Séminaire Lotharingien de Combinatoire **91B** (2024)
Article #43, 12 pp.

Proceedings of the 36th Conference on Formal Power Series and Algebraic Combinatorics (Bochum)

On the f-vectors of poset associahedra

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Abstract. For any finite connected poset P, Galashin introduced a simple convex $(|P|-2)$-dimensional polytope $\mathscr{A}(P)$ called the poset associahedron. First, we show that the f-vector of $\mathscr{A}(P)$ only depends on the comparability graph of P. Additionally, for a family of posets called broom posets, whose poset associahedra interpolate between permutohedra and associahedra, we give a simple combinatorial interpretation of the h-vector. The interpretation relates to the theory of stack-sorting and allows us to prove the real-rootedness of some of their h-polynomials.

Keywords: poset associahedra, stack-sorting, real-rootedness

1 Introduction

For a finite connected poset P, Galashin introduced the *poset associahedron* $\mathscr{A}(P)$ (see [4]). The faces of $\mathscr{A}(P)$ correspond to *tubings* of P, and the vertices of $\mathscr{A}(P)$ correspond to *maximal tubings* of P; see Section 2.2 for the definitions. $\mathscr{A}(P)$ can also be described as a compactification of the configuration space of order-preserving maps $P \to \mathbb{R}$.

The *comparability graph* of a poset P is a graph $\mathscr{C}(P)$ whose vertices are the elements of P and where i and j are connected by an edge if i and j are comparable. A property of P is said to be *comparability invariant* if it only depends on $\mathscr{C}(P)$. Properties of finite posets known to be comparability invariant include the order polynomial and number of linear extensions [10], the fixed point property [3], and the Dushnik–Miller dimension [11]. Our first main result is the following.

Theorem 3.6. *The f-vector of $\mathscr{A}(P)$ is a comparability invariant.*

In our study of the f-vectors of poset associahedra, we also consider a rich class of examples whose poset associahedra interpolate between associahedra and permutohedra. A *broom poset* is a poset of the form $A_{n,k} := C_{n+1} \oplus A_k$ where C_n is a chain of n elements,

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†andrewsack@math.ucla.edu. This material is based upon work supported by the National Science Foundation Graduate Research Fellowship Program under Grant No. DGE-2034835 and National Science Foundation Grants No. DMS-1954121 and DMS-2046915. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

A_k is an antichain of k elements, and $\oplus$ denotes ordinal sum. In particular, $A_{0,k}$ is a *claw poset* where $\mathscr{A}(A_{0,k})$ is a permutohedron, and $A_{n,0}$ is a chain where $\mathscr{A}(A_{n,0})$ is an associahedron. Our second main result is to give a combinatorial interpretation of the h-vector of $A_{n,k}$, giving a common interpretation for both permutohedra and associahedra. Our interpretation involves the theory of stack-sorting.

West's stack-sorting map is a function $s : \mathfrak{S}_n \to \mathfrak{S}_n$ which attempts to sort the permutations w in $\mathfrak{S}_n$ in linear time, not always sorting them completely (see Definition 4.1). It is well-known that for the associahedron, h_i counts the number of permutations in $s^{-1}(1\ldots n)$ with exactly i descents. We give a generalization of this result for all broom poset associahedra. Define

$$\mathfrak{S}_{n,k} := \{w \mid w \in \mathfrak{S}_{n+k}, w_i = i \text{ for all } i > k\}.$$

We prove the following:

Theorem 4.2. *Let $h = (h_0, h_1, \ldots, h_{n+k-1})$ be the h-vector of $\mathscr{A}(A_{n,k})$. Then h_i counts the number of permutations in $s^{-1}(\mathfrak{S}_{n,k})$ with exactly i descents.*

An immediate corollary of Theorem 4.2 is γ-nonnegativity of $\mathscr{A}(A_{n,k})$. In particular, we recall the following result of Bränden.

Theorem 4.4 ([2])**.** *For $A \subseteq \mathfrak{S}_n$, we have*

$$\sum_{\sigma \in s^{-1}(A)} t^{\operatorname{des}(\sigma)} = \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{|\{\sigma \in s^{-1}(A) \ : \ \operatorname{peak}(\sigma) = m\}|}{2^{n-1-2m}} t^m (1+t)^{n-1-2m},$$

where $\operatorname{peak}(\sigma)$ *is the number of index i such that $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$.*

Thus, we have the following corollary.

Corollary 4.5. *The γ-vector of $\mathscr{A}(A_{n,k})$ is nonnegative.*

In addition, in the process of proving Theorem 4.2, we find the size of $s^{-1}(\mathfrak{S}_{n,k})$ in terms of $k!$ and the Catalan convolution $C_n^{(k)}$, which will be introduced in Section 4.2.

Corollary 4.3. *For all $n, k \geq 0$, we have*

$$|s^{-1}(\mathfrak{S}_{n,k})| = k! \cdot C_n^{(k)}.$$

Finally, in Section 4.4, we prove the following strengthening of Corollary 4.5:

Theorem 4.10. *Let $H_n(t)$ be the h-polynomial of $\mathscr{A}(A_{n,2})$. Then, $H_n(t)$ is real-rooted.*

This paper is an extended abstract to [5] and [6].

2 Background

2.1 Face numbers

For a d-dimensional polytope P, the sequence $(f_0(P), \ldots, f_d(P))$ is called the *f-vector* of P, where $f_i(P)$ is the number of i-dimensional faces of P and

$$f_P(t) = \sum_{i=0}^{d} f_i(P) t^i$$

is called the *f-polynomial* of P. When P is *simple*, recall that the h-polynomial and γ-polynomial are defined by

$$f_P(t) = h_P(t+1),$$
$$h_P(t) = (1+t)^d \gamma\left(\frac{t}{(1+t)^2}\right).$$

2.2 Poset associahedra

We recall the following definitions.

Definition 2.1. Let $(P, \preceq)$ be a finite poset and let $\sigma, \tau \subseteq P$.

- τ is *connected* if it is connected as an induced subgraph of the Hasse diagram of P.
- τ is *convex* if whenever $x, z \in \tau$ and $y \in P$ such that $x \preceq y \preceq z$, then $y \in \tau$.
- τ is a *tube* of P if it is connected, convex, and $|\tau| > 1$. We say τ is a *proper tube* if additionally $|\tau| < |P|$.
- τ and σ are *nested* if $\tau \subseteq \sigma$ or $\sigma \subseteq \tau$ and they are *disjoint* if $\tau \cap \sigma = \emptyset$.
- We say $\sigma \prec \tau$ if $\sigma \cap \tau = \emptyset$ and there exists $x \in \sigma$ and $y \in \tau$ such that $x \preceq y$.
- A *tubing* T of P is a set of proper tubes such that any pair of tubes in T is either nested or disjoint and there is no subset $\{\tau_1, \tau_2, \ldots, \tau_k\} \subseteq T$ such that $\tau_1 \prec \tau_2 \prec \ldots \prec \tau_k \prec \tau_1$.
- A tubing T is *maximal* if it is maximal under inclusion, i.e. T is not a proper subset of any other tubing.

Definition 2.2 ([4, Theorem 1.2])**.** For a finite, connected poset P, there exists a simple, convex polytope $\mathscr{A}(P)$ of dimension $|P| - 2$ whose face lattice is isomorphic to the set of tubings ordered by reverse inclusion. The faces of $\mathscr{A}(P)$ correspond to tubings of P, and the vertices of $\mathscr{A}(P)$ correspond to maximal tubings of P. This polytope is called the *poset associahedron* of P.

3 Comparability invariance

The *comparability graph* of a poset P is the graph $\mathscr{C}(P)$ whose vertices are the elements of P and where i and j are connected by an edge if i and j are comparable. A property of a poset is said to be *comparability invariant* if it only depends on $\mathscr{C}(P)$. In [3], Dreesen, Poguntke, and Winkler give a powerful characterization of comparability invariance which we recall in this section.

Definition 3.1. Let P and S be posets and let $a \in P$. The *substitution* of a for S is the poset $P(a \to S)$ on the set $(P - \{a\}) \sqcup S$ formed by replacing a with S.

More formally, $x \preceq_{P(a\to S)} y$ if and only if one of the following holds:

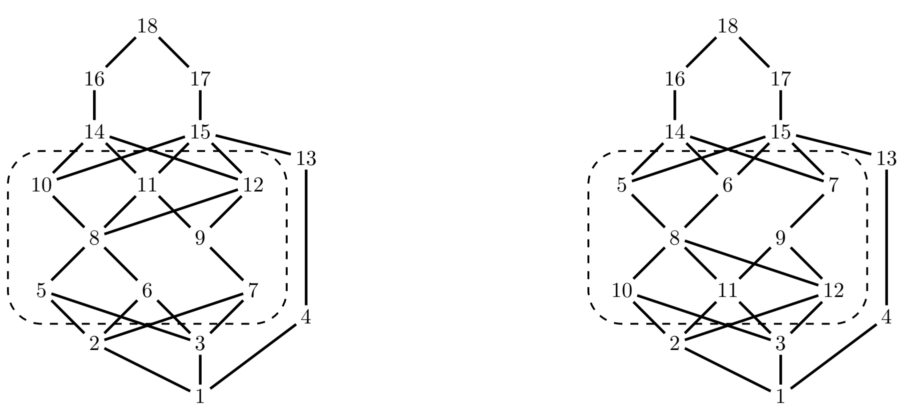
- $x, y \in P - \{a\}$ and $x \preceq_P y$;
- $x, y \in S$ and $x \preceq_S y$;
- $x \in S, y \in P - \{a\}$ and $a \preceq_P y$;
- $y \in S, x \in P - \{a\}$ and $y \preceq_P a$.

Definition 3.2. Let P be a poset and let $S \subseteq P$. S is called *autonomous* if there exists a poset Q and $a \in Q$ such that $P = Q(a \to S)$.

Equivalently, S is autonomous if for all $x, y \in S$ and $z \in P - S$, we have

$$(x \preceq z \Leftrightarrow y \preceq z) \text{ and } (z \preceq x \Leftrightarrow z \preceq y).$$

Definition 3.3. For a poset S, the *dual poset* S^{op} is defined on the same ground set where $x \preceq_S y$ if and only if $y \preceq_{S^{\mathrm{op}}} x$. A *flip* of S in $P = Q(a \to S)$ is the replacement of P by $Q(a \to S^{\mathrm{op}})$.



(a) An autonomous subset S of a poset P.

(b) A flip of S.

Figure 1

See Figure 1a for an example of an autonomous subset and Figure 1b for an example of a flip.

Lemma 3.4 ([3, Theorem 1])**.** *If P and P' are finite posets such that $\mathscr{C}(P) = \mathscr{C}(P')$ then P and P' are connected by a sequence of flips of autonomous subsets.*

Our main technical lemma is the following.

Lemma 3.5. *Let P be a poset and let $S \subseteq P$ be autonomous, and let P' be the poset obtained by flipping S in P. Then $\mathscr{A}(P)$ and $\mathscr{A}(P')$ have the same f-vector.*

Lemma 3.5 immediately gives our first theorem.

Theorem 3.6. *The f-vector of $\mathscr{A}(P)$ is a comparability invariant.*

Theorem 3.6 may lead one to ask if $C(P) \simeq C(P')$, then are $\mathscr{A}(P)$ and $\mathscr{A}(P')$ necessarily combinatorially equivalent? We answer this in the negative with the following example:

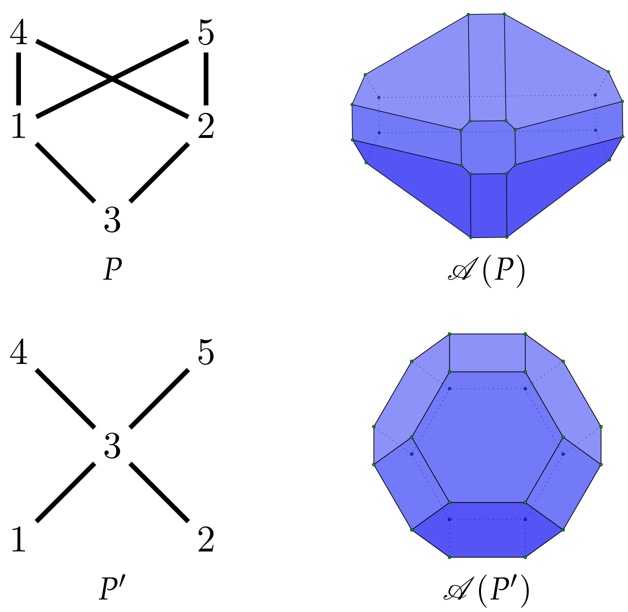


Figure 2: $\mathscr{A}(P)$ has an octagonal face, but $\mathscr{A}(P')$ does not.

3.1 Proof sketch of Lemma 3.5

Let $P = Q(a \to S)$ and $P' = Q(a \to S^{\text{op}})$. By an abuse of notation, we let $\mathscr{A}(P)$ also refer to the set of tubings of P. Our goal is to build a bijection $\Phi_{P,S} : \mathscr{A}(P) \to \mathscr{A}(P')$ such that for any tubing $T \in \mathscr{A}(P)$, we have $|T| = |\Phi_{P,S}(T)|$. Let $T \in \mathscr{A}(P)$. We will describe how to construct $T' := \Phi_{P,S}(T)$.

Definition 3.7. A tube $\tau \in T$ is *good* if $\tau \subseteq P - S$, $\tau \subseteq S$, or $S \subseteq \tau$ and is *bad* otherwise. We denote the set of good tubes by T_{good} and the set of bad tubes by T_{bad}.

The key idea of defining $\Phi_{P,S}$ is to decompose T_{bad} into a triple $(\mathcal{L}, \mathcal{M}, \mathcal{U})$ where $\mathcal{L}$ and $\mathcal{U}$ are nested sequences of sets, some of which may be marked, contained in $P - S$ and M is an ordered set partition of S. We build the decomposition in such a way so that we can uniquely recover T_{bad} from $(\mathcal{L}, \mathcal{M}, \mathcal{U})$. Then, we construct T' by keeping T_{good} and replacing T_{bad} by T'_{bad}, which is obtained from $(\mathcal{L}, \overline{\mathcal{M}}, \mathcal{U})$ where $\overline{\mathcal{M}}$ is the reverse of $\mathcal{M}$. We decompose T_{bad} as follows.

Definition 3.8. A tube $\tau \in T_{\text{bad}}$ is called *lower* (resp. *upper*) if there exist $x \in \tau - S$ and $y \in \tau \cap S$ such that $x \preceq y$ (resp. $y \preceq x$). We denote the set of lower tubes by T_L and the set of upper tubes by T_U.

Lemma 3.9 (Structure Lemma)**.** *T_{bad} is the disjoint union of T_L and T_U. Furthermore, T_L and T_U each form a nested sequence.*

Definition 3.10 (Tubing decomposition)**.** Let $T_L = \{\tau_1, \tau_2, \ldots\}$ where $\tau_i \subset \tau_{i+1}$ for all i. For convenience, we define $\tau_0 = \emptyset$. We define a nested sequence $\mathcal{L} = (L_1, L_2, \ldots)$ and a sequence of disjoint sets $\mathcal{M}_L = (M_L^1, M_L^2, \ldots)$ as follows.

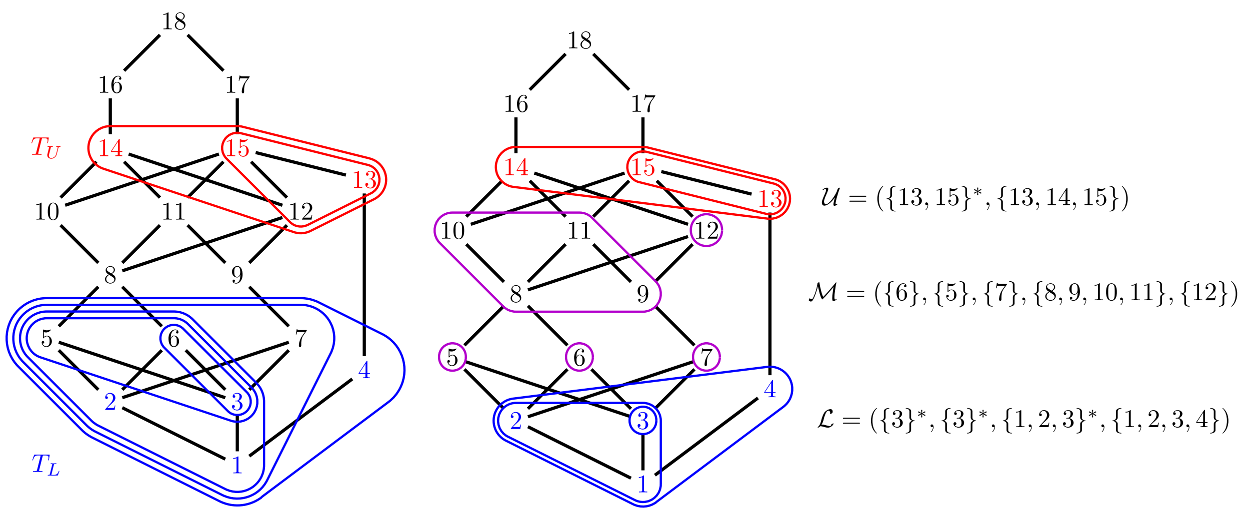
- For each $i \geq 1$, let $L_i = \tau_i - S$, and mark L_i with a star if $(\tau_i - \tau_{i-1}) \cap S \neq \emptyset$.
- If L_i is the j-th starred set, let $M_L^j = (\tau_i - \tau_{i-1}) \cap S$.

We define the sequences $\mathcal{U}$ and $\mathcal{M}_U$ analogously. We make the following definitions.

- Let $\hat{M} := S - \bigcup_{\tau \in T_{\text{bad}}} \tau$.
- For sequences $\mathbf{a}$ and $\mathbf{b}$, let the sequence $\mathbf{a} \cdot \mathbf{b}$ be $\mathbf{b}$ appended to $\mathbf{a}$, and let $\overline{\mathbf{a}}$ be the reverse of $\mathbf{a}$.
- We define
$$\mathcal{M} := \begin{cases} \mathcal{M}_L \cdot \overline{\mathcal{M}}_U & \text{if } \hat{M} = \emptyset \\ \mathcal{M}_L \cdot (\hat{M}) \cdot \overline{\mathcal{M}}_U & \text{if } \hat{M} \neq \emptyset \end{cases}$$
where $(\hat{M})$ is the sequence containing $\hat{M}$.
- The *decomposition* of T_{bad} is the triple $(\mathcal{L}, \mathcal{M}, \mathcal{U})$.

Figure 3 gives an example of a decomposition.

Lemma 3.11 (Reconstruction algorithm)**.** *T_{bad} can be reconstructed from its decomposition.*



(a) T_L is blue and T_U is red. (b) $\mathcal{L}$ is blue, $\mathcal{M}$ is purple, and $\mathcal{U}$ is red.

Figure 3: The decomposition of T_{bad}.

Proof. Let $\mathcal{M} = (M_1, \ldots, M_n)$. To reconstruct T_L, we set $\tau_1 = L_1 \cup M_1$ and take

$$\tau_i = \begin{cases} \tau_{i-1} \cup L_i & \text{if } L_i \text{ is not starred} \\ \tau_{i-1} \cup L_i \cup M_j & \text{if } L_i \text{ is marked with the } j\text{-th star.} \end{cases}$$

For T_U, we set $\tau_1 = U_1 \cup M_n$ and

$$\tau_i = \begin{cases} \tau_{i-1} \cup U_i & \text{if } U_i \text{ is not starred} \\ \tau_{i-1} \cup U_i \cup M_{n-j+1} & \text{if } U_i \text{ is marked with the } j\text{-th star.} \end{cases}$$

□

Lemma 3.12. *Applying the reconstruction algorithm to $(\mathcal{L}, \overline{\mathcal{M}}, \mathcal{U})$ yields a proper tubing T'_{bad} of P' with exactly $|T_{bad}|$ tubes.*

We define $T' := T'_{\text{bad}} \sqcup T_{\text{good}}$ and take $\Phi_{P,S}(T) := T'$.

Lemma 3.13. *T' is a proper tubing of P'. Furthermore, $\Phi_{P',S}(T') = T$ and $|\Phi_{P,S}(T)| = |T|$.*

4 Broom posets

Recall that the *ordinal sum* of two posets $(P, <_P)$ and $(Q, <_Q)$ is the poset $(R, <_R)$ whose elements are those in $P \cup Q$, and $a \leq_R b$ if and only if

- $a, b \in P$ and $a \leq_P b$ or

- $a, b \in Q$ and $a \leq_Q b$ or
- $a \in P$ and $b \in Q$.

We denote the ordinal sum of P and Q as $P \oplus Q$. Let C_n be the chain poset of size n and A_k be the antichain of size k. In this section, we study the *broom posets* $A_{n,k} = C_{n+1} \oplus A_k$. In particular, $A_{n,0}$ is the chain poset C_{n+1}, and $A_{0,k}$ is the claw poset $C_1 \oplus A_k$. Recall that $\mathscr{A}(A_{n,0})$ is the associahedron and $\mathscr{A}(A_{0,k})$ is the permutohedron. We show that the h-vectors of broom posets have a simple combinatorial interpretation in terms of descents of stack-sorting preimages.

4.1 Stack-sorting

In [12], West defined a deterministic version of Knuth's stack-sorting algorithm, which we call the *stack-sorting map* and denote by s. The stack-sorting map is defined as follows.

Definition 4.1 (Stack-sorting)**.** Given a permutation $\pi \in \mathfrak{S}_n$, $s(\pi)$ is obtained through the following procedure. Iterate through the entries of π. In each iteration,

- if the stack is empty or the next entry is smaller than the entry at the top of the stack, push the next entry to the top of the stack;
- otherwise, pop the entry at the top of the stack to the end of the output permutation.

Figure 4 illustrates the stack-sorting process on $\pi = 3142$.

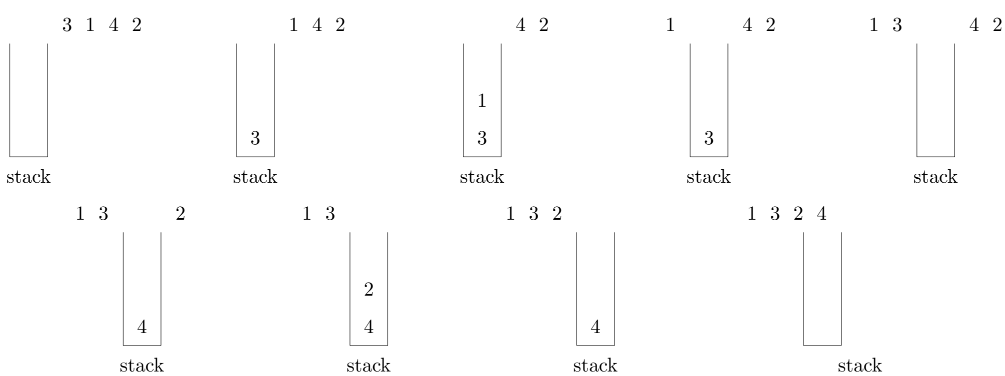


Figure 4: Example of $s(3142)$

4.2 Catalan convolution

Recall that the *Catalan numbers* $C_n = \frac{1}{n+1}\binom{2n}{n}$ have generating function $\mathcal{C}(t) = \frac{1-\sqrt{1-4t}}{2t}$. The *k-th Catalan convolution* is the sequence with generating function $\mathcal{C}(t)^k$. For convenience, we denote $[t^n]\mathcal{C}(t)^k$ by $C_n^{(k)}$.

The explicit formula for $C_n^{(k)}$ is

$$C_n^{(k)} = \frac{k+1}{n+k+1}\binom{2n+k}{n}.$$

4.3 h-vector

Recall that we defined $\mathfrak{S}_{n,k} = \{w \mid w \in \mathfrak{S}_{n+k}, w_i = i \text{ for all } i > k\}$. In this section, our main theorem is:

Theorem 4.2. *Let $h = (h_0, h_1, \ldots, h_{n+k-1})$ be the h-vector of $\mathscr{A}(A_{n,k})$. Then h_i counts the number of permutations in $s^{-1}(\mathfrak{S}_{n,k})$ with exactly i descents.*

As a corollary, we obtain the following result.

Corollary 4.3. *For all $n, k \geq 0$, we have*

$$|s^{-1}(\mathfrak{S}_{n,k})| = k! \cdot C_n^{(k)}.$$

Recall also the following result by Brändén.

Theorem 4.4 ([2])**.** *For $A \subseteq \mathfrak{S}_n$, we have*

$$\sum_{\sigma \in s^{-1}(A)} t^{\operatorname{des}(\sigma)} = \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{|\{\sigma \in s^{-1}(A) \ : \ \operatorname{peak}(\sigma) = m\}|}{2^{n-1-2m}} t^m (1+t)^{n-1-2m},$$

where $\operatorname{peak}(\sigma)$ *is the number of index i such that $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$.*

This gives the following corollary.

Corollary 4.5. *The γ-vector of $\mathscr{A}(A_{n,k})$ is nonnegative.*

Remark 4.6. Corollary 4.5 also follows from the fact that $\mathscr{A}(A_{n,k})$ is isomorphic to the graph associahedra of lollipop graphs, which are chordal. It was shown in [7] that graph associahedra of chordal graphs are γ-nonnegative.

4.4 Real-rootedness

In this section, we will sketch the proof of real-rootedness of the h-polynomial of $\mathscr{A}(A_{n,2})$. We say a polynomial $a_0 + a_1 t + \ldots + a_n t^n$ is real-rooted if all of its zeros are real. We say a sequence $(a_0, a_1, \ldots, a_n)$ is real-rooted if the polynomial $a_0 + a_1 t + \ldots + a_n t^n$ is real-rooted.

Let f and g be real-rooted polynomials with positive leading coefficients and real roots $\{f_i\}$ and $\{g_i\}$, respectively. We say that f *interlaces* g if

$$g_1 \leq f_1 \leq g_2 \leq f_2 \leq \ldots \leq f_{d-1} \leq g_d$$

where $d = \deg g = \deg f + 1$. We say that f *alternates left of* g if

$$f_1 \leq g_1 \leq f_2 \leq g_2 \leq \ldots \leq f_d \leq g_d$$

where $d = \deg g = \deg f$. Finally, we say f *interleaves* g, denoted $f \ll g$, if f either interlaces or alternates left of g.

Recall that the Narayana polynomial $N_n(t)$ is defined by

$$N_n(t) = \sum_{i=0}^{n-1} a_i t^i$$

where a_i counts the number of permutations in $s^{-1}(1 \ldots n)$ with exactly i descents. In other words, $N_n(t)$ is the h-polynomial of $\mathscr{A}(A_{n,0})$ and $\mathscr{A}(A_{n-1,1})$. We have the following result.

Theorem 4.7 ([1]). *For all n, $N_n(t)$ is real-rooted. Furthermore, $N_{n-1}(t) \ll N_n(t)$.*

To prove real-rootedness of the h-polynomial of $\mathscr{A}(A_{n,2})$, we will need the following "happy coincidence".

Proposition 4.8. *The number of permutations in $s^{-1}(2134 \ldots n)$ with exactly i descents is the same as the number of permutations w in $s^{-1}(1 \ldots n)$ with exactly i descents such that $w_1, w_n < n$.*

Proposition 4.8 leads to the following important recurrence.

Proposition 4.9. *Let $H_n(t)$ be the h-polynomial of $\mathscr{A}(A_{n,2})$, and recall that $N_{n+2}(t)$ and $N_{n+1}(t)$ are the h-polynomials of $\mathscr{A}(A_{n+2,0})$ and $\mathscr{A}(A_{n+1,0})$, respectively. We have*

$$H_n(t) = 2N_{n+2}(t) - (1+t)N_{n+1}(t).$$

This recurrence and Theorem 4.7 allows us to prove the following theorem.

Theorem 4.10. *Let $H_n(t)$ be the h-polynomial of $\mathscr{A}(A_{n,2})$. Then, $H_n(t)$ is real-rooted.*

5 Open Questions

Question 5.1. Can we define $f_{\mathscr{A}(P)}(z)$ purely in terms of $C(P)$? It would also be interesting to answer this question even for f_0.

Question 5.2. It remains open to find an interpretation of the h-vector of $\mathscr{A}(P)$ in terms of the combinatorics of P. Can $h(z)$ be defined purely in terms of $C(P)$?

Question 5.3. The map $\Phi_{P,S}$ can be analogously defined for *affine poset cyclohedra* [4], where an autonomous subset S has at most one representative from each residue class. Again, it preserves the f-vector of the affine poset cyclohedron. Does Lemma 3.4 (and hence Theorem 3.6) hold for affine posets?

We have the following conjectured generalization of Proposition 4.9.

Conjecture 5.4. *Let P be a poset with an autonomous subposet S that is a chain of size 2, i.e. $S = C_2$. Let P_1 be the poset obtained from P by replacing S by a singleton. Let P_2 be the poset obtained from P by replacing S by an antichain of size 2, i.e. A_2. Let $h_P(t)$, $h_{P_1}(t)$, $h_{P_2}(t)$ be the h-polynomials of $\mathscr{A}(P)$, $\mathscr{A}(P_1)$, $\mathscr{A}(P_2)$, respectively. Then,*

$$2h_P(t) = h_{P_2}(t) + (1+t)h_{P_1}(t).$$

As a result, let $\gamma_P(t)$, $\gamma_{P_1}(t)$, $\gamma_{P_2}(t)$ be the γ-polynomials of $\mathscr{A}(P)$, $\mathscr{A}(P_1)$, $\mathscr{A}(P_2)$, respectively. Then,

$$2\gamma_P(t) = \gamma_{P_2}(t) + \gamma_{P_1}(t).$$

Conjecture 5.4 is useful in proving real-rootedness of the h-polynomials, as shown in Theorem 4.10. Furthermore, the resulting recurrence of the γ-polynomial would also be useful in proving γ-positivity. More generally, we have the following recurrence when S is an antichain of size n.

Conjecture 5.5. *Let P be a poset with an autonomous subposet S that is a chain of size n, i.e. $S = C_n$. For $1 \le i \le n$, let P_i be the poset obtained from P by replacing S by an antichain of size i, i.e. A_i. Let $h_P(t)$, $h_{P_1}(t)$, $\ldots$, $h_{P_n}(t)$ be the h-polynomials of $\mathscr{A}(P)$, $\mathscr{A}(P_1)$, $\ldots$, $\mathscr{A}(P_n)$, respectively. Then,*

$$h_P(t) = \frac{1}{n!}\sum_{w\in\mathfrak{S}_n} B_1(t)^{c_1(w)}\ldots B_n(t)^{c_n(w)} h_{P_{\ell(\lambda(w))}}(t) \tag{5.1}$$

where

$$B_k(t) = \sum_{i=0}^{k-1}\binom{k-1}{i}^2 t^i$$

are type B Narayana polynomials, $c_i(w)$ is the number of cycles of size i in w, and $\ell(\lambda(w))$ is the length of the cycle type $\lambda(w)$ of w.

The type B Narayana polynomials above also show up as the rank-generating function of the type B analogue NC_n^B of the lattice of non-crossing partitions (see [8]) and the h-polynomials of type B associahedra (see [9]).

Equation 5.1 bears resemblance to the Frobenius characteristic map. Thus, it is a natural question to ask if there is a representation theory story behind this equation. This is an interesting question for future research.

Acknowledgements

The authors are grateful to Vic Reiner and Pavel Galashin for their guidance. We also thank Colin Defant for helpful discussions about stack-sorting.

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