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Higher Specht Polynomials and Tableaux Bijections for Hessenberg Varieties

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Abstract. The cohomology rings of regular semisimple Hessenberg varieties are only completely understood in some cases. One such case is when the Hessenberg function is $h = (h(1), n, \ldots, n)$, and is described by Abe, Horiguchi, and Masuda in 2017. We define an alternative basis for the cohomology ring in this case, which is a higher Specht basis. We give combinatorial bijections between the monomials in this basis and sets of P-tableaux, motivated by the work of Gasharov in 2008 and Shareshian and Wachs in 2016. This bijection illustrates the connection between the symmetric group action on these cohomology rings and the Schur expansion of chromatic symmetric functions. We further use the inversion formula for P-tableaux to give a new combinatorial proof of the known Poincaré polynomial for these Hessenberg varieties.

Keywords: Hessenberg varieties, P-tableaux, higher Specht bases

1 Introduction

In this extended abstract, we exhibit new connections between the combinatorics of Hessenberg varieties, P-tableaux, and chromatic symmetric functions, and illustrate their use in proving geometric results using combinatorial tools. In particular, when S is a regular semisimple matrix and $h = (h(1), n, \ldots, n)$, we construct a higher Specht basis for the cohomology ring $H^*(\mathrm{Hess}(S, h))$, display combinatorial bijections between these higher Specht basis elements and sets of tableaux, and use a new combinatorial method to find the Poincaré polynomial for $\mathrm{Hess}(S, h)$. Full proofs of the results in this paper are forthcoming in [13].

Hessenberg varieties, initially defined and studied in [11, 12], are linear subvarieties of the full flag variety $Fl(\mathbb{C}^n)$. They are connected to chromatic symmetric and quasisymmetric functions due to the action of the symmetric group S_n on their cohomology rings defined by Tymozcko in [18]. The geometry of Hessenberg varieties has been – and continues to be – extensively studied, including in [1, 9, 7, 8, 10, 17].

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1.1 Background on Hessenberg Varieties and Specht Modules

Given a proper coloring $\kappa : V \to \mathbb{N}$ of a finite simple graph $G = (V, E)$ on a totally-ordered vertex set V, define an **ascent** to be an edge $\{v, w\}$ such that $v < w$ and $\kappa(v) < \kappa(w)$. Define $\operatorname{asc}(\kappa)$ to be the total number of ascents in κ. In this paper, we consider graphs with $V = [n] = \{1, 2, \ldots, n\}$. In [14], Shareshian and Wachs defined the chromatic quasisymmetric function, a graded analogue of Stanley's chromatic symmetric function [15], using the ascent statistic:

Definition 1.1. *Given a finite simple graph $G = (V, E)$, the **chromatic quasisymmetric function** for G is*

$$X_G(\mathbf{x}; q) = \sum_{\kappa : V \to \mathbb{N}} \left(\prod_{i \in V} x_{\kappa(i)} \right) q^{\operatorname{asc}(\kappa)}$$

where the sum ranges over all proper colorings κ of the vertices of G.

Recall that the **(full) flag variety** of $\mathbb{C}^n$ is the variety $Fl(\mathbb{C}^n)$ whose points are flags $F_\bullet = F_0 \subset F_1 \subset \cdots \subset F_n$ such that $\dim(F_i) = i$. We will define Hessenberg varieties to be a subvariety of the flag variety, given a matrix $X : \mathbb{C}^n \to \mathbb{C}^n$ and a function h. First, a **Hessenberg function** is a function $h : [n] \to [n]$ such that for all i, we have $i \leq h(i)$, and $h(i) \leq h(i+1)$. We usually denote this as a vector $h = (h(1), h(2), \ldots, h(n))$.

Definition 1.2. *Given a matrix $X : \mathbb{C}^n \to \mathbb{C}^n$ and a Hessenberg function $h : [n] \to [n]$, define the **Hessenberg variety** to be:*

$$\operatorname{Hess}(X, h) = \left\{ F_\bullet \in Fl(\mathbb{C}^n) \mid X(F_i) \subseteq F_{h(i)} \text{ for all } 1 \leq i \leq n \right\} \tag{1.1}$$

In [18], Tymoczko defined an action of the symmetric group on the cohomology ring $H^*(\operatorname{Hess}(S, h))$ when S is a regular semisimple matrix, allowing us to study the structure of this ring as an S_n-module. For each Hessenberg function, we can also construct a poset P_h on $[n]$, using h to determine which elements are comparable: We say that $i <_{P_h} j$ if and only if $h(i) < j$. Let G_h be the incomparability graph of P_h, which has an edge $\{i, j\}$ whenever i and j are incomparable in P_h. Notably, posets formed by this construction are $(3+1)$- and $(2+2)$-avoiding, so their incomparability graphs are relevant to the following conjecture of Stanley and Stembridge on chromatic symmetric functions.

Conjecture 1.3 ([16] Conjecture 5.1)**.** *If P is a $(3+1)$-free poset, then $X_{\operatorname{inc}(P)}(\mathbf{x})$ is e-positive, that is, it can be written with positive coefficients when expanded in the elementary basis of symmetric functions.*

In [5], Guay-Paquet proved that it suffices to prove the conjecture for posets which are $(3+1)$- and $(2+2)$-avoiding. Shareshian and Wachs conjectured that there is a connection between chromatic quasisymmetric functions and the graded cohomology ring

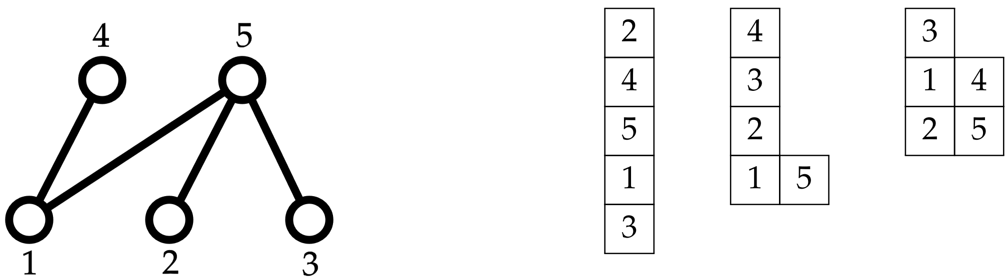


Figure 1: On the left, the poset P_h for $h = (3, 4, 4, 5, 5)$. On the right, three P_h-tableaux (in French notation). The leftmost tableau has 4 inversions given by the pairs $(1, 3), (2, 3), (2, 4)$, and $(4, 5)$, but does not have the inversion $(2, 5)$ since $2 <_{P_h} 5$.

of regular semisimple Hessenberg varieties, using the ascent formula for the incomparability graph G_h. This connection, stated below, was proven by Brosnan and Chow in [2] and separately by Guay-Paquet in [6].

Proposition 1.4 ([2, 6]). *Let S be a regular semisimple $n \times n$ matrix and $h : [n] \to [n]$ be a Hessenberg function. Let G_h be the incomparability graph for P_h. Then*

$$\omega X_{G_h}(\mathbf{x}; q) = \sum_{k=0}^{|E|} \mathrm{Frob}(H^{2k}(\mathrm{Hess}(S, h)))q^k$$

Above, ω is the standard involution on symmetric functions which sends the Schur function s_λ to $s_{\lambda'}$, where λ' is the transpose of λ, and Frob is the Frobenius characteristic map which sends the irreducible S_n-module V_λ to the Schur function s_λ.

Definition 1.5 ([3]). *Let P be a poset and λ be a partition of n. A* ***P-tableau of shape*** *λ is a filling of the Young diagram of λ with entries from P such that:*

- *Each entry in P is used at most once.*
- *Adjacent entries in rows are P-increasing from left to right.*
- *Adjacent entries in columns are P-nondecreasing from bottom to top.*

We say a **P-inversion** in a P-tableau is a pair of entries (i, j) such that $i < j$ as integers, i is in a higher row than j, and i and j are incomparable in P. Define $\mathrm{inv}(T)$ to be the number of P-inversions in T. In [3], Gasharov used P-tableaux to show that the chromatic symmetric functions of incomparability graphs of $(3+1)$-free posets are Schur-positive. Using this inversion statistic on P-tableaux, Shareshian and Wachs extended this result to the chromatic quasisymmetric case, as stated below.

Proposition 1.6 ([14] Theorem 6.3). *Let G be the incomparability graph of a $(3+1)$-free poset P, and let $PT(\lambda)$ be the set of P-tableaux of shape λ. Then we have:*

$$X_G(\mathbf{x};q) = \sum_{\lambda \vdash n} \left(\sum_{T \in PT(\lambda)} q^{\mathrm{inv}(T)} \right) s_\lambda$$

Combining the results of Propositions 1.4 and 1.6, we can connect the graded cohomology of $\mathrm{Hess}(S,h)$ with P-tableaux in the following way. If S is a regular semisimple matrix, and h is a Hessenberg function with poset P_h and incomparability graph G_h, then

$$\sum_{k=0}^{|E|} \mathrm{Frob}(H^{2k}(\mathrm{Hess}(S,h)))q^k = \omega X_{G_h}(\mathbf{x};q) = \sum_{\lambda \vdash n} \left(\sum_{T \in PT(\lambda)} q^{\mathrm{inv}_h(T)} \right) s_{\lambda'} \tag{1.2}$$

where λ' is the transpose partition of λ, and inv_h is the inversion statistic for P_h.

The formula above gives us a nice way of understanding the decomposition of the S_n-module $H^*(\mathrm{Hess}(S,h))$ into irreducible modules. Irreducible S_n-modules are isomorphic to the Specht modules, which have a basis indexed by standard tableaux:

Definition 1.7. *Given a standard tableau T of shape λ, define the* ***Specht polynomial*** *to be*

$$F_T = \prod_{C \in \lambda} \left(\prod_{i<j \in C} (x_j - x_i) \right)$$

where the first product is over all columns in the Young diagram, and the second product is over all pairs of entries $i < j$ in the column C. If $SYT(\lambda)$ is the set of all standard tableaux of shape λ, then the ***Specht module*** *V_λ is the subspace of $\mathbb{Q}[x_1, \ldots, x_n]$ generated by $\{F_T\}_{T \in SYT(\lambda)}$.*

An immediate consequence of this definition is that the dimension of the Specht module V_λ is the number of standard tableaux of shape λ, which we denote $\#SYT(\lambda)$. We define a higher Specht basis for an S_n-module as follows.

Definition 1.8 ([4], Definition 1.5). *If R is an S_n-module which decomposes into irreducible S_n-modules as*

$$R = \bigoplus_\lambda c_\lambda V_\lambda\,,$$

then a ***higher Specht basis of*** *R is a set of elements $\mathcal{B}$ with a decomposition $\mathcal{B} = \bigcup_\lambda \bigcup_{i=1}^{c_\lambda} \mathcal{B}_{i,\lambda}$ such that the elements of $\mathcal{B}_{i,\lambda}$ are a basis of the i-th copy of V_λ in the decomposition of R.*

Hence, higher Specht bases of S_n-modules are a natural way to understand the action of S_n, and allow us to more easily identify the decomposition into irreducible modules.

2 Higher Specht basis for the cohomology ring

In this section, let S be a regular semisimple $n \times n$ matrix, and $h = (h(1), n, \ldots, n)$ be a Hessenberg function. In [9] (Theorem 4.3), Abe, Horiguchi, and Masuda give a presentation of the cohomology ring $H^*(\text{Hess}(S,h))$ as a quotient of a polynomial ring in $2n$ variables. Further, in Remark 4.5, they describe a set of basis elements for this ring. We name the two sets of different types of elements below:

$$B_1 = \left\{ x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \quad \text{not containing the factor } \prod_{\ell=1}^{h(1)} x_\ell \right\} \tag{2.1}$$

$$B_2 = \left\{ x_n^{\ell_1} x_{n-1}^{\ell_2} \cdots x_2^{\ell_{n-1}} y_k \quad \text{not containing the factor } \prod_{\ell=h(1)+1}^{n} x_\ell \right\} \tag{2.2}$$

over all $0 \leq i_j \leq n-j$ in B_1, and over all $0 \leq \ell_j \leq n-1-j$, and $1 \leq k \leq n-1$ in B_2.

The symmetric group S_n acts on the above monomials by fixing the set of x_i and permuting the set of y_i in the natural way. This group action gives a representation of S_n, which decomposes into the direct sum of trivial representations (corresponding to the Specht module $V_{(n)}$) and standard representations (corresponding to the Specht module $V_{(n-1,1)}$).

We define B_3 to be the following set of monomials:

$$B_3 = \left\{ x_n^{\ell_1} x_{n-1}^{\ell_2} \cdots x_2^{\ell_{n-1}} (y_{k+1} - y_1) \quad \text{not containing the factor } \prod_{\ell=h(1)+1}^{n} x_\ell \right\} \tag{2.3}$$

over all $0 \leq \ell_j \leq n-1-j$ and $1 \leq k \leq n-1$.

Notice that there are natural projections from B_1 to the Specht module $V_{(n)}$ and from B_3 to the Specht module $V_{(n-1,1)}$ given by forgetting the x_i variables.

Theorem 2.1 ([13]). *The set $B_1 \cup B_3$ forms a higher Specht basis of $H^*(\text{Hess}(S,h))$.*

The proof (see [13] for full details) uses the fact that $B_1 \cup B_2$ forms a $\mathbb{Z}-$basis of $H^*(\text{Hess}(S,h))$, and constructs the transition matrix from $B_1 \cup B_2$ to $B_1 \cup B_3$ using the relations given in [9] to express the new elements in terms of the old basis. Then, we prove that this transition matrix is invertible, which requires the following lemma.

Lemma 2.2 ([13]). *If $f(x_1, \ldots, x_n)$ is a homogeneous polynomial in the ring $H^*(\text{Hess}(S,h))$, then f can be expressed solely in terms of basis elements from B_1.*

Knowing that $B_1 \cup B_3$ forms a higher Specht basis of $H^*(\text{Hess}(S,h))$ allows us to obtain a more direct proof of the following fact, by counting the number of monomials of each type in B_1 and B_3.

Corollary 2.3. *The dot action of S_n on $H^*(\text{Hess}(S,h))$ decomposes into $h(1)(n-1)!$ copies of the trivial representation, and $(n-h(1))(n-2)!$ copies of the standard representation.*

3 Bijections between basis elements and P-tableaux

As seen in Section 1.1, there is a bijection between the set of standard tableaux of shape λ with basis elements of the Specht module V_λ, given by the construction of the basis elements. Further, from Equation 1.2, we have an explicit connection between the number of basis elements of $H^*(\mathrm{Hess}(S,h))$ of each degree and the set of P_h-tableaux with each number of inversions. In particular, there should be bijections between the higher Specht basis elements and the sets of P_h-tableaux with shape corresponding to the Specht polynomials in the basis.

3.1 Regular Nilpotent Hessenberg Varieties

In the case of regular nilpotent Hessenberg varieties, a polynomial presentation of the cohomology ring is known for any Hessenberg function h.

Proposition 3.1 ([7], Corollary 7.3)**.** *Let N be a regular nilpotent matrix, and let $h : [n] \to [n]$ be a Hessenberg function. Then the following set of monomials form an additive basis for $H^*(Hess(N,h))$:*

$$\mathcal{N}_h := \left\{ x_1^{i_1} \cdots x_n^{i_n} \mid 0 \le i_k \le h(k) - k \text{ for } 1 \le k < n \right\}$$

In [1], Abe et al. show that the cohomology rings of regular nilpotent Hessenberg varieties are isomorphic to the fixed points of the cohomology rings for regular semisimple Hessenberg varieties. In particular, these are the pieces of $H^*(\mathrm{Hess}(S,h))$ which decompose into trivial S_n-modules, corresponding to the Specht module $V_{(n)}$.

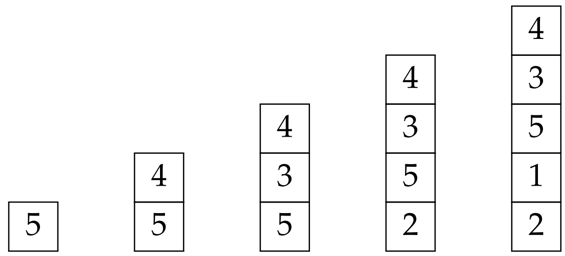
Define $PT(h,\lambda)$ to be the set of P_h-tableaux of shape λ. Taking the transpose partition for $\lambda = (n)$ (because of Equation 1.2), we form a map φ between $\mathcal{N}_h$ and $PT(h,(1^n))$ for any Hessenberg function h.

Definition 3.2. *Let $x_1^{i_1} \cdots x_n^{i_n} \in \mathcal{N}_h$.*

- *Begin with a P_h-tableau T of a single box whose entry is n.*
- *For each $k = n-1, \ldots 1$:*
  - *If $i_k = 0$, insert k into a new box at the bottom of T, so that k occurs directly below some $\ell > k$.*
  - *If $i_k > 0$, then since $i_k \le h(k) - k$, we have $k < h(k)$. List the entries $k+1, \ldots, h(k)$, which already exist in T, in order from the lowest to highest row position in T. Insert k in a new box directly above the i_k-th lowest entry of this list.*

Define $\varphi(x_1^{i_1} \cdots x_n^{i_n}) \in PT(h,(1^n))$ to be the resulting tableau from this process.

Example 3.3. *Let $h = (2, 3, 5, 5, 5)$, and consider the monomial $x_1x_3x_4 \in \mathcal{N}_h$. We construct $\varphi(x_1x_3x_4)$ as follows.*



We start with a single box containing a 5. Then, to insert 4 with $i_4 = 1$ P_h-inversion, we insert the 4 above the 5. To insert 3 with $i_3 = 1$ P_h-inversion, we insert the 3 above the 5 but below the 4. Notice that at each step, the number of elements in P_h greater than k that are incomparable to k is $h(k) - k$, which is also the largest possible power i_k.

In [13], we prove that φ is a bijection, which is weight preserving in the following way: If m is a monomial of degree d, then $\varphi(m)$ has d P_h-inversions.

Theorem 3.4 ([13])**.** *The map φ is a well-defined, weight-preserving bijection.*

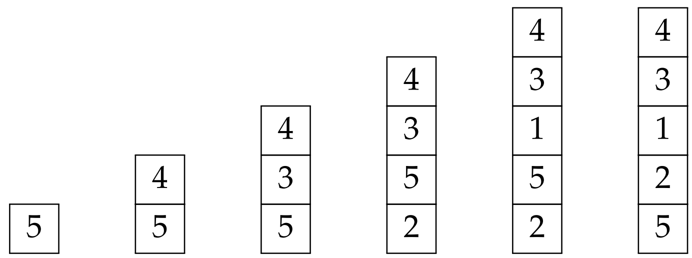
3.2 Regular Semisimple Hessenberg Varieties

Now we turn our attention to regular semisimple Hessenberg varieties. In this section, S is a regular semisimple matrix and $h = (h(1), n, \ldots, n)$. Recall the partial set of basis elements B_1 defined in Equation 2.1, and recall that S_n fixes these monomials, since they contain no y_i variable. These correspond to basis elements of the trivial Specht module $V_{(n)}$. Again, we take the transpose partition, and define a map ψ between B_1 and $PT(h, (1^n))$ as follows.

Definition 3.5. *Let $x_1^{i_1} \cdots x_n^{i_n} \in B_1$.*

- *Begin with a P_h-tableau T of a single box whose entry is n.*
- *For each $k = n-1, \ldots, 1$, insert k into T above exactly i_k of the existing entries.*
- *Let k' be the smallest index in $1, \ldots, h(1)$ such that $i_{k'} = 0$, which exists by the definition of B_1. By this construction, after inserting n through 1, k' will be on the bottom of T.*
    - *If $k' = 1$, then define $\psi(x_1^{i_1} \cdots x_n^{i_n})$ to be T.*
    - *If $1 < k' \le h(1)$, then slide the entry k' up until it is directly below the 1, and define $\psi(x_1^{i_1} \cdots x_n^{i_n})$ to be T after this slide.*

Example 3.6. *Let $h = (3,5,5,5,5)$, and consider the monomial $x_1^2x_3x_4 \in B_1$. We construct $\varphi(x_1^2x_3x_4)$ as follows.*



We start with a single box containing a 5. Then we insert the 4 above one existing entry, the 3 above one existing entry, the 2 above no existing entries, and the 1 above two existing entries. Since $1 <_{P_h} 5$ are comparable, the resulting tableau is not a P_h-tableau, so we shift the 2 (which is incomparable to the 1) to be directly below the 1. In the second-to-last tableau, the number of P_N-inversions with each k as the smaller entry is exactly i_k, so reading these inversions returns the monomial $x_1^2x_3x_4$.

In [13], we find the inverse map of ψ to prove the following theorem.

Theorem 3.7 ([13])**.** *The map ψ is a well-defined bijection.*

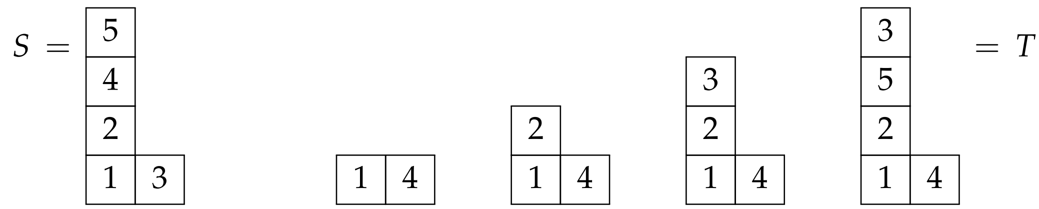
We now construct a map for the set of basis elements B_3 defined in Equation 2.3. Define $PSPT(h, \lambda)$ to be the set of pairs (S, T) where S is a standard tableau and T is a P_h-tableau, both of shape λ. Since the monomials in B_3 correspond to the Specht polynomials in the Specht module $V_{(n-1,1)}$, we construct the map τ to the set $PSPT(h, (2, 1^{n-2}))$, with the x_i variables corresponding to the P_h-tableau and the y_i variables corresponding to the standard tableau.

Definition 3.8. *Let $x_n^{\ell_1} \cdots x_2^{\ell_{n-1}}(y_k - y_1) \in B_3$.*

- *Define S to be the unique standard tableau of shape $(2, 1^{n-2})$ with entries 1 and k in the bottom row.*
- *Let j be the largest entry among $\{h(1) + 1, \ldots, n\}$ such that the exponent ℓ_{n-j+1} on x_j is zero, which exists by the definition of B_3.*
- *Initialize a tableau T with a single row of two boxes, containing a 1 and j.*
- *For each $i = 2, \ldots, n$, other than j, insert i into the left column so that it is under exactly $(i - 2) - \ell_{n-i+1}$ of the current entries in the left column.*

Define $\tau(x_n^{\ell_1} \cdots x_2^{\ell_{n-1}}(y_k - y_1)) \in PSPT(h, (2, 1^{n-2}))$ to be the pair (S, T) resulting from this construction.

Example 3.9. *Let $h = (3,5,5,5,5)$, and consider the monomial $x_5^2 x_3(y_3 - y_1) \in B_3$. Note that $j = 4$ is the largest index where x_j has an exponent of zero. We construct $\tau(x_5^2 x_3(y_2 - y_1))$ as follows.*



S is defined to be the unique standard Young tableaux of shape $(2, 1^{n-2})$ with a 1 and 3 in the bottom row. Then, to form T, we start with a single row containing a 1 and a 4. We then insert a 2 in the left column underneath $(2-2) - 0 = 0$ entries, a 3 in the left column underneath $(3-2) - 1 = 0$ entries, and a 5 in the left column underneath $(5-2) - 2 = 1$ entry.

In [13], we again find the inverse map to prove the following theorem.

Theorem 3.10 ([13])**.** *The map τ is a well-defined bijection.*

This map is almost weight-reversing, since the exponents on the x_i terms correspond to inversions that are missing in the tableaux, since we insert i so that it forms $(i-2) - \ell_{n-i+1})$ inversions. In future work, we hope to use bijections like these to extrapolate potential bases for $H^*(\text{Hess}(S,h))$ in other cases.

4 Poincaré polynomials of Hessenberg varieties

Given a graded vector space V over a field k, if $V = \bigoplus_{i \in \mathbb{N}} V_i$ with each subspace V_i consisting of vectors of degree i being finite dimensional, then the **Poincaré polynomial** of V is

$$\text{Poin}(V,q) = \sum_{i \in \mathbb{N}} \dim_k(V_i) q^i$$

Then, for an algebraic variety X with graded cohomology ring $H^*(X)$, we define the Poincaré polynomial of X to be $\text{Poin}(X,q) := \text{Poin}(H^*(X),q)$. From Equation 1.2, we can write the Poincaré polynomial of a regular semisimple Hessenberg variety in the following way:

$$\text{Poin}(\text{Hess}(S,h),q) = \sum_{\lambda \vdash n} \left(\sum_{T \in \text{PT}(h,\lambda)} q^{\text{inv}_h(T)} \right) \#\text{SYT}(\lambda) \tag{4.1}$$

since the dimension of the irreducible Specht module V_λ is the number of standard tableaux of shape λ. We use this formula to provide an alternate proof of the formula of the Poincaré polynomial for $\mathrm{Hess}(S,h)$ when $h = (h(1), n, \ldots, n)$, originally calculated by Abe, Horiguchi, and Masuda in [9]. Recall that the q-analogue of n is $(n)_q = (1+q+\cdots+q^{n-1})$, and the q-analogue of $n!$ is $(n)_q! = (n)_q(n-1)_q\cdots(1)_q$. We present the full proof here, as it illustrates the new combinatorial method using P_h-tableaux.

Theorem 4.1 ([9], Lemma 3.2)**.** *If $h = (h(1), n, \ldots, n)$, then the Poincaré polynomial of $\mathrm{Hess}(S,h)$ is given by*

$$\begin{aligned}\mathrm{Poin}(\mathrm{Hess}(S,h),q) &= \frac{1-q^{h(1)}}{1-q}\prod_{j=1}^{n-1}\frac{1-q^j}{1-q} + (n-1)q^{h(1)-1}\frac{1-q^{n-h(1)}}{1-q}\prod_{j=1}^{n-2}\frac{1-q^j}{1-q}\\ &= h(1)_q(n-1)_q! + (n-1)q^{h(1)-1}(n-h(1))_q(n-2)_q!\end{aligned}$$

Proof. From above, we know that

$$\mathrm{Poin}(\mathrm{Hess}(S,h),q) = \sum_{\lambda\vdash n}\left(\sum_{T\in\mathrm{PT}(h,\lambda)} q^{\mathrm{inv}_h(T)}\right)\#\mathrm{SYT}(\lambda).$$

Let $h = (h(1), n, \ldots, n)$. All chains in P_h have length two and include the element 1. Since distinct rows in a P_h tableaux need to contain entries from distinct chains in P_h, the only shapes λ with a nonzero number of P_h-tableaux are $\lambda = (1^n)$ and $\mu = (2, 1^{n-2})$. Further, we have that $\#\mathrm{SYT}(\lambda) = 1$ and $\#\mathrm{SYT}(\mu) = n-1$.

For $\lambda = (1^n)$, we need to count the P_h-inversions in the P_h tableaux of this shape. Since the element 1 is incomparable to 2 through $h(1)$, it can form between 0 and $h(1)-1$ inversions as the smaller entry. For each $i = 2, \ldots, n$, the entry i can form up to $n-i$ inversions as the smaller entry. Hence, we get that

$$\sum_{T\in\mathrm{PT}(h,\lambda)} q^{\mathrm{inv}_h(T)} = (1+q+\cdots+q^{h(1)-1})(1+q+\cdots+q^{n-2})! = h(1)_q(n-1)_q!.$$

For $\mu = (2, 1^{n-2})$, the bottom row of any P_h-tableaux of shape μ must be filled with entries from a chain in P_h, so it contains a 1 and an i for some $i = h(1)+1, \ldots, n$. Then, since $i > 1$, it is incomparable with all other $j \neq 1$, so the entry i in the bottom row forms inversions as the larger entry with the entries $2, \ldots, i-1$, of which there are $i-2$. So this entry contributes between $h(1)-1$ and $n-2$ inversions to the P_h-tableaux as the larger entry. Then, for the column entries of $j = 2, \ldots, n$ and $j \neq i$, if $j < i$, then j forms an inversion with i where j is the smaller entry (which was already counted), and can form an inversion as the smaller entry with the other $n-j-1$ entries larger than j. If $j > i$, then j does not form an inversion with i, and can form an inversion as the smaller entry with any of the $n-j$ entries larger than j. In each case, there is a unique placement for j giving each set of inversions. Hence we have

$$\sum_{T\in\mathrm{PT}(h,\mu)} q^{\mathrm{inv}_h(T)} = (q^{h(1)-1}+\cdots+q^{n-1})(1+\cdots+q^{n-3})! = q^{h(1)-1}(n-h(1))_q(n-2)_q!$$

Therefore, for $\lambda = (1^n)$ and $\mu = (2, 1^{n-2})$, we have the Poincaré polynomial of $\mathrm{Hess}(S, h)$ as follows:

$$\begin{aligned}\mathrm{Poin}(\mathrm{Hess}(S,h),q) &= \sum_{T\in \mathrm{PT}(h,\lambda)} q^{\mathrm{inv}_h(T)} + (n-1)\sum_{T\in \mathrm{PT}(h,\mu)} q^{\mathrm{inv}_h(T)} \\ &= h(1)_q(n-1)_q! + (n-1)q^{h(1)-1}(n-h(1))_q(n-2)_q!\end{aligned}$$

This completes the proof. □

These methods provide a new, combinatorial means of finding Poincaré polynomials of regular semisimple Hessenberg varieties, which may be useful in further understanding the basis decomposition of their cohomology rings.

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References

[1] H. Abe, M. Harada, T. Horiguchi, and M. Masuda. "The cohomology rings of regular nilpotent Hessenberg varieties in Lie type A". 2016. arXiv:1512.09072.

[2] P. Brosnan and T. Y. Chow. "Unit interval orders and the dot action on the cohomology of regular semisimple Hessenberg varieties". *Advances in Mathematics* **329** (2018), pp. 955–1001. DOI.

[3] V. Gasharov. "Incomparability graphs of (3 + 1)-free posets are s-positive". *Discrete Mathematics* **157**.1 (1996), pp. 193–197. DOI.

[4] M. Gillespie and B. Rhoades. "Higher Specht Bases for Generalizations of the Coinvariant Ring". *Annals of Combinatorics* **25** (Mar. 2021), pp. 1–27. DOI.

[5] M. Guay-Paquet. "A modular relation for the chromatic symmetric functions of (3+1)-free posets". 2013. arXiv:1306.2400.

[6] M. Guay-Paquet. "A second proof of the Shareshian–Wachs conjecture, by way of a new Hopf algebra". 2016. arXiv:1601.05498.

[7] M. Harada, T. Horiguchi, S. Murai, M. Precup, and J. Tymoczko. "A filtration on the cohomology rings of regular nilpotent Hessenberg varieties". *Mathematische Zeitschrift* **298**.3 (2021), pp. 1345–1382. DOI.

[8] M. Harada and M. Precup. "The cohomology of abelian Hessenberg varieties and the Stanley–Stembridge conjecture". *Algebraic Combinatorics* **2**.6 (2019), pp. 1059–1108. DOI.

[9] A. Hiraku, H. Tatsuya, and M. Mikiya. "The cohomology rings of regular semisimple Hessenberg varieties for h=(h(1),n,...,n)". *Journal of Combinatorics* **10**.1 (2019), pp. 27–59. DOI.

[10] E. Insko and M. Precup. "The singular locus of semisimple Hessenberg varieties". *Journal of Algebra* **521** (2019), pp. 65–96. DOI.

[11] F. de Mari, C. Procesi, and M. A. Shayman. "Hessenberg Varieties". *Transactions of the American Mathematical Society* **332**.2 (1992), pp. 529–534. Link.

[12] F. de Mari and M. A. Shayman. "Generalized Eulerian numbers and the topology of the Hessenberg variety of a matrix". *Acta Applicandae Mathematica* **12**.3 (1988), pp. 213–235. DOI.

[13] K. Salois. "Higher Specht Polynomials and Tableaux Bijections for Hessenberg Varieties". Ph.D. thesis, in preparation.

[14] J. Shareshian and M. L. Wachs. "Chromatic quasisymmetric functions". *Advances in Mathematics* **295** (2016), pp. 497–551. DOI.

[15] R. P. Stanley. "A Symmetric Function Generalization of the Chromatic Polynomial of a Graph". *Advances in Mathematics* **111**.1 (1995), pp. 166–194. DOI.

[16] R. P. Stanley and J. R. Stembridge. "On immanants of Jacobi-Trudi matrices and permutations with restricted position". *Journal of Combinatorial Theory, Series A* **62**.2 (1993), pp. 261–279. DOI.

[17] J. S. Tymoczko. "Linear Conditions Imposed on Flag Varieties". *American Journal of Mathematics* **128**.6 (2006), pp. 1587–1604. Link.

[18] J. S. Tymoczko. "Permutation actions on equivariant cohomology of flag varieties". *Toric topology*. Vol. 460. Contemp. Math. Amer. Math. Soc., Providence, RI, 2008, pp. 365–384. DOI.