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A signed e-expansion of the chromatic symmetric function and some new e-positive graphs

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Abstract. We prove a new signed elementary symmetric function expansion of the chromatic symmetric function of any unit interval graph. We then use sign-reversing involutions to prove new combinatorial formulas for many families of graphs, including the K-chains studied by Gebhard and Sagan, formed by joining cliques at single vertices, and for graphs obtained from them by removing any number of edges from any of the cut vertices. We also introduce a version for the quasisymmetric refinement of Shareshian and Wachs.

Keywords: chromatic symmetric function, elementary symmetric function, Stanley–Stembridge conjecture, unit interval graph

The Stanley–Stembridge conjecture [26, 27] is one of the most actively researched open problems in algebraic combinatorics today. It asserts that if G is the incomparability graph of a $(\mathbf{3}+\mathbf{1})$-free poset, then G is *e-positive*, meaning that the chromatic symmetric function $X_G(\boldsymbol{x})$ defined by Stanley [26] is a nonnegative linear combination of elementary symmetric functions. Several authors have proven that certain graphs are e-positive [3, 6, 9, 11, 14, 16, 21, 29, 30, 31], studied other positivity properties of $X_G(\boldsymbol{x})$, [4, 12, 13, 15, 17, 19, 22], defined generalizations of the chromatic symmetric function, [10, 18, 25], and explored implications of the Stanley–Stembridge conjecture to immanants of Jacobi–Trudi matrices [27], cohomology of Hessenberg varieties [1, 5, 7, 20, 24], and characters of Hecke algebras [8].

In this extended abstract, we give a signed elementary function expansion of $X_G(\boldsymbol{x})$ for any unit interval graph G, in terms of objects called *forest triples*. We then show how sign-reversing involutions on forest triples can be used to prove combinatorial formulas for many classes of unit interval graphs, including the K-chains proven to be e-positive by Gebhard and Sagan [18] and *melting K-chains* obtained from them by removing any number of edges from any of the cut vertices. Melting K-chains were not previously known to be e-positive. We also present a generalization of our forest triple formula for the chromatic quasisymmetric function of Shareshian and Wachs [25].

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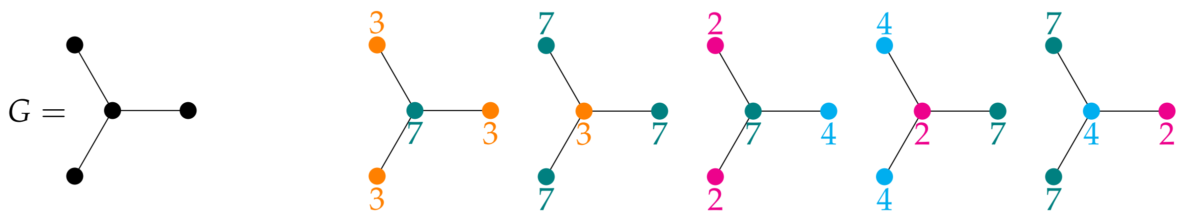
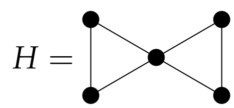
Figure 1: The claw graph G, five proper colourings of G, the corresponding monomials, the chromatic symmetric function $X_G(\boldsymbol{x})$, the bowtie graph H, and the chromatic symmetric function $X_H(\boldsymbol{x})$



$$X_G(\boldsymbol{x}) = \cdots + x_3^3 x_7 + x_3 x_7^3 + \cdots + x_2^2 x_4 x_7 + x_2 x_4^2 x_7 + x_2 x_4 x_7^2 + \cdots \tag{1.3}$$
$$= e_{211} - 2e_{22} + 5e_{31} + 4e_4 \tag{1.4}$$



$$X_H(\boldsymbol{x}) = 4e_{32} + 12e_{41} + 20e_5 \tag{1.5}$$

1 Background

Let $G = (V, E)$ be a graph. A *colouring* of G is a function $\kappa : V \to \mathbb{P} = \{1, 2, 3, \ldots\}$ and we say that κ is *proper* if $\kappa(i) \neq \kappa(j)$ whenever $(i, j) \in E$. The *chromatic symmetric function* of G is the formal power series in infinitely many variables $\boldsymbol{x} = (x_1, x_2, x_3, \ldots)$ given by [26, Definition 2.1]

$$X_G(\boldsymbol{x}) = \sum_{\kappa: V \to \mathbb{P} \text{ proper}} \boldsymbol{x}^\kappa, \text{ where } \boldsymbol{x}^\kappa = \prod_{v \in V} x_{\kappa(v)}. \tag{1.1}$$

We are interested in expanding the symmetric function $X_G(\boldsymbol{x})$ in the basis $\{e_\lambda\}$ of *elementary symmetric functions* indexed by integer partitions $\lambda = \lambda_1 \cdots \lambda_\ell$, defined by

$$e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}, \text{ where } e_n = \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}. \tag{1.2}$$

We say that G is *e-positive* if the chromatic symmetric function $X_G(\boldsymbol{x})$ is a nonnegative linear combination of elementary symmetric functions.

Example 1. *Figure 1 shows the claw graph G. Some proper colourings of G, the corresponding monomials of $X_G(\boldsymbol{x})$, and the e-expansion of $X_G(\boldsymbol{x})$ are given. Because of the negative coefficient* *on the term* $-2e_{22}$*, the graph* G *is not e-positive. By contrast, the bowtie graph* H *is e-positive. For the complete graph* K_n*, proper colourings must use* n *distinct colours and given* n *distinct colours there are* $n!$ *proper colourings, so* $X_{K_n}(\boldsymbol{x}) = n!e_n$ *and* K_n *is e-positive.*

There has been considerable interest in characterizing e-positive graphs. The most prominent open problem in this direction is the Stanley–Stembridge conjecture [26, Corollary 5.1], equivalently [27, Corollary 5.5], which by a result of Guay-Paquet [19, Theorem 5.1] can be equivalently stated for *unit interval graphs* G, which are graphs whose vertices can be labelled 1 through n so that

$$\text{for } i < j < k, \text{ if } (i,k) \in E(G), \text{ then } (i,j) \in E(G) \text{ and } (j,k) \in E(G). \tag{1.6}$$

Conjecture 1. *(Stanley–Stembridge conjecture) All unit interval graphs are e-positive.*

2 A signed formula

Let $G = ([n], E)$ be a *natural unit interval graph*, meaning it satisfies (1.6). We give a signed combinatorial formula for the elementary symmetric function expansion of $X_G(\boldsymbol{x})$.

Definition 1. *A subtree* T *of* G *is* decreasing *if every vertex* $v \in V(T)$ *has at most one larger neighbour. A subforest* F *of* G *is* decreasing *if all of its trees are decreasing.*

Definition 2. *A* tree triple *of* G *is an object* $\mathcal{T} = (T, \alpha, r)$ *consisting of the following data.*

- T *is a decreasing subtree of* G*.*
- α *is an integer composition with size* $|\alpha| = |V(T)|$*.*
- r *is a positive integer with* $1 \le r \le \alpha_1$*, the first part of* α*.*

A forest triple *of* G *is a set of tree triples* $\mathcal{F} = \{\mathcal{T}_i = (T_i, \alpha^{(i)}, r_i)\}_{i=1}^m$ *with* $\sqcup_{i=1}^m V(T_i) = [n]$*, so the set of trees is a decreasing spanning forest of* G*. The* type *of* $\mathcal{F}$ *is the integer partition*

$$type(\mathcal{F}) = sort(\alpha^{(1)} \cdots \alpha^{(m)}) \tag{2.1}$$

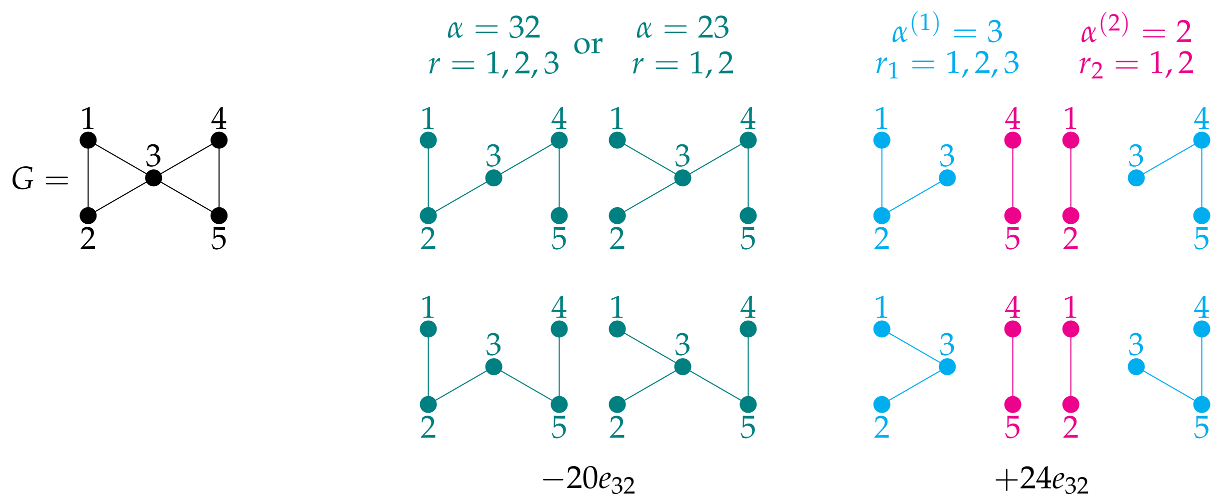
formed by concatenating the compositions and then sorting to form a partition. The sign *of* $\mathcal{F}$ *is the integer*

$$sign(\mathcal{F}) = (-1)^{\sum_{i=1}^m (\ell(\alpha^{(i)})-1)} = (-1)^{\ell(type(\mathcal{F}))-m}, \tag{2.2}$$

where $\ell(\alpha)$ *is the length of a composition* α*. We denote by* $FT(G)$ *the set of forest triples of* G *and by* $FT_\mu(G)$ *the set of forest triples of* G *of type* μ*.*

We now state our combinatorial formula. It was proven by first expanding $X_G(\boldsymbol{x})$ in the power sum basis and then applying a change-of-basis to the elementary symmetric function basis. The technique of studying properties of $X_G(\boldsymbol{x})$ by converting between different bases is explored in upcoming work of Sagan and the author [23].

Figure 2: The bowtie graph G and the forest triples of G of type 32



Theorem 1. *[28, Theorem 4.3] Let G be a natural unit interval graph. The chromatic symmetric function $X_G(\boldsymbol{x})$ satisfies*

$$X_G(\boldsymbol{x}) = \sum_{\mathcal{F}\in FT(G)} sign(\mathcal{F}) e_{type(\mathcal{F})} = \sum_{\mu} \left(\sum_{\mathcal{F}\in FT_\mu(G)} sign(\mathcal{F}) \right) e_\mu. \tag{2.3}$$

Example 2. *Figure 2 shows the forest triples of type 32 for the bowtie graph. We can have a single tree triple $\mathcal{T} = (T, \alpha, r)$, in which case α is either 32 or 23 and there are either 3 or 2 choices for r. Alternatively, we can have two tree triples $\mathcal{T}_1 = (T_1, \alpha^{(1)}, r_1)$ and $\mathcal{T}_2 = (T_2, \alpha^{(2)}, r_2)$ with $\alpha^{(1)} = 3 = |V(T_1)|$ and $\alpha^{(2)} = 2 = |V(T_2)|$, and there are 3 choices of r_1 and 2 choices of r_2.*

Example 3. *For the case of $\mu = n$, forest triples $\mathcal{F} \in FT_n(G)$ consist of a single tree triple $\mathcal{T} = (T, \alpha, r)$, where T is a decreasing spanning tree, we must have $\alpha = n$ so $sign(\mathcal{F}) = 1$, and we can have any value of $1 \le r \le n$. Because a decreasing spanning tree can be identified by specifying the unique larger neighbour of each vertex $1 \le i \le n-1$, we have that the coefficient of e_n in $X_G(\boldsymbol{x})$ is $nd_1 \cdots d_{n-1}$, where d_i is the number of larger neighbours of vertex i in G.*

Now our goal is to find a sign-reversing involution on forest triples of G to combinatorially prove that G is e-positive. The structure of forest triples suggests the following approach. Let us say that a tree triple $\mathcal{T} = (T, \alpha, r)$ is *breakable* if $\ell(\alpha) \ge 2$. In this case, we would like to somehow define a pair of forest triples break$(\mathcal{T}) = (\mathcal{S}_1, \mathcal{S}_2)$ of the form

$$\mathcal{S}_1 = (S_1, \alpha \setminus \alpha_\ell, r) \text{ and } \mathcal{S}_2 = (S_2, \alpha_\ell, r_2) \tag{2.4}$$

for some decreasing trees S_1 and S_2 with $V(S_1) \sqcup V(S_2) = V(T)$ and some integer $1 \le r_2 \le \alpha_\ell$, where α_ℓ is the last part of α and $\alpha \setminus \alpha_\ell$ denotes the composition with α_ℓ removed. Let us say that the pair $(\mathcal{S}_1, \mathcal{S}_2)$ is *joinable* if it is of the form break$(\mathcal{T})$ for some unique $\mathcal{T}$, which we will denote join$(\mathcal{S}_1, \mathcal{S}_2)$. Then we would like to somehow define a map φ on FT(G) by either replacing some breakable tree triple $\mathcal{T}$ by break$(\mathcal{T})$ or by replacing some joinable pair of tree triples $(\mathcal{S}_1, \mathcal{S}_2)$ by join$(\mathcal{S}_1, \mathcal{S}_2)$, if one exists.

If we can systematically choose which tree triples to replace so that φ is an involution, then it would reverse sign because it changes the total number of tree triples by one, it would preserve type by construction, and fixed points $\mathcal{F}$ must have no breakable tree triples or joinable pairs of tree triples so in particular sign$(\mathcal{F}) = (-1)^{\sum_{i=1}^{m}(1-1)} = 1$. Therefore, we would prove that G is e-positive, and we would also get a combinatorial formula for the chromatic symmetric function $X_G(\boldsymbol{x})$ in terms of the fixed points of φ.

We now demonstrate this method in the case of paths. More general results are known [25, Section 5], [26, Proposition 5.3] but this proof technique is new.

Proposition 1. *The chromatic symmetric function of a path P_n is given by*

$$X_{P_n}(\boldsymbol{x}) = \sum_{\alpha \vDash n} \alpha_1(\alpha_2 - 1) \cdots (\alpha_\ell - 1) e_{sort(\alpha)}, \tag{2.5}$$

where the notation $\alpha \vDash n$ means that α is a composition with size n. In particular, P_n is e-positive.

Proof. We label the vertices of P_n so that its edges are of the form $(i, i+1)$, so decreasing subtrees of P_n are paths from some i to some $j > i$, which we will denote $P_{i \to j}$. For a breakable tree triple $\mathcal{T} = (P_{i \to j}, \alpha, r)$ of P_n, we define break$(\mathcal{T}) = (\mathcal{S}_1, \mathcal{S}_2)$, where

$$\mathcal{S}_1 = (P_{i \to j - \alpha_\ell}, \alpha \setminus \alpha_\ell, r) \text{ and } S_2 = (P_{j-\alpha_\ell+1 \to j}, \alpha_\ell, 1), \tag{2.6}$$

and we define a pair of tree triples $(\mathcal{S}_1 = (P_{i \to j}, \alpha^{(1)}, r_1), \mathcal{S}_2 = (P_{i' \to j'}, \alpha^{(2)}, r_2))$ to be *joinable* if $\ell(\alpha^{(2)}) = 1$, $i' = j + 1$, and $r_2 = 1$, in which case we define the tree triple

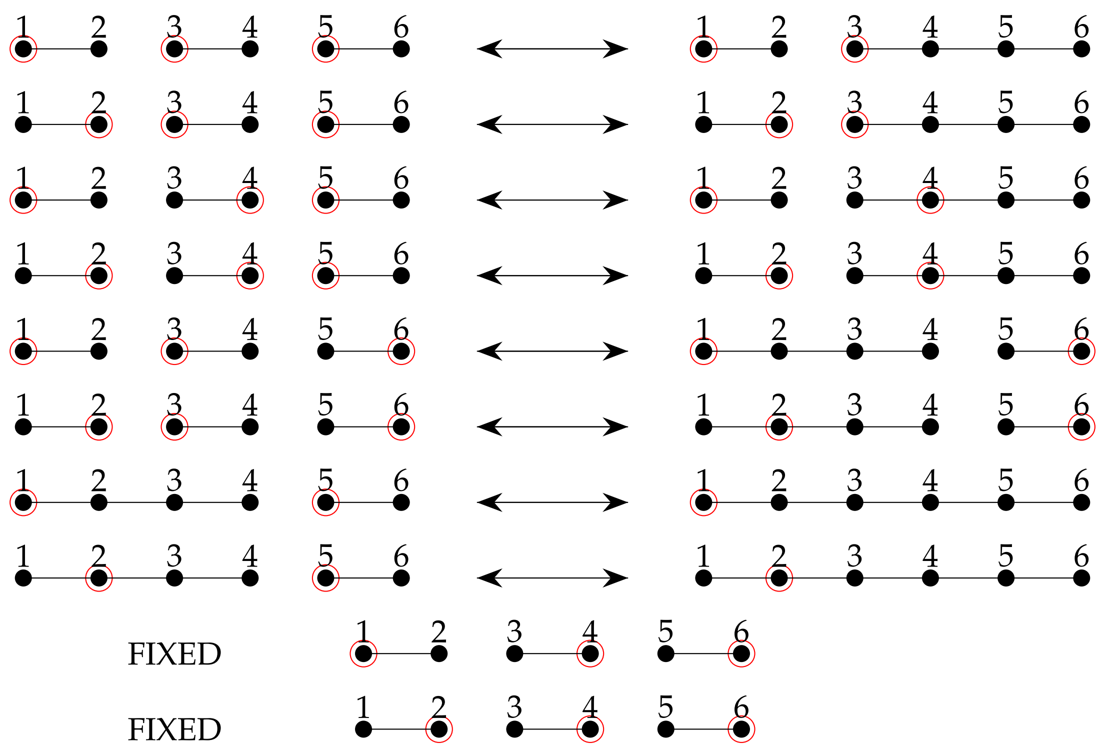
$$\text{join}(\mathcal{S}_1, \mathcal{S}_2) = (P_{i \to j'}, \alpha^{(1)} \cdot \alpha^{(2)}, r_1). \tag{2.7}$$

Note that $\mathcal{T}$ is breakable with break$(\mathcal{T}) = (\mathcal{S}_1, \mathcal{S}_2)$ if and only if $(\mathcal{S}_1, \mathcal{S}_2)$ is joinable with join$(\mathcal{S}_1, \mathcal{S}_2) = \mathcal{T}$. Now given a forest triple

$$\mathcal{F} = \{\mathcal{T}_1 = (P_{i_1 \to i_2 - 1}, \alpha^{(1)}, r_1), \mathcal{T}_2 = (P_{i_2 \to i_3 - 1}, \alpha^{(2)}, r_2), \ldots, \mathcal{T}_\ell = (P_{i_\ell \to n}, \alpha^{(\ell)}, r_\ell)\}, \tag{2.8}$$

we let j be maximal so that either $\mathcal{T}_j$ is breakable or the pair $(\mathcal{T}_{j-1}, \mathcal{T}_j)$ is joinable, if such a j exists, and we define $\varphi(\mathcal{F})$ by either breaking $\mathcal{T}_j$, joining $(\mathcal{T}_{j-1}, \mathcal{T}_j)$, or doing nothing if no such j exists. We can check that φ is a sign-reversing involution, the fixed points can be associated with a composition $\alpha \vDash n$ by reading the tree sizes from left to right, they have type sort(α), and in order for no pair to be joinable, we must have $r_i \ge 2$ for every $i \ge 2$, so there are $\alpha_1(\alpha_2 - 1) \cdots (\alpha_\ell - 1)$ choices of the r_i. □

Figure 3: The forest triples of P_6 of type 222 paired under our sign-reversing involution φ



Example 4. *Figure 3 shows all of the forest triples of P_6 of type 222 paired under our sign-reversing involution φ. The compositions are not written but they are 2, 22, or 222. We have indicated whether each r_i is 1 or 2 by circling the r_i-th smallest vertex of the corresponding tree. There are $2(2-1)(2-1) = 2$ fixed points, so the coefficient of e_{222} in $X_{P_6}(\boldsymbol{x})$ is 2.*

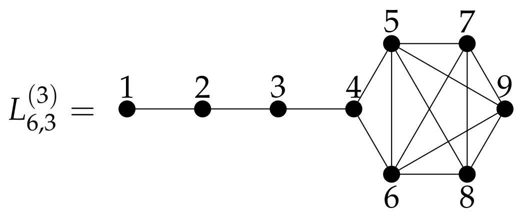
3 Positive formulas

We are able to use sign-reversing involutions on forest triples to prove several combinatorial e-positive expansions of unit interval graphs.

Definition 3. *Let $L_{a,b}^{(t)}$ denote the unit interval graph where a path of length b is joined to a clique of size a, and then t edges incident to the joined vertex are removed from the clique. An example is given in Figure 4. Such graphs are called* melting lollipops.

Huh, Nam, and Yoo showed that melting lollipops are e-positive [21, Theorem 4.9]. A result of Aliniaeifard, Wang, and van Willigenburg [3, Proposition 3.1] implies that

Figure 4: The melting lollipop graph $L_{6,3}^{(3)}$



the sequence $(X_{L_{a,b}^{(t)}}(\boldsymbol{x}))_{t=0}^{a-1}$ forms an arithmetic progression. We were able to use forest triples to get a new explicit formula and see this arithmetic progression directly.

Theorem 2. *[28, Theorem 5.2] The chromatic symmetric function of a melting lollipop $L_{a,b}^{(t)}$ is given by*

$$X_{L_{a,b}^{(t)}}(\boldsymbol{x}) = t(a-2)! \sum_{\substack{\alpha \models n \\ \alpha_\ell = a-1}} \alpha_1(\alpha_2 - 1) \cdots (\alpha_{\ell-1} - 1) e_{sort(\alpha)} \\ + (a-t-1)(a-2)! \sum_{\substack{\alpha \models n \\ \alpha_\ell \geq a}} \alpha_1(\alpha_2 - 1) \cdots (\alpha_\ell - 1) e_{sort(\alpha)}. \tag{3.1}$$

In particular, $L_{a,b}^{(t)}$ is e-positive.

Definition 4. *For a composition γ with all parts at least 2, let K_γ denote the unit interval graph where cliques of sizes $\gamma_1, \ldots, \gamma_\ell$ are successively joined end to end at single vertices. An example is given in Figure 5. Such graphs are called K-*chains.

Gebhard and Sagan showed that K-chains are e-positive [18, Corollary 7.7] by using a generalization of the chromatic symmetric function in noncommuting variables. We were able to use forest triples to get a new explicit formula as a sum over a certain set A_γ of weak compositions.

Theorem 3. *[28, Theorem 6.13] The chromatic symmetric function of a K-chain K_γ is given by*

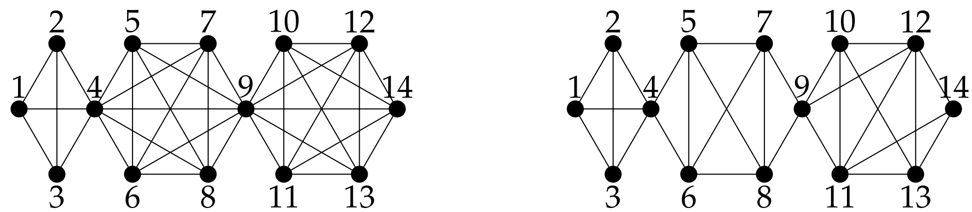
$$X_{K_\gamma}(\boldsymbol{x}) = (\gamma_1 - 2)! \cdots (\gamma_{\ell-1} - 2)!(\gamma_\ell - 1)! \sum_{\alpha \in A_\gamma} \left(\alpha_1 \prod_{i=2}^{\ell(\gamma)} |\alpha_i - (\gamma_{i-1} - 1)| \right) e_{sort(\alpha)} \tag{3.2}$$

In particular, K_γ is e-positive.

Example 5. *If $\gamma = ab$ has length 2, we get that*

$$X_{K_{ab}}(\boldsymbol{x}) = (a-1)!(b-1)! \sum_{k=\max\{a,b\}}^{n} (2k-n) e_{k,n-k}. \tag{3.3}$$

Figure 5: The K-chain K_{466} and the melting K-chain $K_{466}^{(032,032)}$



Definition 5. *Let γ be a composition with all parts at least 2, and let ϵ and ζ be weak compositions with $\ell(\epsilon) = \ell(\zeta) = \ell(\gamma)$ such that $0 \leq \epsilon_t, \zeta_t \leq \gamma_t - 2$ for all t and $\epsilon_t = 0$ if and only if $\zeta_t = 0$. Let $K_\gamma^{(\epsilon,\zeta)}$ denote the unit interval graph formed by removing edges from the K-chain K_γ so that for all t, the t-th clique has ϵ_t edges absent from the smallest vertex and ζ_t edges absent from the largest vertex. An example is given in Figure 5. Such graphs are called* melting K-chains, *and if every $\epsilon_t, \zeta_t \in \{0, 1\}$ (so $\epsilon = \zeta$), they are called* slightly melting K-chains.

Aliniaeifard, Wang, and van Willigenburg showed that slightly melting K-chains are e-positive [3, Proposition 5.5]. We were able to use forest triples to get a new explicit formula as a sum over a certain set $A_\gamma^{(\epsilon)}$ of weak compositions.

Theorem 4. *[28, Theorem 7.9] The chromatic symmetric function of a slightly melting K-chain $K_\gamma^{(\epsilon,\epsilon)}(\boldsymbol{x})$ is given by*

$$X_{K_\gamma^{(\epsilon,\epsilon)}}(\boldsymbol{x}) = (\gamma_1 - 2)! \cdots (\gamma_\ell - 2)! \sum_{\alpha \in A_\gamma^\epsilon} \left(\alpha_1 \prod_{i=1}^{\ell(\gamma)} |\alpha_{i+1} - (\gamma_i - 1 - \epsilon_i)| \right) e_{sort(\alpha)}. \tag{3.4}$$

In particular, $K_\gamma^{(\epsilon,\epsilon)}$ is e-positive.

We also proved the new result that all melting K-chains are e-positive. We have a combinatorial description of the fixed points but they are much more complicated to describe and enumerate.

Theorem 5. *[28, Theorem 8.3] All melting K-chains $K_\gamma^{(\epsilon,\zeta)}$ are e-positive.*

It would be interesting to see whether sign-reversing involutions on forest triples could be used to show e-positivity of other unit interval graphs. Alternatively, we could take a dual approach where we fix μ and show that the coefficient of e_μ is nonnegative for every unit interval graph. This is done by Hwang [22, Theorem 5.13] if $\mu_2 = 1$, by Abreu and Nigro [1, Corollary 1.10] if $\ell(\mu) = 2$, and in upcoming work by Sagan and the author [23] if $\mu_1 \leq 3$. If we can prove the following inequality, the forest triple formula would give another proof of nonnegativity for all two-part partitions.

Problem 1. *Let $G = ([n], E)$ be a natural unit interval graph and let $1 \le k \le n-1$. Let $s_k(G)$ be the number of decreasing spanning forests (T_1, T_2) of G, where $|V(T_2)| = k$ and $1 \in V(T_1)$. Let $s(G)$ be the number of decreasing spanning trees of G. Prove that $ks_k(G) \ge s(G)$.*

The author checked by computer that this inequality holds for all unit interval graphs G with $n \le 10$ vertices.

4 A quasisymmetric generalization

We also generalize our forest triple formula for the *chromatic quasisymmetric function* defined by Shareshian and Wachs [25, Definition 1.2].

Definition 6. *The* chromatic quasisymmetric function *of a labelled graph $G = ([n], E)$ is the formal power series*

$$X_G(\boldsymbol{x}; q) = \sum_{\substack{\kappa:[n]\to\mathbb{P} \\ \kappa \text{ proper}}} q^{asc(\kappa)} \boldsymbol{x}^{\kappa}, \tag{4.1}$$

where $asc(\kappa) = |\{(i,j) \in E(G) : i < j,\ \kappa(i) < \kappa(j)\}|$.

Alexandersson used the following idea to study e-positivity of LLT polynomials [2].

Definition 7. *Let $\theta \subseteq E(G)$. For a vertex $u \in [n]$, let $hrv_\theta(u)$ be the highest $v \in [n]$ reachable from u by an increasing path in $([n], \theta)$ and let $\{[u_1]_\theta, \ldots, [u_m]_\theta\}$ be the set of equivalence classes of $[n]$ under the relation $u \sim_\theta u'$ if $hrv_\theta(u) = hrv_\theta(u')$. Let $\theta' \subseteq \theta$ be the subset of edges used by the increasing paths from every u to $hrv_\theta(u)$ that go to the largest possible vertex at each step. We let $U(\theta) = \theta \setminus \theta'$ and the elements of $U(\theta)$ are called* unnecessary edges.

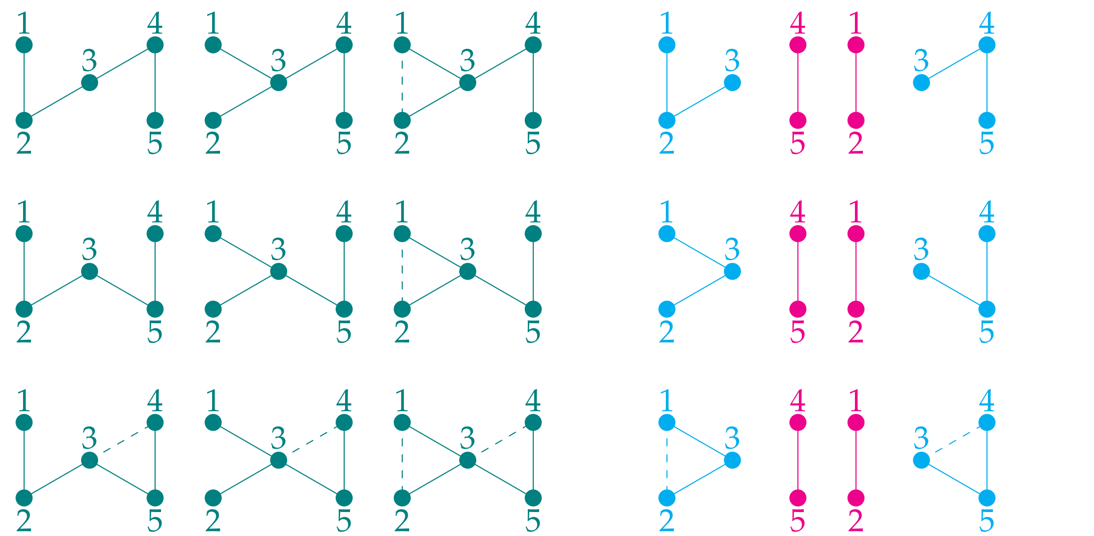
Definition 8. *A* subgraph quadruple *of G is an object $\mathcal{S} = (\theta, f, \boldsymbol{\alpha}, \boldsymbol{r})$ consisting of the following data.*

- *$\theta \subseteq E(G)$ is a subset of the edges of G.*
- *$f : U(\theta) \to \{q, -1\}$ is a function that assigns either a q or a (-1) to each unnecessary edge.*
- *$\boldsymbol{\alpha} = (\alpha^{(1)}, \ldots, \alpha^{(m)})$ is a sequence of compositions such that each $|\alpha^{(i)}| = |[u_i]_\theta|$.*
- *$\boldsymbol{r} = (r_1, \ldots, r_m)$ is a sequence of positive integers such that each $1 \le r_i \le \alpha_1^{(i)}$.*

The type *of $\mathcal{S}$ is the partition*

$$type(\mathcal{S}) = sort(\alpha^{(1)} \cdots \alpha^{(m)}), \tag{4.2}$$

Figure 6: The edges of the subgraph quadruples of type 32 for the bowtie graph G



$-(4+4(q-1)+(q-1)^2)(1+q+q^2+1+q)e_{32}$ $\quad$ $(4+2(q-1))(1+q+q^2)(1+q)e_{32}$

the sign *of* $\mathcal{S}$ *is the integer*

$$sign(\mathcal{S}) = (-1)^{\sum_{i=1}^{m}(\ell(\alpha^{(i)})-1)}(-1)^{|\{e\in U(\theta):f(e)=-1\}|}, \tag{4.3}$$

and the weight *of* $\mathcal{S}$ *is the integer*

$$weight(\mathcal{S}) = \sum_{i=1}^{m}(r_i-1) + |\{e \in U(\theta) : f(e) = q\}|. \tag{4.4}$$

We denote by $SQ(G)$ *the set of subgraph quadruples of* G.

Theorem 6. *[28, Theorem 9.5] The chromatic quasisymmetric function* $X_G(\boldsymbol{x};q)$ *of a natural unit interval graph* G *satisfies*

$$X_G(\boldsymbol{x};q) = \sum_{\mathcal{S}\in SQ(G)} sign(\mathcal{S})q^{weight(\mathcal{S})}e_{type(\mathcal{S})}. \tag{4.5}$$

Example 6. *Figure 6 shows the edges of the subgraph quadruples of type* 32 *for the bowtie graph, where the unnecessary edges are shown by dotted lines and would each be assigned a* q *or a* (-1). *If we have a single equivalence class, then* α *is either* 32 *or* 23 *and there are either* 3 *or* 2 *choices for* r. *If we have two, then* $\alpha^{(1)} = 3$, $\alpha^{(2)} = 2$, *there are* 3 *choices of* r_1, *and* 2 *choices of* r_2. *We have written the contributions to the coefficient of* e_{32}, *taking into account the choices of* f *and* $\boldsymbol{r}$.

We can apply the earlier sign-reversing involution to subgraph quadruples to prove the following combinatorial e-expansion of Shareshian and Wachs [25, Section 5].

Proposition 2. *The chromatic quasisymmetric function of the path P_n is given by*

$$X_{P_n}(\boldsymbol{x};q) = \sum_{\alpha \vDash n} q^{\ell(\alpha)-1}[\alpha_1]_q[\alpha_2-1]_q \cdots [\alpha_\ell - 1]_q e_{sort(\alpha)}, \tag{4.6}$$

where the vertices of P_n are labelled so that (1.6) holds and we define $[k]_q = 1 + q + \cdots + q^{k-1}$.

We could try to adapt our other sign-reversing involutions to subgraph quadruples.

Problem 2. *Use subgraph quadruples to prove combinatorial e-positive expansions for the chromatic quasisymmetric functions $X_{K_\gamma}(\boldsymbol{x};q)$ and $X_{K_\gamma^{(\epsilon,\zeta)}}(\boldsymbol{x};q)$.*

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