3} & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{3} \\ \hline \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{123} & \multicolumn{1}{c|}{234} \\ \hline \end{array}$ then $\mathsf{revrow}(T) = 432321431$ and $\mathrm{wt}(T) = (2, 2, 3, 2)$.

Fix $i \in [n-1]$. Mark each i in $\mathsf{revrow}(T)$ by a right parenthesis ")" and each $i+1$ by a left parenthesis "(". A letter in $\mathsf{revrow}(T)$ is *i-unpaired* if it is equal to i or $i+1$ but does not belong to a matching pair of parentheses.

Definition 8.4. Given $i \in \mathbb{P}$ and a set-valued decomposition tableau T, construct $e_i(T)$ in the following way. Define $e_i(T) = 0$ if there are no i-unpaired letters equal to $i+1$. Otherwise, suppose the first i-unpaired $i+1$ in $\mathsf{revrow}(T)$ occurs in box (x, y) of T.

(a) Form $e_i(T)$ from T by changing the $i+1$ in box (x, y) to i if this yields a set-valued decomposition tableau. For example, $e_2 : \begin{array}{|c|c|c|} \hline & 1 & 2 \\ \hline 3 & 13 & 123 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline & 1 & 2 \\ \hline 3 & 12 & 123 \\ \hline \end{array}$.

(b) Otherwise, some box (a, b) preceding (x, y) in the reverse row reading word order has $\{i, i+1\} \subseteq T_{ab}$. If (a, b) is the last such box, then form $e_i(T)$ by removing $i+1$ from T_{ab} and adding i to T_{xy}. One can show that the box (a, b) must either have $a = x$ and $b > y$, or $a = x - 1$ and $b < y$, as in the examples

$$e_3 : \begin{array}{|c|c|c|} \hline & 1 & 2 \\ \hline 4 & 1 & 34 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline & 1 & 2 \\ \hline 34 & 1 & 3 \\ \hline \end{array} \quad \text{and} \quad e_2 : \begin{array}{|c|c|c|} \hline & 1 & 3 \\ \hline 4 & 123 & 234 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline & 1 & 23 \\ \hline 4 & 12 & 234 \\ \hline \end{array}.$$

Theorem 8.5. For each strict partition λ with at most n parts, $\mathsf{SetDecTab}_n(\lambda)$ has a $\mathfrak{gl}_n$-crystal structure for the raising operators $e_1, e_2, \ldots, e_{n-1}$ given in Definition 8.4.

Corollary 8.6. The power series $\sum_{T \in \mathsf{SetDecTab}(\lambda)} x^T$ is symmetric.

We can slightly extend this partial progress on Ikeda's conjecture. A power series $f \in \mathbb{Z}[\![x_1, x_2, \ldots]\!]$ satisfies the *K-theoretic Q-cancelation property* if for all $1 \leq i < j$ the power series $f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_{j-1}, \frac{-t}{1+t}, x_{j+1}, \ldots)$ does not depend on t, that is, belongs to $\mathbb{Z}[\![x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots]\!]$. The symmetric functions satisfying the K-theoretic Q-cancelation property are exactly the ones that may be (uniquely) expressed as formal (i.e., possibly infinite) $\mathbb{Z}$-linear combinations of GP-functions [7, Prop. 3.4].

Proposition 8.7. The symmetric power series $\sum_{T \in \mathsf{SetDecTab}(\lambda)} x^T$ has the K-theoretic Q-cancelation property and lowest degree term P_λ, so is equal to GP_λ plus a (possibly infinite) $\mathbb{Z}$-linear combination of GP_μ's with $|\mu| > |\lambda|$.

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