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Weak Bruhat interval 0-Hecke modules in finite type

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Abstract. We extend the weak Bruhat interval modules, defined for 0-Hecke algebras in type A, to arbitrary finite types. We determine structural properties, with a main focus on projective covers and injective hulls, for certain general families of these modules in a type-independent way. As an application, we recover a number of results for type A 0-Hecke modules in a uniform manner. We also obtain some further results relating to recently-introduced type A 0-Hecke modules.

Keywords: 0-Hecke algebra, Coxeter group, projective cover, quasisymmetric functions

1 Introduction

The 0-Hecke algebra $H_W(0)$ associated to a finite Coxeter group W is a certain deformation of the group algebra of W. Norton [20] classified the projective indecomposable modules and the simple modules over $H_W(0)$ up to isomorphism. Subsequently, Fayers [12] proved that $H_W(0)$ is a Frobenius algebra, and Huang [14] provided combinatorial interpretations of the projective indecomposable modules for each classical type.

In type A, the quasisymmetric characteristic [11] provides an isomorphism between the Grothendieck group of type A 0-Hecke modules and the ring of quasisymmetric functions. Due to this connection, the past decade has seen significant activity related to constructing 0-Hecke modules in type A that correspond to various notable bases of quasisymmetric functions; examples include [2, 5, 19, 21, 22]. There has also been a focus on understanding the structure of such modules, especially regarding indecomposability, projective covers and injective hulls, for example, Choi, Kim, Nam and Oh [8] applied the ribbon tableau model of [14] to obtain the projective covers of modules in [5, 21, 22, 23]. A notable development in this regard was the introduction of *weak Bruhat interval modules* by Jung, Kim, Lee and Oh [15]. These modules, defined in terms of intervals in the left weak Bruhat order on symmetric groups, provided a uniform approach to understanding modules associated to quasisymmetric functions. It was also determined

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in [15] how these modules behave under certain equivalences of categories introduced in [12]. This was applied in [15] to determine structural properties of certain families of modules by realising them as images under these functors of other families of modules for which corresponding properties were known.

In this extended abstract, we summarise results from [3]. We extend the definition of weak Bruhat interval modules to all Coxeter types, and we show the projective indecomposable $H_W(0)$-modules are weak Bruhat interval modules, as proven in type A in [15]. Throughout, we use a type-independent description of the projective indecomposable $H_W(0)$-modules in terms of right descents of elements of W. We extend certain results of [15] regarding equivalences of categories to all types and to quotients and submodules of weak Bruhat interval modules, and identify a type-independent indecomposability criterion that covers a number of the type A modules associated to quasisymmetric functions. We determine projective covers of a significant family of $H_W(0)$-modules in a type-independent way, and use our results concerning the equivalences of categories to obtain injective hulls of a related family of $H_W(0)$-modules. Our approach works directly with elements of W, and our results are stated in terms of right descent sets. Finally, we apply this approach to recover a number of results on indecomposability, projective covers and injective hulls for type A families of $H_W(0)$-modules in a uniform manner. We also obtain some new results for certain modules recently introduced in [19].

2 0-Hecke algebras and weak Bruhat interval modules

A *finite Coxeter system* (W, S) is a finite group W with generating set S satisfying the relations $s^2 = 1$ for all $s \in S$, and $(st)_{m(s,t)} = (ts)_{m(s,t)}$ for all pairs of distinct elements $s, t \in S$, where $m(s,t) = m(t,s) \in \mathbb{Z}_{\geq 2}$ and $(st)_{m(s,t)}$ denotes the alternating product of s and t with $m(s,t)$ factors. For $w \in W$, the length $\ell(w)$ of w is the minimal number of terms appearing in a product of elements of S equal to w; any such product with minimal number of terms is a *reduced word* for W.

An element $s \in S$ is a *left descent* of $w \in W$ if $\ell(sw) = \ell(w) - 1$, and a *right descent* of w if $\ell(ws) = \ell(w) - 1$. Let $\mathrm{D}_L(w)$ denote the set of left descents of w, and $\mathrm{D}_R(w)$ the set of right descents of w. For $I \subseteq S$, the *right descent class* $\mathcal{D}_I$ consists of the elements $w \in W$ such that $\mathrm{D}_R(w) = I$. Denote the union of right descent classes $\mathcal{D}_X$ such that $I \subseteq X \subseteq J$ by $\mathcal{D}_I^J$.

The *parabolic subgroup* W_I is the subgroup of W generated by I. Let $w_0(I)$ denote the longest element in W_I, that is, $\ell(w) < \ell(w_0(I))$ for all $w \in W_I \setminus \{w_0(I)\}$. Let w_0 denote the longest element in W, i.e., $w_0 = w_0(S)$.

2.1 0-Hecke algebras

Let $\mathbb{K}$ be any field. The *0-Hecke algebra* $H_W(0)$ of a finite Coxeter system (W,S) is the associative $\mathbb{K}$-algebra generated by $\{\pi_s : s \in S\}$ with relations

$$\pi_s^2 = \pi_s \quad \text{and} \quad (\pi_s\pi_t)_{m(s,t)} = (\pi_t\pi_s)_{m(s,t)}$$

for all distinct $s,t \in S$.

Let $\overline{\pi}_s$ denote $\pi_s - 1$. The algebra $H_W(0)$ is also generated by $\{\overline{\pi}_s : s \in S\}$. Given $w \in W$ with reduced word $w = s_1 \ldots s_k$, define π_w to be the product $\pi_{s_1} \cdots \pi_{s_k}$, and define $\overline{\pi}_w$ to be $\overline{\pi}_{s_1} \cdots \overline{\pi}_{s_k}$; note π_w and $\overline{\pi}_w$ are well-defined. The projective indecomposable $H_W(0)$-modules have the following description due to Norton [20].

Theorem 2.1. *[20, Theorem 4.12(2)] Let (W,S) be a finite Coxeter system and let $I \subseteq S$. The left ideal $\mathcal{P}_I := H_W(0)\pi_{w_0(I)}\overline{\pi}_{w_0(S\setminus I)}$ is a projective indecomposable $H_W(0)$-module with $\mathbb{K}$-basis $\{\pi_w\overline{\pi}_{w_0(S\setminus I)} : w \in \mathcal{D}_I\}$.*

The set $\{\mathcal{P}_I : I \subseteq S\}$ is a complete list of non-isomorphic projective indecomposable $H_W(0)$-modules. For $I \subseteq J \subseteq S$, let $\mathcal{P}_I^J$ denote the $H_W(0)$-module $H_W(0)\pi_{w_0(I)}\overline{\pi}_{w_0(S\setminus J)}$. The following result on $\mathcal{P}_I^J$ is analogous to [14, Theorem 3.2], and proved similarly. Modules isomorphic to $\mathcal{P}_I^J$ will play a significant role in our work.

Theorem 2.2. *Let $I \subseteq J \subseteq S$. Then $\mathcal{P}_I^J$ has a $\mathbb{K}$-basis*

$$\{\pi_w\overline{\pi}_{w_0(S\setminus J)} : w \in W \text{ with } I \subseteq D_R(w) \subseteq J\},$$

and decomposes as a direct sum of projective indecomposable modules via the formula

$$\mathcal{P}_I^J \cong \bigoplus_{I\subseteq X\subseteq J} \mathcal{P}_X.$$

2.2 Weak Bruhat interval modules

The *left weak Bruhat order* $\leq_L$ on W is the partial order defined by $u \leq_L v$ if some reduced word for u appears as a terminal segment in some reduced word for v. Given $u, v \in W$ with $u \leq_L v$, the *left weak Bruhat interval* is the set $[u,v]_L = \{w \in W : u \leq_L w \leq_L v\}$.

Definition 2.3. Let $[u,v]_L \subseteq W$. The *weak Bruhat interval module* $\mathrm{B}(u,v)$ is the vector space $\mathbb{K}[u,v]_L$ with $H_W(0)$-action defined by

$$\pi_s w = \begin{cases} w & \text{if } s \in \mathrm{D}_L(w), \\ sw & \text{if } s \notin \mathrm{D}_L(w) \text{ and } sw \in [u,v]_L, \\ 0 & \text{if } s \notin \mathrm{D}_L(w) \text{ and } sw \notin [u,v]_L \end{cases} \tag{2.1}$$

for all $s \in S$ and $w \in [u,v]_L$.

The type A case of Definition 2.3 is precisely [15, Definition 1(1)]. That (2.1) defines an action of $H_W(0)$ in finite type follows from Theorems 3.1 and 3.3 in [10].

Example 2.4. Let $W = \mathfrak{S}_4$; we write elements of symmetric groups in one-line/list notation, e.g. $s_2 = 1324$ and $s_2 s_3 s_2 = 1432$. Figure 1 shows the action of π_1, π_2 and π_3 (where π_i denotes π_{s_i}) on the basis $[1324, 1432]_L$ of $\mathrm{B}(1324, 1432)$ and the basis $[3142, 4231]_L$ of $\mathrm{B}(3142, 4231)$. Following [15] we draw Hasse diagrams from top to bottom, so the 0-Hecke operators move elements downwards (or send them to zero).

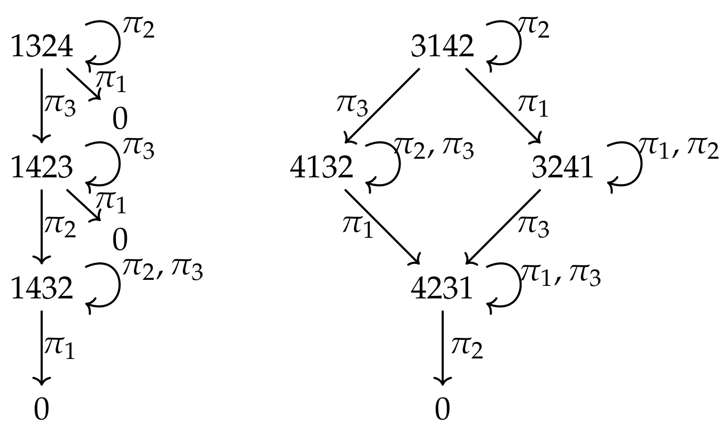

Figure 1: The $H_{\mathfrak{S}_4}(0)$-action on $\mathbb{K}$-bases for $\mathrm{B}(1324, 1432)$ and $\mathrm{B}(3142, 4231)$.

For $I \subseteq J \subseteq S$, the union $\mathcal{D}_I^J$ of right descent classes is an interval in left weak Bruhat order [6, Theorem 6.2]. In particular, each right descent class $\mathcal{D}_I$ itself is an interval in left weak Bruhat order.

Example 2.5. Figure 2 shows the poset $(\mathfrak{S}_4, \leq_L)$, in which we colour $\mathcal{D}_I$ using distinct colours for each I other than $I = \varnothing$ and $I = \{1, 2, 3\}$.

We now realise the $H_W(0)$-modules $\mathcal{P}_I$ and $\mathcal{P}_I^J$ as weak Bruhat interval modules. Denote the shortest element in $\mathcal{D}_I$ by u_I and the longest element in $\mathcal{D}_I$ by v_I. Note that $u_I = w_0(I)$ and $v_I = w_0\, w_0(S \setminus I)$.

Theorem 2.6. *Let $I \subseteq J \subseteq S$. Then $\mathcal{P}_I^J \cong \mathrm{B}(u_I, v_J)$.*

We henceforth denote $\mathrm{B}(u_I, v_J)$ by P_I^J and $\mathrm{B}(u_I, v_I)$ by P_I, to emphasise their nature as (direct sums of) projective indecomposable $H_W(0)$-modules.

Example 2.7. Consider the $H_{\mathfrak{S}_4}(0)$-module $\mathrm{B}(2134, 4231)$, and let i denote s_i. Since $2134 = u_{\{1\}}$ and $4231 = v_{\{1,3\}}$, we have $\mathcal{P}_{\{1\}}^{\{1,3\}} \cong \mathrm{B}(2134, 4231) = \mathrm{P}_{\{1\}}^{\{1,3\}} \cong \mathrm{P}_{\{1\}} \oplus \mathrm{P}_{\{1,3\}}$ by Theorems 2.2 and 2.6. Figure 3 depicts the basis elements for $\mathrm{P}_{\{1\}}^{\{1,3\}}$; the orange/pink

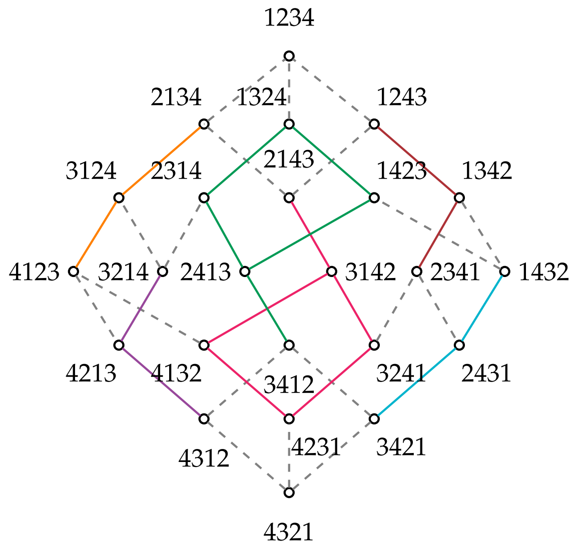

Figure 2: The poset $(\mathfrak{S}_4, \leq_L)$ and the right descent classes $\mathcal{D}_I$.

colour (cf. Figure 2) indicates $\mathrm{P}_{\{1\}}^{\{1,3\}}$ is isomorphic to the direct sum of $\mathrm{P}_{\{1\}}$ and respectively $\mathrm{P}_{\{1,3\}}$. Note however the basis elements of $\mathrm{P}_{\{1\}}$ do not span a submodule of $\mathrm{P}_{\{1\}}^{\{1,3\}}$.

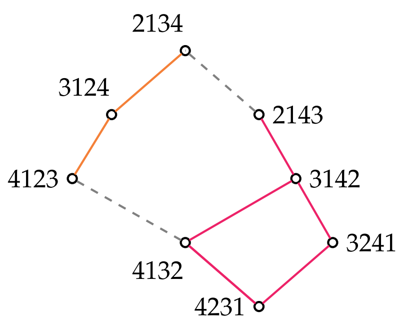

Figure 3: The $H_{\mathfrak{S}_4}(0)$-module $\mathrm{B}(2134, 4231) = \mathrm{P}_{\{1\}}^{\{1,3\}} \cong \mathrm{P}_{\{1\}} \oplus \mathrm{P}_{\{1,3\}}$.

The following indecomposability criterion follows from the algebraic structure of $H_W(0)$; in particular since $H_W(0)$ is a Frobenius algebra [12] it is self-injective, and so the projective indecomposable modules are also injective indecomposable.

Proposition 2.8. *Every submodule and quotient of* P_I *is indecomposable.*

Specialising Proposition 2.8 to weak Bruhat interval modules obtains the following.

Proposition 2.9. *The weak Bruhat interval modules* $\mathrm{B}(w, v_I)$ *and* $\mathrm{B}(u_I, w)$ *are indecomposable for all* $w \in \mathcal{D}_I$*, and all submodules of* $\mathrm{B}(w, v_I)$ *and quotients of* $\mathrm{B}(u_I, w)$ *are also indecomposable.*

Several of the families of 0-Hecke modules for bases of quasisymmetric functions are isomorphic to weak Bruhat interval modules that are either submodules or quotient modules of some P_I. Applications of Proposition 2.9 will be given in Section 4.

2.3 Equivalences of categories

Fayers [12] introduced certain (dual) equivalences of the category $H_W(0)$-mod, as follows. Define involutions ϕ, θ and an anti-involution χ on $H_W(0)$ by

$$\phi : \pi_s \mapsto \pi_{w_0 s w_0}, \qquad \theta : \pi_s \mapsto 1 - \pi_s, \qquad \chi : \pi_s \mapsto \pi_s.$$

Let M be a $H_W(0)$-module. Define $\phi[M]$ and $\theta[M]$ to be the $H_W(0)$-modules whose underlying space is M, and whose actions $\cdot_\phi$ and $\cdot_\theta$ are given by $\pi_s \cdot_\phi m = \phi(\pi_s) \cdot m$ and $\pi_s \cdot_\theta m = \theta(\pi_s) \cdot m$, for $m \in M$. Define $\chi[M]$ to be the $H_W(0)$-module whose underlying space is the dual space M^* of M, with action given by $(\pi_s \cdot^\chi f)(m) = f(\chi(\pi_s) \cdot m)$, for $f \in M^*$ and $m \in M$. The functors $M \mapsto \phi[M]$ and $M \mapsto \theta[M]$ are self-equivalences of $H_W(0)$-mod, and the functor $M \mapsto \chi[M]$ is a dual equivalence of $H_W(0)$-mod.

Jung, Kim, Lee and Oh [15] determined the images of type A weak Bruhat interval modules under ϕ, θ and χ and their compositions; see [15, Table 1] for a summary. We extend this result on ϕ, $\hat{\theta} := \theta \circ \chi$, and $\hat{\omega} := \phi \circ \theta \circ \chi$ to arbitrary finite type, and moreover to quotients and submodules of weak Bruhat interval modules defined by upper order ideals in intervals in weak Bruhat order. The cases for the other compositions can be extended similarly by introducing *negative weak Bruhat interval modules* in arbitrary type: the type A definition is given in [15, Definition 1(2)], and the natural extension of this to finite type is well-defined by [10]. In this work, we do not use the negative analogue of weak Bruhat interval modules. For $Y \subseteq W$, let $w_0 Y w_0$ denote the set $\{w_0 y w_0 : y \in Y\}$. Similarly, $Y w_0 := \{y w_0 : y \in Y\}$, and $w_0 Y := \{w_0 y : y \in Y\}$. Note that if Y is an upper order ideal in $[u, v]_L$, then $\mathbb{K}Y$ is a submodule of $\mathrm{B}(u, v)$.

Theorem 2.10. *Let Y be an upper order ideal in $[u, v]_L$. Then we have the following isomorphisms of $H_W(0)$-modules.*

$$\phi[\mathrm{B}(u,v)/\mathbb{K}Y] \cong \mathbb{K}([w_0 u w_0, w_0 v w_0]_L \setminus w_0 Y w_0),$$
$$\hat{\theta}[\mathrm{B}(u,v)/\mathbb{K}Y] \cong \mathbb{K}([v w_0, u w_0]_L \setminus Y w_0),$$
$$\hat{\omega}[\mathrm{B}(u,v)/\mathbb{K}Y] \cong \mathbb{K}([w_0 v, w_0 u]_L \setminus w_0 Y).$$

We use Theorem 2.10 to determine the images of the modules P_I^J.

Corollary 2.11. *Let $I \subseteq J \subseteq S$. Then*

$$\phi[\mathrm{P}_I^J] \cong \mathrm{P}_{w_0 I w_0}^{w_0 J w_0}, \quad \hat{\theta}[\mathrm{P}_I^J] \cong \mathrm{P}_{S \setminus w_0 J w_0}^{S \setminus w_0 I w_0} \quad \textit{and} \quad \hat{\omega}[\mathrm{P}_I^J] \cong \mathrm{P}_{S \setminus J}^{S \setminus I}.$$

Corollary 2.11 will be applied in Sections 3 and 4.

3 Projective covers and injective hulls

In this section, we determine the projective covers and injective hulls for significant families of $H_W(0)$-modules. A *projective cover* of a $H_W(0)$-module M is a projective $H_W(0)$-module P such that there is an epimorphism $f : P \to M$ whose kernel is contained in the radical of P. Projective covers exist since $H_W(0)$ is Artinian, and the projective module P is unique up to isomorphism. In [8, Section 5], Choi, Kim, Nam and Oh constructed projective covers for the 0-Hecke modules introduced by Tewari and van Willigenburg in [23], in terms of *generalised compositions*, using the ribbon tableau model of [14]. Our approach, similarly to [8], involves directly establishing radical membership; we work with and state results in terms of right descent sets.

The morphism $f : \mathrm{P}_I^J \to \mathrm{P}_I^J/\mathbb{K}Y$ given by $f(w) = w + \mathbb{K}Y$ is an epimorphism with kernel equal to $\mathbb{K}Y$.

Theorem 3.1. *Let Y be an upper order ideal in $\mathcal{D}_I^J$ with $u_J \notin Y$. Then P_I^J is the projective cover of $\mathrm{P}_I^J/\mathbb{K}Y$.*

Specialising Theorem 3.1 to weak Bruhat interval modules obtains the following.

Corollary 3.2. *Let $I \subseteq J$ and $w \in \mathcal{D}_J$. Then P_I^J is the projective cover of $\mathrm{B}(u_I, w)$.*

Remark 3.3. The type A case of Corollary 3.2 has been obtained independently, in the language of generalised compositions, by Kim, Lee and Oh in [16, Lemma 5.2].

Example 3.4. Consider the $H_{\mathfrak{S}_4}(0)$-module $\mathrm{B}(2134, 4132)$, and let i denote s_i. Since $2134 = u_{\{1\}}$ and $4132 \in \mathcal{D}_{\{1,3\}}$, by Corollary 3.2 we have that $\mathrm{P}_{\{1\}}^{\{1,3\}}$ is the projective cover of $\mathrm{B}(2134, 4132)$. The projective module $\mathrm{P}_{\{1\}}^{\{1,3\}}$ is depicted in Figure 3; note the appearance of the interval $[2134, 4132]_L$ in this figure.

An *injective hull* of a $H_W(0)$-module M is an injective $H_W(0)$-module Q together with a monomorphism $g : M \to Q$ such that the image of g has nontrivial intersection with every non-zero submodule of Q. The injective module Q is unique up to isomorphism.

Since $M \mapsto \hat{\omega}[M]$ is a dual equivalence of categories, P is the projective cover of M if and only if $\hat{\omega}[P]$ is the injective hull of $\hat{\omega}[M]$. The analogous statement holds for $M \mapsto \hat{\theta}[M]$. As an application of Theorem 2.10, we obtain the injective hulls of another significant family of $H_W(0)$-modules from Theorem 3.1.

Theorem 3.5. *Let Y be an upper order ideal in $\mathcal{D}_I^J$ with $v_I \in Y$. Then P_I^J is the injective hull of $\mathbb{K}Y$.*

The specialisation of Theorem 3.5 to weak Bruhat interval modules is as follows.

Corollary 3.6. *Let $I \subseteq J$ and $w \in \mathcal{D}_I$. Then P_I^J is the injective hull of $\mathrm{B}(w, v_J)$.*

4 Applications to modules for quasisymmetric functions

Much recent work has been devoted to constructing $H_{\mathfrak{S}_n}(0)$-modules whose images under the quasisymmetric characteristic [11] are important families of quasisymmetric functions. In this section, we apply results from Sections 2 and 3 to uniformly recover a number of results on indecomposability, projective covers, and injective hulls for various such modules. We also obtain new results concerning the modules associated to the recently-introduced row-strict dual immaculate functions and row-strict extended Schur functions of Niese, Sundaram, van Willigenburg, Vega, and Wang [18].

The $H_{\mathfrak{S}_n}(0)$-modules associated to quasisymmetric functions are usually stated in terms of *compositions of n*: sequences of positive integers that sum to n. Compositions of n are in bijection with subsets of $[n-1]$: if $\alpha = (\alpha_1, \dots, \alpha_k)$ is a composition of n, then the associated subset $\mathrm{set}(\alpha)$ is $\{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\}$. We denote the complement of $\mathrm{set}(\alpha)$ by $\mathrm{set}(\alpha)^c$. The *reversal* of α, denoted by α^r, is the composition obtained by reversing the sequence α.

Example 4.1. Let $\alpha = (1,3,2)$. Then $\mathrm{set}(\alpha) = \{1,4\}$ and $\alpha^r = (2,3,1)$.

As done in previous examples, we index projective indecomposable $H_{\mathfrak{S}_n}(0)$-modules by subsets of $[n-1]$, where i is understood to denote s_i. We first consider modules for the dual immaculate [4] and extended Schur [1] bases of quasisymmetric functions, and their row-strict analogues [19]. These modules are defined in terms of certain families of tableaux of shape α.

The *diagram* $D(\alpha)$ associated to a composition α is the left-justified array of boxes with α_i boxes in the ith row from the top. A *standard immaculate tableau* of shape α is a labelling of the boxes of $D(\alpha)$ by the integers $1, \dots, n$, each used once, such that entries increase from left to right along rows and from top to bottom in the first column. A standard immaculate tableau is a *standard extended tableau* if the entries increase from top to bottom in every column. The set of standard immaculate tableaux of shape α, and its subset of standard extended tableaux, are denoted by $\mathrm{SIT}(\alpha)$ and $\mathrm{SET}(\alpha)$ respectively.

For $T \in \mathrm{SIT}(\alpha)$, the *reading word* $\mathrm{rw}(T)$ of T is the permutation obtained from reading the entries in each row in T from right to left, starting with the topmost row and iterating downwards. Let T_0^α and T_1^α be the elements of $\mathrm{SIT}(\alpha)$ with shortest, respectively, longest (Coxeter length) reading words, and let $\mathscr{T}_1^\alpha$ be the element of $\mathrm{SET}(\alpha)$ with longest reading word.

Example 4.2. The standard immaculate tableaux $\mathrm{SIT}(2,2)$ are shown in Figure 4. The standard extended tableaux $\mathrm{SET}(2,2)$ are the middle and rightmost tableaux. The leftmost tableau is T_1^α, the middle tableau is $\mathscr{T}_1^\alpha$, and the rightmost tableau is T_0^α. Their reading words, from left to right, are 4132, 3142, and 2143.

In [5], Berg, Bergeron, Saliola, Serrano and Zabrocki define a $H_{\mathfrak{S}_n}(0)$-action on the $\mathbb{K}$-span of $\mathrm{SIT}(\alpha)$, and show the quasisymmetric characteristics of the resulting modules

1	4
2	3

1	3
2	4

1	2
3	4

Figure 4: The three standard immaculate tableaux of shape $(2,2)$.

$\mathcal{V}_\alpha$ are the dual immaculate functions of [4]. In [21], Searles defines an $H_{\mathfrak{S}_n}(0)$-action on the $\mathbb{K}$-span of $\mathrm{SET}(\alpha)$, and shows the quasisymmetric characteristics of the resulting modules X_α are the extended Schur functions of [1].

In [15, Theorem 5], Jung, Kim, Lee and Oh prove the isomorphisms

$$\mathcal{V}_\alpha \cong \mathrm{B}(\mathrm{rw}(T_0^\alpha), \mathrm{rw}(T_1^\alpha)) \quad \text{and} \quad X_\alpha \cong \mathrm{B}(\mathrm{rw}(T_0^\alpha), \mathrm{rw}(\mathscr{T}_1^\alpha)). \tag{4.1}$$

It is also shown in the proof of [15, Theorem 5] that reading words of $\mathrm{SIT}(\alpha)$ belong to $\mathcal{D}_{\mathrm{set}(\alpha)^c}$, and that $\mathrm{rw}(T_0^\alpha)$ is the shortest element of $\mathcal{D}_{\mathrm{set}(\alpha)^c}$.

Indecomposability of $\mathcal{V}_\alpha$ and X_α were proved in [5, Theorem 3.12] and, respectively, [21, Theorem 3.13]. Combining (4.1) and Proposition 2.9 we recover these results, and additionally that all quotients of these modules are also indecomposable.

Theorem 4.3. *For any composition α, the modules $\mathcal{V}_\alpha$, X_α, and all quotients of these modules are indecomposable.*

The projective covers for $\mathcal{V}_\alpha$ and X_α were established in [8, Theorem 3.2] and [8, Theorem 3.5]. One can recover these results by combining (4.1) and Corollary 3.2.

Theorem 4.4. *For any composition α, the projective cover of $\mathcal{V}_\alpha$ and of X_α is $\mathrm{P}_{\mathrm{set}(\alpha)^c}$.*

In [19], Niese, Sundaram, van Willigenburg, Vega and Wang define a $H_{\mathfrak{S}_n}(0)$-action on the $\mathbb{K}$-span of $\mathrm{SIT}(\alpha)$ (different from that of [5]), and obtain $H_{\mathfrak{S}_n}(0)$-modules $\mathcal{W}_\alpha$ whose quasisymmetric characteristics are the row-strict dual immaculate functions of [18]. The same action is defined on the $\mathbb{K}$-span of $\mathrm{SET}(\alpha)$ in [19], obtaining $H_{\mathfrak{S}_n}(0)$-modules $\mathcal{Z}_\alpha$ whose quasisymmetric characteristics are the row-strict extended Schur functions of [18].

Remark 4.5. We use $\mathcal{V}_\alpha$ to denote the modules for dual immaculate functions, following [5] and [15]. On the other hand, in [19], these modules are denoted $\mathcal{W}_\alpha$ and the modules for row-strict dual immaculate functions are denoted $\mathcal{V}_\alpha$. Therefore, our use of $\mathcal{V}_\alpha$ and $\mathcal{W}_\alpha$ is the reverse of that in [19].

To apply the results of Sections 2 and 3, we need to identify $\mathcal{W}_\alpha$ and $\mathcal{Z}_\alpha$ as weak Bruhat interval modules. For $T \in \mathrm{SIT}(\alpha)$, define the *row-strict reading word* $\mathrm{rw}_{\mathcal{R}}(T)$ of T to be the permutation obtained by reading the entries of T from left to right along rows, beginning at the bottom row and proceeding to the top row. For example, the row-strict reading words of the tableaux in Example 4.2 from left to right are $2314, 2413$ and 3412.

Theorem 4.6. *For any composition α,*

$$\mathcal{W}_\alpha \cong \mathrm{B}(\mathrm{rw}_{\mathcal{R}}(T_1^\alpha), \mathrm{rw}_{\mathcal{R}}(T_0^\alpha)) \quad \text{and} \quad \mathcal{Z}_\alpha \cong \mathrm{B}(\mathrm{rw}_{\mathcal{R}}(\mathscr{T}_1^\alpha), \mathrm{rw}_{\mathcal{R}}(T_0^\alpha)),$$

and these Bruhat interval modules are submodules of $\mathrm{P}_{\mathrm{set}(\alpha^r)}$.

The indecomposability of $\mathcal{W}_\alpha$ and $\mathcal{Z}_\alpha$ were proved in [19, Theorem 6.15] and [19, Theorem 7.13]. One can recover this via Theorem 4.6 together with Proposition 2.9, which additionally shows that any submodule of these modules is indecomposable.

Corollary 4.7. *For any composition α, the modules $\mathcal{W}_\alpha$, $\mathcal{Z}_\alpha$, and all submodules of these modules are indecomposable.*

Using Corollary 3.2, we determine the injective hulls of $\mathcal{W}_\alpha$ and $\mathcal{Z}_\alpha$.

Corollary 4.8. *For any composition α, the injective hull of $\mathcal{W}_\alpha$ and $\mathcal{Z}_\alpha$ is* $\mathrm{P}_{\mathrm{set}(\alpha^r)}$.

Remark 4.9. In [3], we prove Theorem 4.6 by showing directly that $\{\mathrm{rw}_{\mathcal{R}}(T) : T \in \mathrm{SIT}(\alpha)\}$ is an interval in left weak Bruhat order, and then showing the action on $\mathrm{SIT}(\alpha)$ defined in [19] agrees with the action on this weak Bruhat interval module. Alternatively one can show these permutations form an interval in left weak Bruhat order by noting that $\mathrm{rw}_{\mathcal{R}}(T) = \mathrm{rw}(T)w_0$ and appealing to (4.1). It also follows that $\mathcal{W}_\alpha \cong \hat{\theta}[\mathcal{V}_\alpha]$ and $\mathcal{Z}_\alpha \cong \hat{\theta}[\mathcal{X}_\alpha]$. We note the fact that the modules $\mathcal{W}_\alpha$ and $\mathcal{Z}_\alpha$ can be obtained by applying $\hat{\theta}$ to $\mathcal{V}_\alpha$ and $\mathcal{X}_\alpha$ is observed in [9, Table 4.1].

For completeness, we also provide the projective cover of $\mathcal{W}_\alpha$. Choi, Kim, Nam, and Oh showed that the injective hull of $\mathcal{V}_\alpha$ is $\oplus_{\beta \in [\underline{\boldsymbol{\alpha}}]} \mathrm{P}_{\mathrm{set}(\beta)^c}$ [7, Theorem 4.1], where $[\underline{\boldsymbol{\alpha}}]$ is a particular set of compositions obtained from α; see [7, Section 4] for a full definition of $[\underline{\boldsymbol{\alpha}}]$. Applying $\hat{\theta}$ to this formula obtains the following.

Theorem 4.10. *For any composition α, the projective cover of $\mathcal{W}_\alpha$ is* $\oplus_{\beta \in [\underline{\boldsymbol{\alpha}}]} \mathrm{P}_{\mathrm{set}(\beta^r)}$.

As far as we are aware, the injective hull of X_α and projective cover of $\mathcal{Z}_\alpha$ are not currently known.

Finally, we consider a family of modules defined on *standard permuted composition tableaux* by Tewari and van Willigenburg in [23]. These modules are denoted $\mathbf{S}_\alpha^\sigma$, where α is a composition and σ a permutation (see [23, Section 3] for a full definition); and when σ is the identity it was shown in [22] that these correspond to the quasisymmetric Schur functions of [13]. These modules have a direct sum decomposition $\mathbf{S}_\alpha^\sigma = \oplus_E \mathbf{S}_{\alpha,E}^\sigma$, where each E is an equivalence class of standard permuted composition tableaux. Jung, Kim, Lee, and Oh define a reading word $\mathrm{rw}_{\mathcal{S}}$ on the standard permuted composition tableaux ([15, Definition 6]). Let τ_E (respectively, τ_E') denote the standard permuted composition tableau in E that has shortest (respectively, longest) reading word. It is proved in [15, Theorem 6] that

$$\mathbf{S}_{\alpha,E}^\sigma \cong \mathrm{B}(\mathrm{rw}_{\mathcal{S}}(\tau_E), \mathrm{rw}_{\mathcal{S}}(\tau_E')), \tag{4.2}$$

and that $\mathrm{rw}_{\mathcal{S}}(\tau_E)$ is the shortest element of some right descent class.

The projective cover of $\mathbf{S}^{\sigma}_{\alpha,E}$ was determined in [8, Theorem 5.11] in terms of a generalised composition associated to E. Combining (4.2) with Corollary 3.2 recovers this result, with a different statement in terms of right descent sets.

Theorem 4.11. *Let $rw_{\mathcal{S}}(\tau_E) \in \mathcal{D}_I$ and $rw_{\mathcal{S}}(\tau'_E) \in \mathcal{D}_J$. Then P_I^J is the projective cover of $\mathbf{S}^{\sigma}_{\alpha,E}$.*

The images of the modules $\mathbf{S}^{\sigma}_{\alpha}$ and $\mathbf{S}^{\sigma}_{\alpha,E}$ under $\hat{\omega}$ are a family of modules that generalise the modules introduced in [2] for the Young row-strict dual immaculate functions of [17]. Specifically, denoting these modules by $\mathbf{R}^{\sigma}_{\alpha}$ and $\mathbf{R}^{\sigma}_{\alpha,E}$, one has $\mathbf{R}^{\sigma}_{\alpha} \cong \hat{\omega}[\mathbf{S}^{w_0\sigma w_0}_{\alpha^r}]$, and for E an equivalence class of standard permuted composition tableaux corresponding to α_r and $w_0\sigma w_0$, $\mathbf{R}^{\sigma}_{\alpha,E} \cong \hat{\omega}[\mathbf{S}^{w_0\sigma w_0}_{\alpha^r,E}]$ ([15, Proposition 1]). The injective hull of $\mathbf{R}^{\sigma}_{\alpha,E}$ was determined in [15, Corollary 2], using $\hat{\omega}$. Applying $\hat{\omega}$ to Theorem 4.11 gives a description of the injective hull in terms of right descent sets.

Corollary 4.12. *Let E be an equivalence class of standard permuted composition tableaux corresponding to α^r and $w_0\sigma w_0$. Suppose $rw_{\mathcal{S}}(\tau_E) \in \mathcal{D}_I$ and $rw_{\mathcal{S}}(\tau'_E) \in \mathcal{D}_J$. Then $\mathrm{P}^{S\setminus I}_{S\setminus J}$ is the injective hull of $\boldsymbol{R}^{\sigma}_{\alpha,E}$.*

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