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Diagram model for the Okada algebra and monoid

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Abstract.

It is well known that the Young lattice is the Bratelli diagram of the symmetric groups expressing how irreducible representations restrict from $\mathfrak{S}_N$ to $\mathfrak{S}_{N-1}$. In 1988, Stanley discovered a similar lattice called the Young-Fibonacci lattice which was realized as the Bratelli diagram of a family of algebras by Okada in 1994.

In this paper, we realize the Okada algebra and its associated monoid using a labeled version of Temperley-Lieb arc-diagrams. We prove in full generality that the dimension of the Okada algebra is $n!$. In particular, we interpret a natural bijection between permutations and labeled arc-diagrams as an instance of Fomin's Robinson-Schensted correspondence for the Young-Fibonacci lattice. We prove that the Okada monoid is aperiodic and describe its Green relations. Lifting those results to the algebra allows us to construct a cellular basis of the Okada algebra.

Résumé. Il est bien connu que le treillis de Young peut s'interpréter comme le diagramme de Bratelli des groupes symétriques, décrivant, par exemple, comment les représentations irréductibles se restreignent de $\mathfrak{S}_n$ à $\mathfrak{S}_{n-1}$. En 1975, Stanley a découvert un treillis similaire appelée treillis de Young-Fibonacci qui a été interprété comme le diagramme de Bratelli d'une famille d'algèbres par Okada en 1994.

Dans cet article, nous réalisons l'algèbre d'Okada et le monoïde associé grâce à une version étiquetée des diagrammes d'arcs du monoïde de Jones et de l'algèbre de Tempeley-Lieb. Nous prouvons en toute généralité que l'algèbre d'Okada est de dimension $n!$. En particulier, nous interprétons la bijection naturelle entre les permutations et les diagrammes d'arcs comme une instance de la correspondance de Robinson-Schensted-Fomin associée au treillis de Young-Fibonacci. Nous prouvons que le monoïde est apériodique et décrivons ses relations de Green. En relevant, ces dernières à l'algèbre nous en construisant une base cellulaire.

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1 Introduction

The theory of 1-differential posets was developed by R. Stanley [7] as a framework for generalizing the Robinson-Schensted correspondence beyond the combinatorics of the Young lattice $\mathbb{Y}$ of integer partitions. A similar undertaking was made by S. Fomin in his work on dual graded graphs and growth processes, where the later technique was used to construct an explicit RS-correspondence for Stanley's *Young-Fibonacci* lattice $\mathbb{YF}$ [1, 7]. Both $\mathbb{Y}$ and $\mathbb{YF}$ are 1-differential and they are the only lattices having this property. Fomin's approach involves a Fibonacci version of standard tableaux; a notion later examined independently by T. Roby, K. Killpatrick, and J. Nzeutchap (whose formulation by-passes the growth construction altogether), see [5] and the references therein.

S. Okada [6] showed that the $\mathbb{YF}$-lattice supports a theory of *clone symmetric functions* with analogues of the classical bases (e.g. complete homogeneous, Schur, and power-sum symmetric functions) as well as a $\mathbb{YF}$-variant of the Littlewood-Richardson rule. The clone theory appears in Goodman-Kerov's determination of the *Martin boundary* of the $\mathbb{YF}$-lattice [2] and is related to various random processes.

The Okada algebras $\{\mathsf{O}_N(X,Y)\}_{N\geq 0}$ were introduced by S. Okada as a counterpart to the clone theory, and occupy a role similar to that played by the symmetric groups in the classical theory of symmetric functions. Okada algebras are finite dimensional, associative, and depend on parameters $X = (x_1, \ldots, x_{N-1})$ and $Y = (y_1, \ldots, y_{N-2})$. When those parameters are generic, they are semi-simple and their branching rule, which describes how irreducible representations restrict from $\mathsf{O}_N(X,Y)$ to $\mathsf{O}_{N-1}(X,Y)$, is expressed by the covering relations of the $\mathbb{YF}$-lattice.

In this paper we realize the Okada algebra $\mathsf{O}_N(X,Y)$ as a diagram algebra with a multiplicative/monoidal basis expressed in terms of certain arc-labeled, non-crossing perfect matchings (as appear in both the Temperley-Lieb and Martin-Saleur Blob algebras [4]). Like most diagram algebras, this basis is cellular and affords us with a novel, diagrammatic presentation of the irreducible representations of $\mathsf{O}_N(X,Y)$ (i.e. as *cell modules*). We interpret Fomin's RS-correspondence diagrammatically. This involves constructing a bijection between saturated chains in the $\mathbb{YF}$-lattice (presented in terms of sequences of *Fibonacci sets*) and *Okada half arc-diagrams*. In addition we examine the structure theory of the Okada algebra and monoid via a *dominance order* on Fibonacci sets.

2 Background

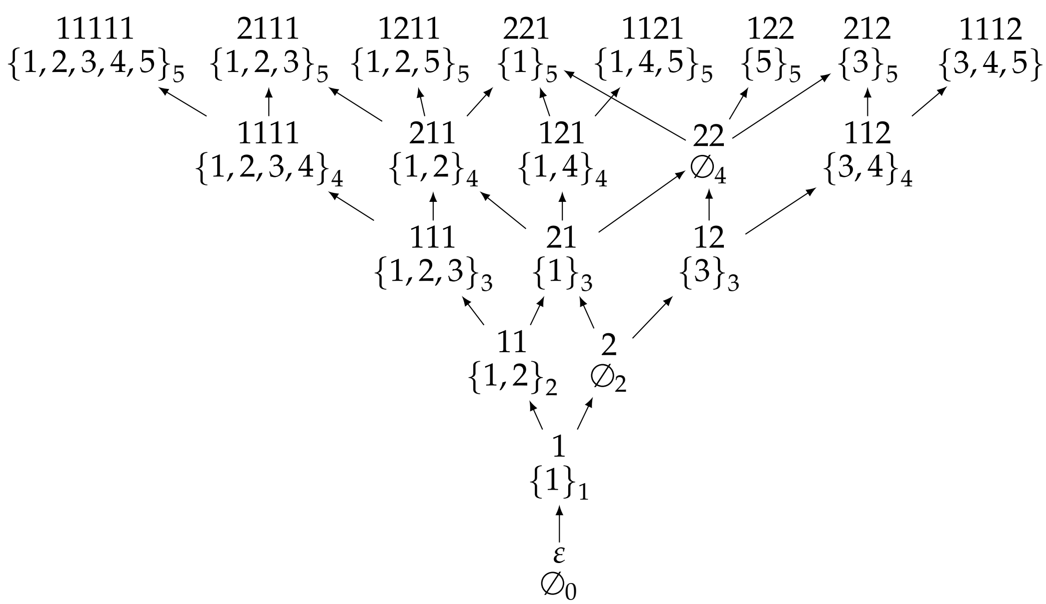
Throughout this paper N denotes a non-negative integer. We denote by $[N]$ the set $\{1, \ldots, N\}$. We often write negative numbers as overlined numbers such as $\overline{4}$. The cardinality of a set S is denoted $\# S$. For a non-negative integer N we endow $[N] \cup [\overline{N}]$ with the total order $\{1 < 2 < \cdots < N < \overline{N} < \cdots < \overline{2} < \overline{1}\}$. Overlining numbers which are negative should also help the reader remember this unusual ordering.

Stanley's original construction of the Young-Fibonacci lattice [7] involves endowing the set of *Fibonacci words*, i.e. binary words in the alphabet $\{1,2\}$, with a partial order. We present an alternative description using *Fibonacci sets*.

Definition 2.1. *A* Fibonacci set of rank N *is a subset* $S = \{s_1 < s_2 < \cdots < s_k\}$ *of* $[N]$ *whose size* k *has the same parity as* N *and such that* s_ℓ *have the same parity as* ℓ*. We write* $\mathbb{YFS}_N$ *for the collection of all rank* N *Fibonacci sets and* $\mathbb{YFS}$ *for the* disjoint *union of* $\mathbb{YFS}_N$ *as* N *varies.*

The entire interval $[N]$ itself is always a Fibonacci set of rank N, while $\varnothing$ is a Fibonacci set only when N is even. We emphasize on the fact that in $\mathbb{YFS}$ the set $\{1,2,5\}$ of rank 5 is not the same Fibonacci set as $\{1,2,5\}$ of rank 7. When they need to be distinguished we include N as a subscript, as in $\{1,2,5\}_5$ and $\{1,2,5\}_7$.

The covering relations which generate the lattice structure on $\mathbb{YFS}$ are defined by $S \lhd T$ if and only if $S \in \mathbb{YFS}_{N-1}$ and $T \in \mathbb{YFS}_N$ and one of these two sets can be obtained from the other one by removing its largest element. Stanley's description is equivalent to ours through the bijection sending a binary word w to the set of the sums of its suffixes whose first digit is 1. The Hasse diagram of $\mathbb{YFS}$ upto rank 5 is illustrated below.



Definition 2.2. *Fix a positive integer* N*. Given a field* $\mathbb{K}$*, fix also* $X = (x_1, \ldots, x_{N-1})$ *and* $Y = (y_1, \ldots, y_{N-2})$ *two sequences of parameters in* $\mathbb{K}$*. The* Okaka algebra $\mathsf{O}_N(X,Y)$ *is the algebra generated by* $\{\mathsf{E}_i \mid i = 1 \ldots N-1\}$ *with the relations*

$$\mathsf{E}_i^2 = x_i \mathsf{E}_i \qquad 1 \le i \le N-1, \tag{$I(X,Y)$}$$

$$\mathsf{E}_i \mathsf{E}_j = \mathsf{E}_j \mathsf{E}_i \qquad |i-j| \ge 2, \tag{C(X,Y)}$$

$$\mathsf{E}_{i+1}\mathsf{E}_i\mathsf{E}_{i+1} = y_i \mathsf{E}_{i+1} \qquad 1 \le i \le N-2, \tag{S(X,Y)}$$

If all the X's and the Y's are equal to 1, the Okada algebra is actually the algebra of a monoid; we call this the *Okaka Monoid* and denote it O_N. Recall that setting all $y_i := 1$ and all $x_i := q$ and adding the extra relation $\mathsf{E}_i\mathsf{E}_{i+1}\mathsf{E}_{i+1} = \mathsf{E}_i$ defines the Temperley-Lieb algebra which is also a deformation of the algebra of a monoid called the Jones monoid (obtained when $q = 1$).

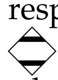
We now review some of Okada's results [6]: For generic values of the X and Y parameters $\mathsf{O}_N(X,Y)$ is semi-simple and its irreducible representations V_T correspond to rank N Fibonacci sets T. When V_T is restricted to the subalgebra $\mathsf{O}_{N-1}(X,Y) \subset \mathsf{O}_N(X,Y)$ it decomposes as a direct sum of irreducible representations V_S of $\mathsf{O}_{N-1}(X,Y)$ where $S \lhd T$ is a covering relation in $\mathbb{YFS}$.

The dimension of $\mathsf{O}_N(X,Y)$ is $N!$ and a basis for $\mathsf{O}_N(X,Y)$ can be constructed from permutations in the following way. Recall that the *code* of a permutation $\sigma \in \mathfrak{S}_N$ is $\mathsf{code}(\sigma) = (c_1, \dots c_N)$ where $c_i := \#\left\{j < i \,\middle|\, \sigma^{-1}(j) > \sigma^{-1}(i)\right\}$. It is well known that the product $\prod_{i=1}^{n} \sigma_{i-1}\sigma_{i-2}\cdots\sigma_{i-c_i}$ taken from left to right, increasing with i, is the lexicographically minimal reduced factorization of σ into simple transpositions $\sigma_i = (i, i+1)$. Define $\mathsf{E}_\sigma := \prod_{i=1}^{n} \mathsf{E}_{i-1}\mathsf{E}_{i-2}\cdots\mathsf{E}_{i-c_i}$. Okada showed in [6] that the family $\{\mathsf{E}_\sigma \,|\, \sigma \in \mathfrak{S}_n\}$ is, generically, a basis of the Okada algebra. His proof, however, requires semi-simplicity and doesn't apply to degenerate specializations, such as the monoid case.

3 Diagram models for the Okada Monoid and Algebra

The goal of this section is to build a basis of the Okada algebra in full generality using rewriting techniques. Inspired by Viennot's theory of heaps of dimers [8], we use diagram rewriting rather than word rewriting.

A *diamond diagram* of rank-N is a trapezoidal arrangement of boxes with $N-1$ rows starting with a north-east diagonal and ending with south-east diagonal, where each box can be either black or white. The rows are indexed from bottom to top. The *reading* of such a diagram is the sequence $\underline{\mathbf{i}} = (i_1, \dots, i_\ell)$ obtained by recording the row index i_k of the k-th black box, starting on the left and reading each south-east diagonal from top to bottom. Associated to the reading $\underline{\mathbf{i}}$ is the monomial $\mathsf{E}_{\underline{\mathbf{i}}} := \mathsf{E}_{i_1}\cdots\mathsf{E}_{i_\ell}$ in the Okada algebra $\mathsf{O}_N(X,Y)$. We identify diagrams differing by empty south-east diagonals on the right. This identification is compatible with the reading and the associated monomial. See Figure 1 for some examples. Using rewriting techniques on such diagrams, one can show right away that the Okada algebra has dimension $N!$.

The relevant combinatorics becomes more transparent after we re-encode a diamond diagram as a *fully packed loop configuration* (FPLC). This is done by replacing black and white squares respectively with double U-turn and double horizontal squares: ◆ $\to$  and ◇ $\to$ . The paths fragments at the top and bottom of the trapezoid are completed by adding horizontal lines. The result is a set $\mathcal{C}$ of non-crossing planar loops and arcs. The endpoints of the arcs are situated on the left and right boundaries of the trapezoid and we number these endpoints, from bottom to top, with positive indices (on the left) and negative indices (overlined, on the right). See Figure 1 where we've colored some of the arcs in order to make the picture more legible.

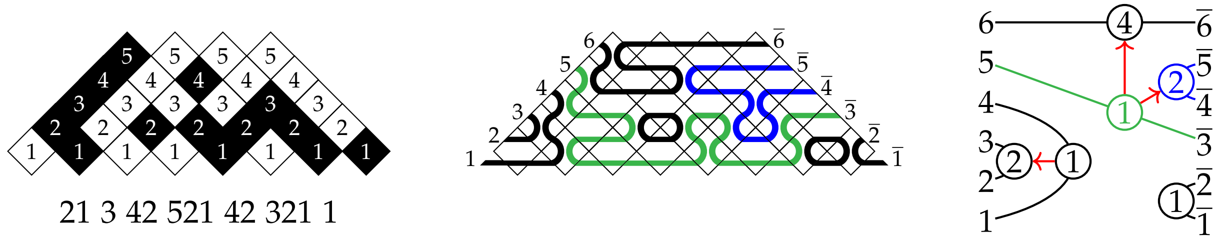
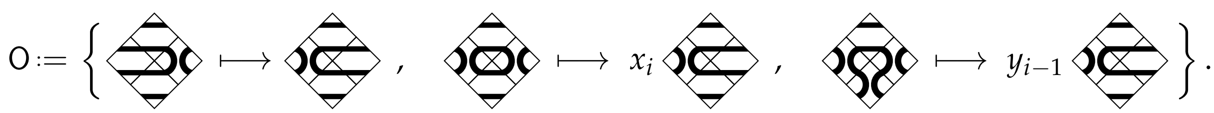


Figure 1: A diamond diagram, its reading together with the associated loop configuration and arc diagram

The horizontal arc segments in the -boxes occupy levels $1, \ldots, N$ starting from the bottom of the trapezoid. The *height* of an arc/loop in an FPLC is the minimal level of the horizontal segments which form it. The height statistic of an arc/loop is invariant under the following local moves which implement the Okada relations:



The first and third moves can be viewed as restricted isotopies which transform arcs and loops horizontally and downward, while the second move erases loops. By repeatedly applying local moves, each FPLC can be brought to a *normal* form (*ie.* a configuration without any possible move). This normal form is independent of the sequence of moves used to obtain it and is therefore uniquely defined. It contains no arcs which go up and then down when followed in any direction; in particular, there are no loops. There is a bijection between permutations and normal forms which shows the following result:

Theorem 3.1. *For any N and for any specialization of the X, Y parameters, the map $\sigma \mapsto \mathsf{E}_\sigma$ is a bijection from $\mathfrak{S}_N$ to the monoid O_N and the family $(\mathsf{E}_\sigma)_{\sigma \in \mathfrak{S}_N}$ is a basis for the Okada algebra $\mathsf{O}_N(X, Y)$. In particular the dimension of the Okada algebra $\mathsf{O}_N(X, Y)$ is always $N!$.*

We abbreviate the structure of a FPLC $\mathcal{C}$ by removing its loops, labeling each arc by its respective height, and taking the isotopy class of what remains. We denote the result $[\mathcal{C}]$; an example is depicted in the third image of Figure 1. In view of the previous remarks $[\mathcal{C}] = [\mathcal{D}]$ whenever $\mathcal{C}$ and $\mathcal{D}$ are two FPLCs of rank N which are related by a sequences of moves. It turns out that this is actually an equivalence, providing us with a diagram model for the Okada algebra which we now examine.

Recall that a rank N *non-crossing arc-diagram* is a visualization of a perfect matching linking vertices $\{1, \ldots, N\}$ and $\{\overline{1}, \ldots, \overline{N}\}$ on the right and left boundaries of a rectangle by non-crossing arcs (drawn in the interior of the rectangle). A pair $\{a, b\}$ in the matching is depicted by an arc joining vertices $a, b \in [N] \cup [\overline{N}]$ and is denoted by $a \longmapsto b$. Either a, b are both positive, both negative, or else have different signs; in the later case we say the arc $a \longmapsto b$ is a *propagating*. Only the incidence relations of the arc-diagram are relevant, and so isotopic diagrams are considered equivalent.

An arc $a \longmapsto b$ is said to be *nested* in another arc $c \longmapsto d$ if $c < a < b < d$. Nesting defines a partial order on the arcs of a non-crossing arc diagram. The reader should be aware that any arc situated above a propagating arc is nested in the later. In particular, given any pair of propagating arcs, one arc must be nested in the other; consequently the nesting order is total when restricted to propagating arcs.

Definition 3.2. *A rank N Okada arc-diagram is a rank N non-crossing arc-diagram where each arc $a \longmapsto b$ is assigned an height-label $\mathbf{h}(a \longmapsto b)$ such that*

1. $\mathbf{h}(a \longmapsto b)$ *must be at least* 1 *and at most* $\min(|a|, |b|)$*,*
2. $\mathbf{h}(a \longmapsto b)$ *must have the same parity as* $\min(|a|, |b|)$*,*
3. $\mathbf{h}(a \longmapsto b) > \mathbf{h}(c \longmapsto d)$ *whenever $a \longmapsto b$ is nested in $c \longmapsto d$.*

The set of all Okada arc-diagrams of rank N is denoted $\mathfrak{A}_N$ and $\mathbb{C}\mathfrak{A}_N$ will denote the vector space spanned by all Okada arc-diagrams of rank N.

Definition 3.3. *Given $C, D \in \mathfrak{A}_N$ their composition $C \circ D$ is the diagram obtained by merging the right boundary nodes of C with the left boundary nodes of D and concatenating their respective arcs. The diagram $C \circ D$ may include loops, created from concatenated arc fragments of diagrams C and D. The height label of an arc/loop in $C \circ D$ is the minimum of the height labels of the arc fragments of C and D which comprise it. Let $[C \circ D]$ denoted the isotopy class of the height labeled, non-crossing arc-diagram obtained by removing all loops from the composition.*

Lemma 3.4. *If $C, D \in \mathfrak{A}_N$ then $[C \circ D] \in \mathfrak{A}_N$. Hence $\mathfrak{A}_N$ acquires the structure of a monoid denoted O_N whose unit is the identity Okada arc-diagram id_N which consists entirely of labeled propagating arcs $\mathbf{h}(a \longmapsto \overline{a}) = a$ for all $1 \leq a \leq N$.*

For simplicity we'll present the following results for arbitrary X-parameters together with the assumption that $y_k = 1$ for all $k \geq 1$. This is sufficiently generic to include the semi-simple case and all but the most degenerate specializations.

Definition 3.5. *Given $C, D \in \mathfrak{A}_N$ their product $C \cdot D$ is the element $X^{\underline{\ell}}[C \circ D]$ in $\mathbb{C}\mathfrak{A}_N$ where $X^{\underline{\ell}} = \prod_{k \geq 0} x_k^{\ell_k}$ and ℓ_k counts the number of loops γ in $C \circ D$ whose height label equals k.*

Lemma 3.6. *The product $C \cdot D$ endows $\mathbb{C}\mathfrak{A}_N$ with the structure of an associative, unital algebra, denoted $\widetilde{\mathsf{O}}_N(X, \mathbf{1})$. Using a rewriting rule, the Y-parameters can also be incorporated in the diagram product and $\widetilde{\mathsf{O}}_N(X, Y)$ will denote the corresponding diagram algebra. (See Figure 2 .)*

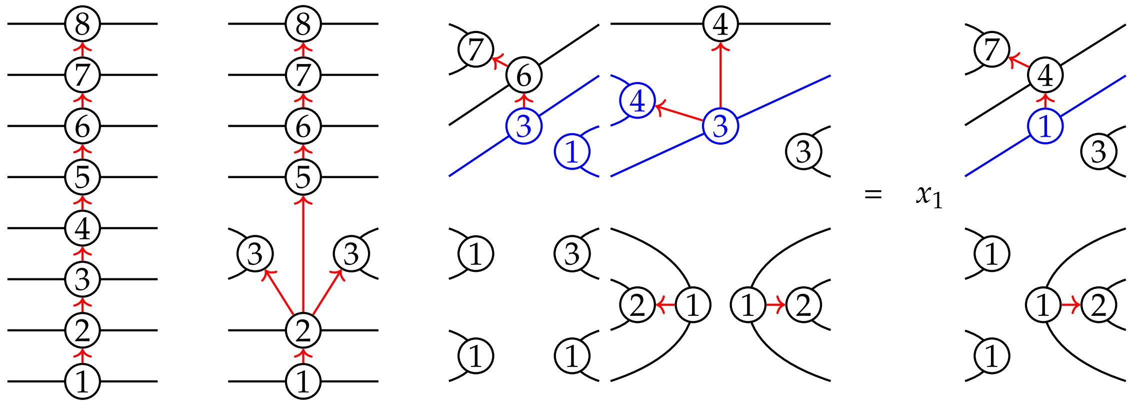


Figure 2: The identity, the generator G_3 and a composition of Okada arc-diagrams of rank 8. The red arrows indicate the Hasse diagram of the nesting order.

The *mirror* $D^\star$ of an Okada arc-diagram, obtained by reflecting D horizontally, is an Okada-arc diagram and the map $D \mapsto D^\star$ extends to an anti-isomorphism of $\widetilde{\mathsf{O}}_N(X,Y)$. Let ι_N denote the map from $\widetilde{\mathsf{O}}_N(X,Y)$ to $\widetilde{\mathsf{O}}_{N+1}(X,Y)$ adding the labeled, propagating arc $\mathbf{h}(N+1 \longmapsto \overline{N+1}) = N+1$ to each arc-diagram. Then ι_N is an injective algebra homomorphism with image the set of diagrams containing $\mathbf{h}(N+1 \longmapsto \overline{N+1}) = N+1$.

Definition 3.7. *For $1 \leq i < N$, let G_i denote the* elementary *Okada arc-diagram containing the labeled arcs $\mathbf{h}(j \longmapsto \bar{j}) = j$ for $j \neq i, i+1$, $\mathbf{h}(i \longmapsto i+1) = i$ and $\mathbf{h}(\bar{i} \longmapsto \overline{i+1}) = i$.*

The elementary Okada diagrams $\mathsf{G}_1, \ldots, \mathsf{G}_{N-1}$ satisfy Okada relations $I(X,Y)$, C(X,Y), and S(X,Y). To construct the isomorphism from $\mathsf{O}_N(X,Y)$ to $\widetilde{\mathsf{O}}_N(X,Y)$, we first need to show that the elements (G_i) generate $\widetilde{\mathsf{O}}_N(X,Y)$. It is clear from the definition that if a product ends with G_i, then the diagram contains the arc $\mathbf{h}(\bar{i} \longmapsto \overline{i+1}) = i$. The converse is actually true: If an element $\mathbf{e} \in \mathfrak{A}_N$ contains the arc $\mathbf{h}(\bar{i} \longmapsto \overline{i+1}) = i$ then it can be factored as $\mathbf{e} = \mathbf{f} \cdot \mathsf{G}_i$.

Proposition 3.8. *Suppose $D \in \mathfrak{A}_N$ doesn't contain the arc $\mathbf{h}(N \longmapsto \overline{N}) = N$. Then there exist an integer i such that D contains the arc $\mathbf{h}(\bar{i} \longmapsto \overline{i+1}) = i$. If I is the largest such integer, then there exists a unique arc diagram $D^\flat \in \mathfrak{A}_{N-1}$ such that D factorize as*

$$D = \iota_{N-1}(D^\flat)\mathsf{G}_{N-1}\mathsf{G}_{N-2}\cdots\mathsf{G}_I\,.$$

By induction, this proves the following theorem:

Theorem 3.9. *The dimension of $\widetilde{\mathsf{O}}_N(X,Y)$ is $N!$ and it is generated by the elementary diagrams $\mathsf{G}_1, \ldots, \mathsf{G}_{N-1}$. Furthermore the map sending E_i to G_i extends multiplicatively to a unique algebra isomorphism $\Theta : \mathsf{O}_N(X,Y) \to \widetilde{\mathsf{O}}_N(X,Y)$.*

We conclude this section by making explicit the relation between fully packed loop configurations and Okada arc-diagrams:

Proposition 3.10. *For simplicity assume $Y = \mathbf{1}$. Let $\mathcal{C}$ be a FPLC of rank N, let $\underline{\mathbf{i}}$ be its reading, and let $\mathsf{E}_{\underline{\mathbf{i}}}$ be the corresponding monomial in the Okada algebra $\mathsf{O}_N(X, \mathbf{1})$. Then $\Theta(\mathsf{E}_{\underline{\mathbf{i}}}) = X^{\underline{\ell}}[\mathcal{C}]$ where $X^{\underline{\ell}} = \prod_{k \geq 1} x_k^{\ell_k}$ and ℓ_k counts the number of loops in $\mathcal{C}$ with height k.*

4 Fomin correspondence and Okada arc-diagrams

We have a bijection between $\mathfrak{S}_N$ and the monoid $\widetilde{\mathsf{O}}_N$ of Okada arc-diagrams, however, it is circuitous: Starting from a permutation, first its code is computed, then the associated FPLC is drawn, from which an Okada arc-diagram is obtained. It is not obvious, for example, that the inverse of a permutation corresponds to the mirror of the associated Okada arc-diagram. The goal of this section is to better explicate this graphical bijection which turns out to be an incarnation of Fomin's Robinson-Schested correspondence for the Young-Fibonacci [7, 1] lattice. Recall that this is a bijection between permutations of $\mathfrak{S}_N$ and pairs of saturated chains in the Young-Fibonacci lattice starting at $\emptyset$ and sharing a common endpoint in $\mathbb{YF}_N$. The reader who is not familiar with Fomin's construction should refer to [1]. See Figure 3a for an example. We will see in this section that Okada arc-diagrams are also in natural bijection with the same pairs of chains.

Cutting a labeled arc-diagram D in the middle gives a natural notion of a *Okada half arc-diagram* containing either a labeled full arc $\mathbf{h}(a \longmapsto b)$ joining two nodes $a, b \in [N]$, or else a labeled half arc $\mathbf{h}(a \longmapsto)$ with a free end. Such a half arc is called *propagating*. The *bra* $\langle D|$ is the Okada half arc-diagram obtained by restricting D to its positive part. The *ket* $|D\rangle$ is defined to be the bra $\langle D^\star|$ of the mirror $D^\star$.

Definition 4.1. *The* propagating label set *of an Okada half arc-diagram H is the subset* $\mathrm{PLab}(H)$ *of $[N]$ consisting of the height labels of its propagating arcs.*

The propagating label set of an Okada half arc-diagram of rank N is always a Fibonacci set of rank N. The following trivial lemma-definition is of great importance:

Lemma 4.2 (Gluing lemma)**.** *For any Okada arc diagram D, the left and right half diagram $\langle D|$ and $|D\rangle$ are two Okada half arc diagrams which have the same propagating labels set. As consequence, it makes sense to define* $\mathrm{PLab}(D) \coloneqq \mathrm{PLab}(\langle D|) = \mathrm{PLab}(|D\rangle)$.

Moreover if L and R are two Okada half arc-diagrams such that $\mathrm{PLab}(L) = \mathrm{PLab}(R)$, *there is a unique Okada arc-diagram $L \bowtie R$ such that $\langle L \bowtie R| = L$ and $|L \bowtie R\rangle = R$.*

To convert half arc diagrams to chains we need to restrict the former:

Definition 4.3. *For $r \leq N$, the* r-restriction *of an Okada half arc diagram H is the Okada half arc-diagram denoted by $H/[r]$ of rank r possessing*

- *a full arc* $\mathbf{h}(a \longmapsto b) = h$ *whenever* H *contains a full arc* $\mathbf{h}(a \longmapsto b) = h$ *with* $a, b \leq r$
- *a half arc* $\mathbf{h}(a \longmapsto) = h$ *whenever* H *contains either a full arc* $\mathbf{h}(a \longmapsto b) = h$ *with* $a \leq r < b$ *or a half arc* $\mathbf{h}(a \longmapsto) = h$ *with* $a \leq r$.

If $r \leq s$ then clearly the r-restriction of the s-restriction of any Okada half arc-diagram H coincides with the r-restriction of H.

Definition 4.4. *To any Okada half arc-diagram* H *of rank* N *we associate the sequence of Fibonacci sets* $\mathrm{Chain}(D) := (C_0, \ldots, C_N)$ *defined by* $C_i := \mathrm{PLab}(H/[i])$.

Proposition 4.5. *The map* Chain *is a bijection between Okada half arc-diagrams and saturated chains of rank* N *in the* $\mathbb{YFS}$*-lattice. See Figure 3a for an example.*

Theorem 4.6. *Given a permutation* $\sigma \in \mathfrak{S}_N$, *let* L_σ *and* R_σ *denote the two Okada half arc-diagrams associated to the pair of saturated chains obtained from Fomin's RS-correspondence. Then* $\Theta(\mathsf{E}_\sigma) = L_\sigma \bowtie R_\sigma$. *Moreover* $\Theta(\mathsf{E}_\sigma)^\star = \Theta(\mathsf{E}_{\sigma^{-1}}) = R_\sigma \bowtie L_\sigma$.

5 Structure of the Okada algebra and monoid

From now on, we identify $\mathsf{O}_N(X,Y)$ and $\widetilde{\mathsf{O}}_N(X,Y)$ through the isomorphism Θ. The goal of this section is to understand the structure of the Okada algebra and its monoid via the $\mathbb{YFS}_N$ *dominance order*. In particular, we describe the stratification of the Okada algebra by two-sided ideals (generated by *free elements*) and the Green relations for the monoid (which allows us show that the monoid is aperiodic). This allows us to prove cellularity of the Okada algebra in the next section.

Definition 5.1. *Let* $S = \{s_1 < \cdots < s_k\}$ *and* $T = \{t_1 < \cdots < t_\ell\}$ *be two Fibonacci sets of the same rank* N. *We says that* S is dominated by T *and write* $S \preceq T$ *if* $k < \ell$ *and* $s_{k-i} \leq t_{\ell-i}$ *for any* $0 \leq i < k$. *We write* $S \prec T$ *if* $S \preceq T$ *but* $S \neq T$.

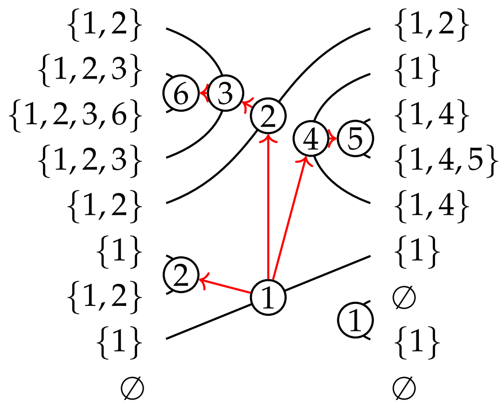
Proposition 5.2. $(\mathbb{YFS}_N, \preceq)$ *is a ranked lattice.*

Definition 5.3. *A* free set *of rank* N *is a subset of* $[N]$ *which does not contain both* i *and* $i+1$ *for all* $1 \leq i < N$. *The map* $S \mapsto \mathfrak{F}(S) := \{i \mid i - \max\{s \in S \mid s < i\} \text{ is odd}\}$ *defines a bijection from the collection of rank* N *Fibonacci sets to the collection of rank* N *free sets.*

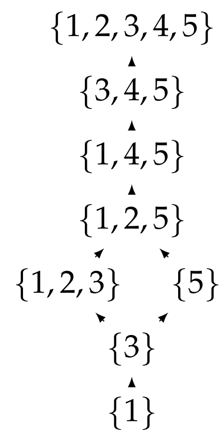
Definition 5.4. *For* $S \in \mathbb{YFS}_N$ *the associated* free element *in* $\mathsf{O}_N(X,Y)$ *is* $\mathsf{E}_S := \prod_{i \in \mathfrak{F}(S)} \mathsf{E}_i$.

Note that $\mathsf{E}_S = \mathsf{E}_{\sigma_S}$ where $\sigma_S := \prod_{i \in \mathfrak{F}(S)} \sigma_i$ is the associated *free involution* in $\mathfrak{S}_N$.

Remark 5.5. The half arc-diagrams $\langle \mathsf{E}_S|$ and $|\mathsf{E}_S\rangle$ coincide for any $S \in \mathbb{YFS}_N$. Furthermore $\langle \mathsf{E}_S|$ consists only of labeled propagating arcs $\mathbf{h}(s \longmapsto) = s$ for $s \in S$ and labeled full arcs $\mathbf{h}(i \longmapsto i+1) = i$ for $i \in \mathfrak{F}(S)$.



(a) An arc-diagram with its associated chains



(b) The dominance order on $\mathbb{YFS}_5$

Proposition 5.6. *Let $\mathfrak{J}_S$ be the two-sided ideal in $\mathsf{O}_N(X,Y)$ generated a free element E_S for $S \in \mathbb{YFS}_N$, then $\mathfrak{J}_S \subseteq \mathfrak{J}_T$ if and only if $T \preceq S$ for any pair $S, T \in \mathbb{YFS}_N$.*

Theorem 5.7 (Triangular Factorization)**.** *For $\sigma \in \mathfrak{S}_N$ there exists a unique pair of permutations $\rho, \tau \in \mathfrak{S}_N$ such that $\mathsf{E}_\sigma = \mathsf{E}_\rho \cdot \mathsf{E}_S \cdot \mathsf{E}_\tau$ where $\ell(\sigma) = \#S + \ell(\rho) + \ell(\tau)$ and $S \preceq \inf(\mathrm{PLab}(\mathsf{E}_\rho), \mathrm{PLab}(\mathsf{E}_\tau))$ and where $S = \mathrm{PLab}(\mathsf{E}_\sigma)$.*

Returning to the Okada monoid, an element $\mathbf{e} \in \mathsf{O}_N$ is said to be *involutive* whenever it equals its mirror $\mathbf{e}^\star$. Involutive elements are always idempotents and thanks to the RS correspondence, these are precisely the basis monomials E_σ where $\sigma \in \mathfrak{S}_N$ is an involution (*i.e.* $\sigma^2 = 1$).

Remark 5.8. The set of idempotents in O_N is not exhausted by the involutive elements. For example in O_3 all element are idempotents, while $\mathsf{E}_1\mathsf{E}_2\mathsf{E}_3$ and $\mathsf{E}_3\mathsf{E}_2\mathsf{E}_1$ are the only non-idempotents in O_4. Computer calculations show that the number of idempotents for $N \le 10$ are: 1, 1, 2, 6, 22, 108, 594, 4116, 30500, 274006, 2560400.

Proposition 5.9. *Let $\mathbf{e}, \mathbf{f} \in \mathsf{O}_N$. Then either $\langle \mathbf{ef}| = \langle \mathbf{e}|$ and thus $\mathrm{PLab}(\mathbf{ef}) = \mathrm{PLab}(\mathbf{e})$ or $\mathrm{PLab}(\mathbf{ef}) \prec \mathrm{PLab}(\mathbf{e})$. As a consequence, $\mathrm{PLab}(\mathbf{ef}) \preceq \inf(\mathrm{PLab}(\mathbf{e}), \mathrm{PLab}(\mathbf{f}))$.*

The previous proposition is the main ingredient of the following theorem which describe the structure of the Okada monoid:

Theorem 5.10. *The monoid O_N is aperiodic, i.e. there exists an integer K such that $\mathbf{e}^K = \mathbf{e}^{K+1}$ for all $\mathbf{e} \in \mathsf{O}_N$. Equivalently, all the groups in O_N are trivial.*

Recall that $\mathcal{R}$ (resp. $\mathcal{J}$) is the equivalence relation on O_N such that $\mathbf{e}\,\mathcal{R}\,\mathbf{f}$ if $\mathbf{e}$ and $\mathbf{f}$ generate the same right (resp. two-sided) ideals.

Theorem 5.11. *Each $\mathcal{R}$-class of O_N contains a unique involutive element. Each $\mathcal{J}$-class of O_N contains a unique free element. Moreover, the free representative of $\mathbf{e} \in \mathsf{O}_N$ is the free element having the same propagating set as $\mathbf{e}$.*

6 Cellular structure of the Okada algebra

Recall that a cellular algebra A is a finite dimensional algebra with distinguished cellular basis which is particularly well-adapted to studying the representation theory of A, especially as the ground ring/field varies. For brevity, we skip a general discussion about cellular algebras and point the reader to [3] for definitions and context.

Definition 6.1. *Let $\mathfrak{H}_N$ and $\mathbb{C}\mathfrak{H}_N$ denote respectively the set and the vector space spanned by all Okada half arc-diagrams of rank N. Likewise $\mathfrak{H}_N^S$ and $\mathbb{C}\mathfrak{H}_N^S$ will denote the set and the vector space spanned by all half diagrams $H \in \mathfrak{H}_N$ for which $\mathrm{PLab}(H) = S$ where $S \in \mathbb{YFS}_N$. We extend the bra map $D \mapsto \langle D|$ by linearity to obtain a map from $\mathsf{O}_N(X,Y)$ to $\mathbb{C}\mathfrak{H}_N$.*

The following result is a consequence of the factorization given in Proposition 5.7:

Theorem 6.2. *The Okada algebra $\mathsf{O}_N(X,Y)$ is cellular with the following data*

1. *A cell-poset is $\Lambda_N = (\mathbb{YFS}_N, \preceq)$.*
2. *An index set $M_S = \mathfrak{H}_N^S$ for each $S \in \mathbb{YFS}_N$*
3. *A cellular basis element $C^S_{L,R} := L \bowtie R$ associated to $L, R \in \mathfrak{H}_N^S$*
4. *An involutive anti-isomorphism given by the mirror map $\star : L \bowtie R \mapsto R \bowtie L$.*

Remark 6.3. The left $\mathsf{O}_N(X,Y)$ *cell module* associated to $S \in \mathbb{YFS}_N$ can be realized by the vector space $\mathbb{C}\mathfrak{H}_N^S$ equipped with the left action defined by

$$D \bullet H \ := \begin{cases} \langle D \cdot (H \bowtie H)| & \text{if } \mathrm{PLab}\left(D \circ (H \bowtie H)\right) = S \\ 0 & \text{otherwise} \end{cases}$$

where $D \in \mathsf{O}_N(X,Y)$ and $H \in \mathfrak{H}_N^S$. For generic values of X and Y, $\mathbb{C}\mathfrak{H}_N^S$ is irreducible.[1]

7 Prospectives

For a fixed choice of a threshold $k \geq 1$, we *truncate* any Okada arc-diagram D, replacing its height labels h by $\min(h,k)$. The k-truncated Okada arc-diagrams form a multiplicative basis for a *higher Blob algebra* $\mathrm{Blob}_N^{(k)}$, which can be realized as a quotient of the Okada algebra $\mathsf{O}_N(X,Y)$ after specializing the X, Y parameters appropriately. In particular the Temperely-Lieb and Martin-Saleur Blob algebras [4] are recovered for $k = 1, 2$ respectively. It seems that the corresponding Bratelli diagram $\mathbb{YF}^{(k)}$ naturally embeds

[1] The cell module $\mathfrak{H}_N^S$ carries an invariant bilinear form φ_S. We conjecture an explicit value for the determinant of the associated Gram matrix G_S, which we express in terms of clone Schur functions.

into the $\mathbb{YF}$-lattice and can be seen as a Fibonacci counterpart of the sublattice of integer partitions with at most k parts. Both the Temperely-Lieb and the Blob algebras are *intertwiner algebras* which raises the question of whether the higher Blob algebras have such a description for $k \geq 3$. If so, this would be indicative of a Fibonacci version of *Schur-Weyl duality*, and would entail, on a combinatorial level, a well-behaved version of the RSK-correspondence.

It should be possible to incorporate height labels into other diagram algebras such as the partition and Brauer algebras. Can one, for example, define a suitable notion of height labeled *braids* together with *skein relations* consistent with these labels? A satisfactory answer might shed light onto the problem of identifying appropriate *Jucys-Murphy* elements for the Okada algebras.

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