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Monomial expansions for q-Whittaker and modified Hall-Littlewood polynomials

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Abstract. We consider the monomial expansion of the q-Whittaker polynomials given by the fermionic formula and via the *inv* and *quinv* statistics. We construct bijections between the parametrizing sets of these three models which preserve the x- and q-weights, and which are compatible with natural projection and branching maps. We apply this to the limit construction of local Weyl modules and obtain a new character formula for the basic representation of $\widehat{\mathfrak{sl}_n}$. Finally, we indicate how our main results generalize to the modified Hall-Littlewood case.

Keywords: q-Whittaker polynomial, modified Hall-Littlewood polynomial, local Weyl modules

1 Introduction

Let λ be a partition. For $n \geq 1$, let X_n denote the tuple of indeterminates $x_1, x_2, \cdots, x_n$. The q-Whittaker polynomial $W_\lambda(X_n; q)$ and the modified Hall-Littlewood polynomial $Q'_\lambda(X_n; q)$ are well-studied specializations of the modified Macdonald polynomial. Several different monomial expansions for these polynomials are known. In this article, our focus will be on three of these: the so-called *fermionic formulas* [13, (0.2), (0.3)] and the inv- and quinv-expansions arising from specializations of the formulas of Haglund-Haiman-Loehr [9] and Ayyer-Mandelshtam-Martin [1].

We recall that the Schur expansion of the $W_\lambda(X_n; q)$ (resp. $Q'_\lambda(X_n; q)$) has certain q-Kostka polynomials as coefficients [13]. In turn, this implies yet another monomial expansion, with the underlying indexing set involving pairs of semistandard Young tableaux of conjugate (resp. equal) shapes. This relates to the inv-expansion via the RSK correspondence [9].

The fermionic formula, expressed as a sum of products of q-binomials, is seemingly of a very different nature from all the other monomial expansions, and should probably viewed as a kind of compression of these formulas. Recently, Garbali-Wheeler [8]

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obtained a general formula of the fermionic kind for the full modified Macdonald polynomial $\widetilde{H}_\lambda(X_n;q,t)$.

The purpose of this article is to bijectively reconcile the fermionic formula with both the inv- and quinv-expansions. We construct bijections between the underlying sets of these three models which (i) preserve the x- and q-weights, and (ii) are compatible with natural projection and branching maps.

As a corollary, we obtain bijections between the inv- and quinv-models in the q-Whittaker and modified Hall-Littlewood specializations, partially answering a question of [1]. We find that the *inv*- and *quinv*-models are related by the simple *box-complementation* map of the fermionic model and that $inv + quinv$ is a constant on fibers of the natural projection. We also apply this to the limit construction for Weyl modules [7, 15] and obtain an apparently new character formula for the basic representation of the affine Lie algeba $\widehat{\mathfrak{sl}_n}$.

In this extended abstract, we describe the q-Whittaker polynomials in greater detail, contenting ourselves with brief remarks about the modified Hall-Littlewood case in §8 due to space limitations. Complete proofs will appear in [3].

2 Specializations of $\widetilde{H}_\lambda(X_n;q,t)$

Given a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$, we will draw its Young diagram $\mathrm{dg}(\lambda)$ following the English convention, as a left-up justified array of boxes, with λ_i boxes in the ith row from the top. The boxes are called the cells of $\mathrm{dg}(\lambda)$. We let $|\lambda| := \sum_i \lambda_i$. Fix $n \geq 1$ and let $\mathcal{F}(\lambda)$ denote the set of all maps ("fillings") $F : \mathrm{dg}(\lambda) \to [n]$ where $[n] = \{1, 2, \cdots, n\}$. If the values of F strictly increase (resp. weakly decrease) as we move down a column, we say F is a *column strict filling* (CSF) (resp. *weakly decreasing filling* (WDF)[1]), and denote the set of such fillings by $\mathrm{CSF}(\lambda)$ (resp. $\mathrm{WDF}(\lambda)$). The x-weight of a filling F is the monomial $x^F := \prod_{c \in \mathrm{dg}(\lambda)} x_{F(c)}$.

We recall that the modified Macdonald polynomial $\widetilde{H}_\lambda(X_n;q,t)$ is a symmetric polynomial in the x_i with $\mathbb{N}[q,t]$ coefficients. We expand this in powers of t; our interest lies in the coefficients of the lowest and highest powers [2, (3.1)]:

$$\widetilde{H}_\lambda(X_n;q,t) = \mathcal{H}_\lambda(X_n;q)t^0 + \cdots + W_\lambda(X_n;q)t^{\eta(\lambda)} \tag{2.1}$$

where $\eta(\lambda) = \sum_{j\geq 1} \binom{\lambda'_j}{2}$ where λ'_j denote the parts of the partition conjugate to λ. The $W_\lambda(X_n;q)$ is the q-Whittaker polynomial. The q-reversal (or reciprocal) polynomial of $\mathcal{H}_\lambda(X_n;q)$ coincides with the modified Hall-Littlewood polynomial $Q'_{\lambda'}(X_n;q)$ where λ'

[1] These latter ones may be easily transformed into the familiar *tabloids* by transposing rows and columns and replacing $i \mapsto n - i + 1$

is the partition conjugate to λ, i.e., $q^{\eta(\lambda')}\mathcal{H}_\lambda(X_n;q^{-1}) = Q'_{\lambda'}(X_n;q)$. These are further related to each other by $\omega W_\lambda(X_n;q) = Q'_{\lambda'}(X_n;q)$ where ω is the classical involution on the ring of symmetric polynomials.

Following Haglund-Haiman-Loehr [9] and Ayyer-Mandelshtam-Martin [1], there are statistics *inv, quinv* and *maj* on $\mathcal{F}(\lambda)$ such that

$$\widetilde{H}_\lambda(X_n;q,t) = \sum_{F\in\mathcal{F}(\lambda)} x^F q^{v(F)} t^{\mathrm{maj}(F)} \tag{2.2}$$

where $v \in \{\mathrm{inv}, \mathrm{quinv}\}$. The next lemma follows directly from the definition of maj [9]:

Lemma 1. *Let $F \in \mathcal{F}(\lambda)$. Then (i) $\mathrm{maj}(F) = \eta(\lambda)$ iff $F \in \mathrm{CSF}(\lambda)$, and (ii) $\mathrm{maj}(F) = 0$ iff $F \in \mathrm{WDF}(\lambda)$.*

Putting together (2.1), (2.2) and Lemma 1, we obtain for $v \in \{\mathrm{inv}, \mathrm{quinv}\}$:

$$W_\lambda(X_n;q) = \sum_{F\in\mathrm{CSF}(\lambda)} x^F q^{v(F)} \tag{2.3}$$

$$Q'_{\lambda'}(X_n;q) = \sum_{F\in\mathrm{WDF}(\lambda)} x^F q^{\eta(\lambda')-v(F)} \tag{2.4}$$

These are in fact symmetric in the x-variables and can be viewed as expansions in terms of the monomial symmetric functions in $x_1, x_2, \cdots, x_n$.

3 Fermionic formula for $W_\lambda(X_n;q)$

Let $n \geq 1$ and $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$ be a partition with at most n nonzero parts. Let $\mathrm{GT}(\lambda)$ denote the set of integral Gelfand-Tsetlin (GT) patterns with bounding row λ. Given $T \in \mathrm{GT}(\lambda)$, we denote its entries by T_i^j for $1 \leq i \leq j \leq n$ as in Figure 1. It will also be convenient to define $T_{j+1}^j = 0$ for all $1 \leq j \leq n$. We define the North-East and South-East differences of T by: $\mathrm{NE}_{ij}(T) = T_i^{j+1} - T_i^j$ and $\mathrm{SE}_{ij}(T) = T_i^j - T_{i+1}^{j+1}$ for $1 \leq i \leq (j+1) \leq n$. The GT inequalities ensure that these differences are non-negative.

We will interchangeably think of a GT pattern as a semistandard Young tableau (SSYT). In this perspective, $(T_1^j, T_2^j, \cdots, T_j^j)$ is the partition formed by the cells of the tableau which contain entries $\leq j$. It follows that $\mathrm{NE}_{ij}(T)$ is the number of cells in the i^{th} row of the tableau which contain the entry $j+1$. We let x^T denote the x-weight of the corresponding tableau. The following fermionic formula for the q-Whittaker polynomial appears in [10, 13] and follows readily from Macdonald's more general formula [14, Chap VI, (7.13)']:

$$W_\lambda(X_n;q) = \sum_{T\in\mathrm{GT}(\lambda)} x^T \prod_{1\leq i\leq j<n} \begin{bmatrix} NE_{ij}(T) + SE_{ij}(T) \\ NE_{ij}(T) \end{bmatrix}_q \tag{3.1}$$

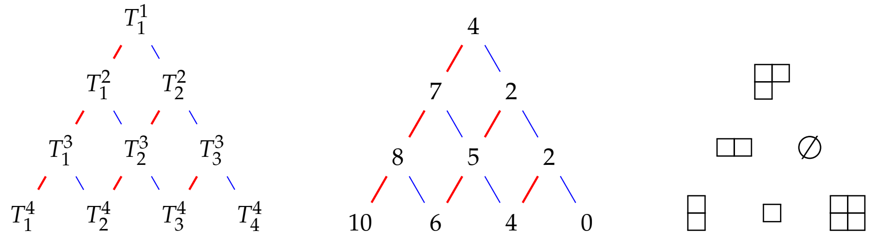


Figure 1: A GT pattern for $n = 4$. The NE and SE differences are those along the red and blue lines. On the right is a partition overlay compatible with this GT pattern.

Following [12], we define $\mathrm{wt}_q(T) = \prod_{1 \leq i \leq j < n} \begin{bmatrix} NE_{ij}(T) + SE_{ij}(T) \\ NE_{ij}(T) \end{bmatrix}_q$.

3.1 Partition overlaid patterns

We recall that the q-binomial $\begin{bmatrix} k+\ell \\ k \end{bmatrix}_q$ is the generating function of partitions that fit into a $k \times \ell$ rectangle, i.e., $\begin{bmatrix} k+\ell \\ k \end{bmatrix}_q = \sum q^{|\gamma|}$ where $\gamma = (\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_k \geq 0)$ with $\ell \geq \gamma_1$. We also identify partitions of the above form with strictly decreasing k-tuples of integers between 0 and $k + \ell - 1$ via the bijection $\gamma \mapsto \overline{\gamma} = \gamma + \delta$ where $\delta = (k-1, k-2, \cdots, 0)$.

As shown in [15], the right-hand side of (3.1) can be interpreted in terms of the so-called *partition overlaid patterns* (POPs). A POP of shape λ is a pair (T, Λ) where $T \in \mathrm{GT}(\lambda)$ and $\Lambda = (\Lambda_{ij} : 1 \leq i \leq j < n)$ is a tuple of partitions such that each Λ_{ij} fits into a rectangle of size $NE_{ij}(T) \times SE_{ij}(T)$. For example, if T is the GT pattern of Figure 1, we could take $\Lambda_{11} = (2,1,0)$, $\Lambda_{12} = (2)$, $\Lambda_{13} = (1,1)$, $\Lambda_{22} = (0,0,0)$, $\Lambda_{23} = (1)$, $\Lambda_{33} = (2,2)$. We imagine the Λ_{ij} as being placed in a triangular array as in Figure 1. We let $\mathrm{POP}(\lambda)$ denote the set of POPs of shape λ. It is now clear from (3.1) that

$$W_\lambda(X_n; q) = \sum_{(T,\Lambda) \in \mathrm{POP}(\lambda)} x^T q^{|\Lambda|} \tag{3.2}$$

where $|\Lambda| = \sum_{i,j} |\Lambda_{ij}|$. We remark that $W_\lambda(X_n; q)$ is the character of the *local Weyl module* $W_{\mathrm{loc}}(\lambda)$ - a module for the current algebra $\mathfrak{sl}_n[t]$ [6, 5]. Further, POPs of shape λ index a special basis of this module with Gelfand-Tsetlin like properties [6, 15].

3.2 Projection and Branching for Partition overlaid patterns

Given $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$, we say that $\mu = (\mu_1, \mu_2, \cdots, \mu_{n-1})$ *interlaces* λ (and write $\mu \prec \lambda$) if $\lambda_i \geq \mu_i \geq \lambda_{i+1}$ for $1 \leq i < n$. The q-Whittaker polynomials have the following important properties which readily follow from (3.2):

(*projection*) $W_\lambda(X_n; q = 0) = s_\lambda(X_n)$, the Schur polynomial, and

$$(\textit{branching})\ W_\lambda(x_1, x_2, \cdots, x_{n-1}, x_n = 1; q) = \sum_{\mu \prec \lambda} \prod_{1 \leq i < n} \begin{bmatrix} \lambda_i - \lambda_{i+1} \\ \lambda_i - \mu_i \end{bmatrix}_q \cdot W_\mu(X_{n-1}; q) \tag{3.3}$$

In fact, Chari-Loktev [6] lift (3.3) to the level of modules, showing that the local Weyl module $W_{\text{loc}}(\lambda)$ when restricted to $\mathfrak{sl}_{n-1}[t]$ admits a filtration whose successive quotients are of the form $W_{\text{loc}}(\mu)$ for $\mu \prec \lambda$; further their graded multiplicities are precisely given by the product of q-binomial coefficients that appear in (3.3).

The combinatorial shadow of projection is the map $\text{pr} : \text{POP}(\lambda) \to \text{GT}(\lambda)$ given by $\text{pr}(T, \Lambda) = T$. Likewise, we define *combinatorial branching* to be the map $\text{br} : \text{POP}(\lambda) \to \bigsqcup_{\mu \prec \lambda} \text{POP}(\mu)$ defined by $\text{br}(T, \Lambda) = (T^\dagger, \Lambda^\dagger)$ where $T^\dagger$ is obtained from T by deleting its bottom row, and $\Lambda^\dagger$ is obtained from Λ by deleting the overlays Λ_{ij} with $j = n - 1$.

3.3 Box complementation

In addition to pr and br, $\text{POP}(\lambda)$ is endowed with another important map, which we term *box complementation*. Observe that given a partition $\pi = (\pi_1 \geq \pi_2 \geq \cdots \geq \pi_k \geq 0)$ fitting into a $k \times \ell$ rectangle, i.e., with $\pi_1 \leq \ell$, we may consider its complement in this rectangle, defined by $\pi^c = (\ell - \pi_k \geq \ell - \pi_{k-1} \geq \cdots \geq \ell - \pi_1)$. Now, for $(T, \Lambda) \in \text{POP}(\lambda)$, define $\text{boxcomp}(T, \Lambda) = (T, \Lambda^c)$ where for each i, j, $(\Lambda^c)_{ij}$ is defined to be the complement of Λ_{ij} in its bounding rectangle of size $\text{NE}_{ij}(T) \times \text{SE}_{ij}(T)$.

We note that since $|\Lambda| \neq |\Lambda^c|$ in general, boxcomp preserves x-weights, but not q-weights. However $|\Lambda| + |\Lambda^c| = \sum_{i,j} \text{NE}_{ij}(T)\,\text{SE}_{ij}(T) =: \text{area}(T)$ (in the terminology of [15]), which depends only on T.

4 Projection and branching for Column strict fillings

Our goal is to construct natural bijections between $\text{CSF}(\lambda)$ and $\text{POP}(\lambda)$ which explain the equality of (2.3) and (3.2) for $v = \text{inv}, \text{quinv}$. In addition to preserving x- and q-weights, we would like our bijections to be compatible with projection and branching. Towards this end, we first define these latter maps in the setting of $\text{CSF}(\lambda)$.

4.1 Projection: rowsort

Given $F \in \text{CSF}(\lambda)$, let $\text{rsort}(F)$ denote the filling obtained from F by sorting entries of each row in ascending order. In light of the following easy lemma, we think of rsort as the projection map in the CSF setting.

Lemma 2. *If* $F \in \text{CSF}(\lambda)$, *then* $\text{rsort}(F) \in \text{SSYT}(\lambda) \cong \text{GT}(\lambda)$.

4.2 Branching: delete-and-splice

A strictly increasing sequence $a = (a_1 < a_2 < \cdots < a_m)$ of positive integers will also be termed a *column tuple* with $\text{len}(a) = m \geq 0$. Let $\ell \geq 1$ and suppose $\sigma = (\sigma_1 < \sigma_2 <$

$\cdots < \sigma_{\ell-1})$ and $\tau = (\tau_1 < \tau_2 < \cdots < \tau_\ell)$ are column tuples of length $\ell - 1$ and ℓ respectively. We set $\sigma_0 = 0$ and let k denote the maximum element of the (non-empty) set $\{1 \le i \le \ell : \sigma_{i-1} < \tau_i\}$. Define $\operatorname{splice}(\sigma, \tau) = (\overline{\sigma}, \overline{\tau})$ where

$$\overline{\sigma}_i = \begin{cases} \sigma_i & 1 \le i < k \\ \tau_i & k \le i \le \ell \end{cases} \qquad \text{and} \qquad \overline{\tau}_i = \begin{cases} \tau_i & 1 \le i < k \\ \sigma_i & k \le i < \ell \end{cases}$$

i.e., $\overline{\sigma}, \overline{\tau}$ are obtained by swapping certain suffix portions of σ, τ. The choice of k ensures that $\overline{\sigma}, \overline{\tau}$ are also column tuples; we also have $\operatorname{len}(\overline{\sigma}) = \operatorname{len}(\tau)$ and $\operatorname{len}(\overline{\tau}) = \operatorname{len}(\sigma)$. For instance, when $(\sigma, \tau) = \left(\begin{array}{|c|}\hline 1 \\ \hline 5 \\ \hline \end{array}, \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} \right)$, we get $(\overline{\sigma}, \overline{\tau}) = \left(\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 4 \\ \hline \end{array}, \begin{array}{|c|}\hline 2 \\ \hline 5 \\ \hline \end{array} \right)$.

We now define the delete-and-splice rectification ("dsplice") map on $F \in \operatorname{CSF}(\lambda)$ as follows: **(1)** delete all cells in F containing the entry n and let $F^\dagger$ denote the resulting filling. While its column entries remain strictly increasing, $F^\dagger$ may no longer be of partition shape. **(2)** Let $\sigma^{(j)}$ $(j \ge 1)$ denote the column tuple obtained by reading the j^{th} column of $F^\dagger$ from top to bottom. If $F^\dagger$ is not of partition shape, there exists $j \ge 1$ such that $\operatorname{len}(\sigma^{(j+1)}) = \operatorname{len}(\sigma^{(j)}) + 1$. Choose any such j and modify $F^\dagger$ by replacing the pair of columns $(\sigma^{(j)}, \sigma^{(j+1)})$ in $F^\dagger$ by $\operatorname{splice}(\sigma^{(j)}, \sigma^{(j+1)})$. This swaps the column lengths and brings the shape of $F^\dagger$ one step closer to being a partition. **(3)** If the shape of $F^\dagger$ is a partition, STOP. Else go back to step 2.

It is clear that this process terminates and finally produces a CSF of partition shape (filled by numbers between 1 and $n - 1$), which we denote $\operatorname{dsplice}(F)$. The following properties hold:

Proposition 1. *With notation as above: (i) $D := \operatorname{dsplice}(F)$ is independent of the intermediate choices of j made in step 2 of the procedure. (ii) $\operatorname{rsort}(D)$ is obtained from $\operatorname{rsort}(F)$ by deleting the cells containing the entry n. (iii) If μ and λ are the shapes of D and F respectively, then $\mu \prec \lambda$.*

We consider dsplice to be the combinatorial branching map in the CSF context. Its key property is its compatibility with the natural branching map br of the POP setting.

5 The main theorem

Theorem 1. *For any $n \ge 1$ and any partition $\lambda : \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0$ with at most n nonzero parts, there exist two bijections ψ_{inv} and ψ_{quinv} from $\operatorname{CSF}(\lambda)$ to $\operatorname{POP}(\lambda)$ with the following properties:*

1. If $\psi_v(F) = (T, \Lambda)$, then $x^F = x^T$ and $v(F) = |\Lambda|$, for $v = \text{inv}$ or quinv.

2. The following diagrams commute ($v = \text{inv}$ or quinv):

(A)

$$\begin{array}{ccc} \mathrm{CSF}(\lambda) & \xrightarrow{\psi_v} & \mathrm{POP}(\lambda) \\ & {\scriptstyle \mathrm{rsort}} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}} & \\ & \mathrm{GT}(\lambda) & \end{array}$$

(B)

$$\begin{array}{ccc} \mathrm{CSF}(\lambda) & \xrightarrow{\psi_v} & \mathrm{POP}(\lambda) \\ {\scriptstyle \mathrm{dsplice}} \downarrow & & \downarrow {\scriptstyle \mathrm{br}} \\ \bigsqcup_{\mu \prec \lambda} \mathrm{CSF}(\mu) & \xrightarrow[\psi_v]{} & \bigsqcup_{\mu \prec \lambda} \mathrm{POP}(\mu) \end{array}$$

3. The two bijections are related via the commutative diagram:

$$\begin{array}{ccc} & \mathrm{CSF}(\lambda) & \\ & {\scriptstyle \psi_{\mathrm{inv}}} \swarrow \quad \searrow {\scriptstyle \psi_{\mathrm{quinv}}} & \\ \mathrm{POP}(\lambda) & \xrightarrow{\mathrm{boxcomp}} & \mathrm{POP}(\lambda) \end{array}$$

□

To summarize, ψ_{inv} and ψ_{quinv} acting on a CSF produce POPs with the same underlying GT pattern, but with complementary overlays. These bijections are compatible with the natural projection and branching maps, and preserve x- and appropriate q-weights (inv or quinv). Note the slight abuse of notation in part 2(B) above: for $\mu \prec \lambda$, $\mathrm{CSF}(\mu)$ denotes the set of column strict fillings $F : \mathrm{dg}(\mu) \to [n-1]$ (rather than $[n]$). Theorem 1, with the exception of part 2(B), can also be formulated in the setting of q-Whittaker functions in infinitely many variables. Next, we obtain the following corollaries:

Corollary 1. *Let $T \in \mathrm{GT}(\lambda)$ and let $\mathrm{rsort}^{-1}(T) = \{F \in \mathrm{CSF}(\lambda) : \mathrm{rsort}(F) = T\}$ be the fiber of* rsort *over T.*

1. $$\sum_{F \in \mathrm{rsort}^{-1}(T)} q^{\mathrm{inv}(F)} = \sum_{F \in \mathrm{rsort}^{-1}(T)} q^{\mathrm{quinv}(F)} = \mathrm{wt}_q(T).$$
2. $\mathrm{inv}(F) + \mathrm{quinv}(F) = \mathrm{area}(T)$ *is constant for* $F \in \mathrm{rsort}^{-1}(T)$.

An interpretation of $\mathrm{wt}_q(T)$ in terms of flags of subspaces compatible with nilpotent operators appears in [12, Theorem 5.8(i)]. In [1], the authors asked for an explicit bijection on $\mathcal{F}(\lambda)$ which interchanges the inv and quinv statistics. We describe this bijection on $\mathrm{CSF}(\lambda)$, thereby partially answering their question.

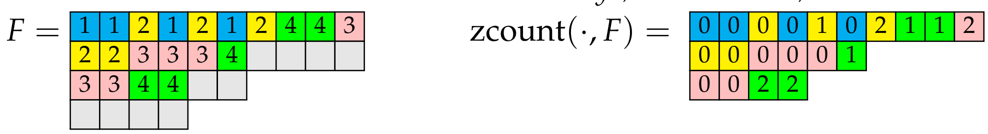


Figure 2: Here $F \in \mathrm{CSF}(\lambda)$ for $\lambda = (10, 6, 4, 0)$ and $n = 4$. Cells of F are coloured according to their entries. The gray cells are the extra cells in the augmented diagram $\widehat{\mathrm{dg}}(\lambda)$. On the right are cellwise zcount values. Here $\mathrm{quinv}(F) = 12$.

Corollary 2. *The map* $\Omega : \psi_{inv}^{-1} \circ \psi_{\mathrm{quinv}} = \psi_{inv}^{-1} \circ \mathrm{boxcomp} \circ \psi_{\mathrm{inv}} : \mathrm{CSF}(\lambda) \to \mathrm{CSF}(\lambda)$ *is an involution satisfying* $\mathrm{inv}(\Omega(F)) = \mathrm{quinv}(F)$ *for all* $F \in \mathrm{CSF}(\lambda)$.

The explicit construction of the ψ_v and their inverses in the next section makes Ω effectively computable.

6 Proof sketch

For a partition λ, the augmented diagram $\widehat{\mathrm{dg}}(\lambda)$ is $\mathrm{dg}(\lambda)$ together with one additional cell below the last cell in each column (see Figure 2). Given $F \in \mathrm{CSF}(\lambda)$, a *quinv-triple* in F is a triple of cells (x, y, z) in $\widehat{\mathrm{dg}}(\lambda)$ such that (i) $x, z \in \mathrm{dg}(\lambda)$ and z is to the right of x in the same row, (ii) y is the cell immediately below x in its column, (iii) $F(x) < F(z) < F(y)$, where we set $F(y) = \infty$ if y lies outside $\mathrm{dg}(\lambda)$. It is easy to see that the quinv-triples considered in [1] for $F \in \mathcal{F}(\lambda)$ reduce to this description when F is a CSF rather than a general filling. Thus, $\mathrm{quinv}(F)$ as defined in [1] equals the number of quinv-triples in F (as defined above) for a CSF F.

Given $F \in \mathrm{CSF}(\lambda)$, we define a function *zcount* which tracks the contributions of individual cells of $\mathrm{dg}(\lambda)$ to $\mathrm{quinv}(F)$ as follows: for each cell $c \in \mathrm{dg}(\lambda)$, let $\mathrm{zcount}(c, F) =$ the number of quinv-triples (x, y, z) in F with $z = c$. Clearly

$$\sum_{c \in \mathrm{dg}(\lambda)} \mathrm{zcount}(c, F) = \mathrm{quinv}(F) \tag{6.1}$$

We next group cells of the filling F row-wise according to the entries they contain. More precisely, let $\mathrm{cells}(i, j, F) = \{c \in \mathrm{dg}(\lambda) : c \text{ is in the } i^{th} \text{ row and } F(c) = j + 1\}$ for $1 \le i \le j + 1 \le n$. Figure 2 shows an example, with these groups colour-coded in each row. It readily follows from §3 that

$$|\,\mathrm{cells}(i, j, F)| = NE_{ij}(T), \text{ where } T = \mathrm{rsort}(F). \tag{6.2}$$

The next proposition brings the SE differences also into play [3]:

Proposition 2. *Let* $F \in \mathrm{CSF}(\lambda)$ *and* $T = \mathrm{rsort}(F)$. *Fix* $1 \le i \le j + 1 \le n$. *(1) If* $c \in \mathrm{cells}(i, j, F)$, *then* $\mathrm{zcount}(c, F) \le \mathrm{SE}_{ij}(T)$. *(2) If* $c, d \in \mathrm{cells}(i, j, F)$ *with* c *lying to the right of* d, *then* $\mathrm{zcount}(c, F) \ge \mathrm{zcount}(d, F)$. *(3) Further, equality holds in (1) for all* i, j *and all cells* $c \in \mathrm{cells}(i, j, F)$ *iff* $F = T$.

Figure 3: (left to right) Configuration of quinv, inv and refinv triples.

6.1 Definition of ψ_{quinv}

We now have all the ingredients in place to define ψ_{quinv}. Let $F \in \text{CSF}(\lambda)$ and $T = \text{rsort}(F)$. For each $1 \le i \le j+1 \le n$, consider the sequence

$$\Lambda_{ij} = (\text{zcount}(c, F) : \ c \in \text{cells}(i, j, F) \text{ traversed right to left in row } i). \tag{6.3}$$

In Figure 2, this amounts to reading the entries of a fixed colour from right to left in a given row of $\text{zcount}(\cdot, F)$. By Proposition 2, this is a weakly decreasing sequence bounded above by $\text{SE}_{ij}(T)$. Together with (6.2), this implies that Λ_{ij} may be viewed as a partition fitting into the $\text{NE}_{ij}(T) \times \text{SE}_{ij}(T)$ rectangle. Since $\text{SE}_{ij} = 0$ for $i = j+1$, Λ_{ij} is the zero sequence in this case. We drop the pairs $(j+1, j)$ to conclude that if $\Lambda = (\Lambda_{ij} : 1 \le i \le j < n)$, then $(T, \Lambda) \in \text{POP}(\lambda)$. We define $\psi_{quinv}(F) = (T, \Lambda)$. Clearly, $x^F = x^T$ and (6.1) implies $\text{quinv}(F) = |\Lambda|$, establishing (1) of Theorem 1 for $v = \text{quinv}$.

6.2 refinv triples

We now turn to the definition of ψ_{inv}. While we may anticipate doing this via a modification of the foregoing arguments, replacing quinv-triples with Haglund-Haiman-Loehr's inv-triples, that turns out not to work out-of-the-box. In place of the latter (see Figure 3), we consider triples (x, y, z) in $\widehat{\text{dg}}(\lambda)$ where (i) $x, z \in \text{dg}(\lambda)$ with z to the left of x in the same row, (ii) y is the cell immediately below x in its column. Given $F \in \text{CSF}(\lambda)$, we call (x, y, z) a *refinv-triple* (or "reflected inv-triple") for F if in addition to (i) and (ii), we also have (iii) $F(x) < F(z) < F(y)$, where $F(y) := \infty$ if $y \notin \text{dg}(\lambda)$. We have [3]:

Proposition 3. *For* $F \in \text{CSF}(\lambda)$, $\text{inv}(F)$ *equals the number of refinv-triples of* F.

Remarks. 1. We may in fact define a new statistic[2] *refinv* on all fillings $F \in \mathcal{F}(\lambda)$ as follows: $\text{refinv}(F) = \text{Inv}(F) - \sum_{u \in \text{Des}\, F} \text{coarm}(u)$, borrowing notation of [9, §2]. This replaces *arm* in HHL's definition by *coarm*. The content of Proposition 3 is that $\text{refinv}(F) = \text{inv}(F)$ for $F \in \text{CSF}(\lambda)$. In fact, this equality holds more generally for all fillings F whose descent set is a union of rows of $\text{dg}(\lambda)$.

2. The refinv triples for $F \in \text{CSF}(\lambda)$ actually make an appearance in [13, §2.2], where they are attributed to Zelevinsky (and their total number denoted $\widetilde{ZEL}$). From this perspective, the content of Proposition 3 is that $\widetilde{ZEL}(F) = \text{inv}(F)$.

[2]In fact, *refquinv* can also be likewise defined on all fillings, and agrees with *quinv* on CSFs. But rephrased in terms of refquinv-triples, this involves counting such triples with signs [3].

6.3 $\overline{\text{zcount}}$, zcount and the proof of the main theorem

Given $F \in \text{CSF}(\lambda)$ and $c \in \text{dg}(\lambda)$, define $\overline{\text{zcount}}(c, F)$ = the number of refinv-triples (x, y, z) in F with $z = c$. In light of Proposition 3, it is clear that

$$\sum_{c \in \text{dg}(\lambda)} \overline{\text{zcount}}(c, F) = \text{inv}(F) \tag{6.4}$$

We have the following relation between $\overline{\text{zcount}}$ and zcount [3]:

Proposition 4. *Let* $F \in \text{CSF}(\lambda)$ *and* $T = \text{rsort}(F)$. *Let* $1 \le i \le j+1 \le n$ *and* $c \in \text{cells}(i, j, F)$. *Then* $\text{zcount}(c, F) + \overline{\text{zcount}}(c, F) = \text{SE}_{ij}(T)$.

We may now define ψ_{inv} following the template of ψ_{quinv}. Given $F \in \text{CSF}(\lambda)$, let $T = \text{rsort}(F)$. For each $1 \le i \le j < n$, consider the sequence:

$$\overline{\Lambda}_{ij} = (\overline{\text{zcount}}(c, F) : \ c \in \text{cells}(i, j, F) \text{ traversed left to right in row } i)$$

Recall also the definition of the partition Λ_{ij} from (6.3). It follows from Propositions 2 and 4 that $\overline{\Lambda}_{ij}$ is the box-complement of Λ_{ij} in the $\text{NE}_{ij}(T) \times \text{SE}_{ij}(T)$ rectangle. Letting $\overline{\Lambda} = (\overline{\Lambda}_{ij} : 1 \le i \le j < n)$, we define $\psi_{\text{inv}}(F) = (T, \overline{\Lambda})$. As in the case of quinv, we have $x^F = x^T$, and $\text{inv}(F) = |\overline{\Lambda}|$ by (6.4). This proves part (1) of Theorem 1 for $v = \text{inv}$.

Since by definition $\text{pr}(\psi_v(F)) = T$ for $v = \text{inv}, \text{quinv}$, Part (2A) of Theorem 1 follows. Part (3) of Theorem 1 follows from the fact that Λ and $\overline{\Lambda}$ are box complements of each other in the appropriate rectangles. That the diagrams in part (2B) of Theorem 1 are commutative follows from an analysis of each elementary splice step of the dsplice map.

Finally, this leaves us with proving that the ψ_v are bijections. We sketch the construction of ψ_{inv}^{-1}. Given $(T, \Lambda) \in \text{POP}(\lambda)$, construct the filling $F := \psi_{\text{inv}}^{-1}(T, \Lambda) \in \text{CSF}(\lambda)$ inductively row-by-row, from the bottom (n^{th}) row to the top as follows: (a) fill all cells of the n^{th} row (if nonempty) with n, (b) let $1 \le i \le j < n$; assuming that all rows of F strictly below row i have been completely determined and that the locations of entries $> (j+1)$ in row i have been determined, we now need to fill $\text{NE}_{ij}(T)$ many cells of row i with the entry $j+1$. It turns out that the number of cells in row i in which we can potentially put a $j+1$ without violating the CSF condition thus far is exactly $k + \ell$ where $k = \text{NE}_{ij}(T)$ and $\ell = \text{SE}_{ij}(T)$. We label these cells $0, 1, \cdots, k+\ell-1$ from right to left (left-to-right when defining ψ_{quinv}^{-1}). We now use the identification from §3.1 of partitions fitting inside a $(k \times \ell)$-box with k-tuples of distinct integers in $0, 1, \cdots, k+\ell-1$. Via this, the partition Λ_{ij} can be viewed as a k-tuple of candidate cells in row i; we put the entry $j+1$ into these, (c) fill the remaining cells of row i with the entry i. The rest of the argument is straightforward [3]. □

For example, let $n = 4$, $\lambda = (10, 6, 4, 0)$ and let T, Λ be the GT pattern and overlay depicted in Figure 1. Then $\psi_{\text{quinv}}^{-1}(\mathcal{T}, \Lambda)$ is precisely the CSF F of Figure 2, while

$\psi_{\text{inv}}^{-1}(\mathcal{T}, \Lambda) =$

2	1	1	1	3	2	1	4	4	2
3	3	2	2	4	3				
4	4	3	3						

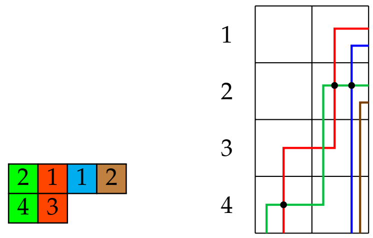


Figure 4: A CSF F with columns colour-coded to match its lattice path representation. The three marked intersections show that $\mathrm{inv}(F) = 3$.

7 Local Weyl modules and limit constructions

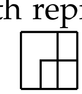
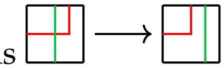
Finally, we can apply these ideas to the study of local Weyl modules, in particular to the *limit constructions* of [7, 15, 16]. Let $L(\Lambda_0)$ denote the basic representation of the affine Lie algebra $\widehat{\mathfrak{sl}_n}$ [11, Prop. 12.13]. Using Theorem 1 to replace POPs with CSFs as our model in [15, Corollary 5.13], we deduce [3]:

Proposition 5. *Fix $n \geq 2$ and consider the partition $\theta = (2, 1, 1, \cdots, 1, 0)$ with $n - 1$ nonzero parts and $|\theta| = n$. For $k \geq 0$, let $\mathcal{C}_k$ denote the set of CSFs F of shape $k\theta$ and entries in $[n]$, with the property that either 1 occurs in the first column of F or 1 does not occur in its last column. Then $\sum_{k\geq 0} \sum_{F \in \mathcal{C}_k} x^F q^{k^2 - \mathrm{inv}(F)}$ equals the character of $L(\Lambda_0)$.*

There is also a more general version with $\lambda + k\theta$ in place of $k\theta$ (for appropriate λ), mirroring [15, Corollary 5.13].

8 Concluding Remarks

For the modified Hall-Littlewood polynomials $Q'_{\lambda'}(X_n; q)$ of (2.4), the fermionic formula appears in [13, (0.2)]. Analogous to (3.2), this can now be recast as a *weighted sum* over *partition overlaid plane-partitions* (POPP) of shape λ. Theorem 1 takes the form of bijections from $\mathrm{WDF}(\lambda)$ to $\mathrm{POPP}(\lambda)$ (or equivalently, from tabloids to partition overlaid reverse-plane-partitions). The subtlety here is that POPPs need to be weighted with an additional power of q (which depends only on the underlying plane-partition, cf [13, (0.2)]). The refinv- or quinv-triples in this case also involve $\leq$ relations (rather than just $<$) and this extra q-power keeps track of certain equalities among the triples [3].

Secondly, the bijections of Theorem 1 (and those indicated above for the modified Hall-Littlewood case) have an attractive interpretation in terms of lattice-path diagrams [8, 4]. Figure 4 shows the lattice path representation of a CSF F; $\mathrm{inv}(F)$ is just the total number of intersections of the form  in the grid, and refining this further to each box of the grid produces the partition overlay as well [3]. Likewise $\mathrm{quinv}(F)$ counts non-intersections of the above form. The dsplice map of §4.2 translates into deletion of the last row of the grid followed by appropriate rectifications 

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