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Hochschild polytopes

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Abstract. The (m, n)-multiplihedron is a polytope whose faces correspond to m-painted n-trees. Deleting certain inequalities from its facet description, we obtain the (m, n)-Hochschild polytope whose faces correspond to m-lighted n-shades. Moreover, there is a natural shadow map from m-painted n-trees to m-lighted n-shades, which defines a meet semilattice morphism of rotation lattices. In particular, when $m = 1$, our Hochschild polytope is a deformed permutahedron realizing the Hochschild lattice.

Résumé. Le (m, n)-multiplièdre est un polytope dont les faces correspondent aux n-arbres m-peints. En retirant certaines inégalités de sa description par facettes, nous obtenons le (m, n)-polytope de Hochschild dont les faces correspondent aux n-ombres m-illuminées. De plus, il existe une fonction d'ombre naturelle des n-arbres m-peints vers les n-ombres m-illuminées, qui définit un morphisme de semi-treillis supérieur entre les treillis de rotations correspondants. En particulier, quand $m = 1$, notre polytope de Hochschild est un permutaèdre déformé qui réalise le treillis de Hochschild.

Keywords: Multiplihedron, Freehedron, Hochschild lattice, Quotient

Introduction

We present a remake of the famous combinatorial, geometric, and algebraic interplay between permutations and binary trees. In the original story, the central character is the surjective map from permutations to binary trees (given by successive binary search tree insertions [19, 9]). This map enables us to construct the Tamari lattice [18] as a lattice quotient of the weak order, the sylvester fan as a quotient fan of the braid fan, Loday's associahedron [10] as a removahedron of the permutahedron, and the Loday–Ronco Hopf algebra as a Hopf subalgebra of the Malvenuto–Reutenauer Hopf algebra. Many variations of this saga have been further investigated, notably for other lattice quotients of the weak order and for generalized associahedra arizing from finite type cluster algebras. See [13] for a recent survey on this topic, in particular for a bibliography.

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In the present remake, permutations are replaced by binary m-painted n-trees (binary trees on n nodes with m horizontal labeled edge cuts), while binary trees are replaced by unary m-lighted n-shades (compositions of n with m labels inside their gaps). The precise definitions are delayed to Section 1, but the reader can already glance at Figure 8 for $m = 1$ and $n = 3$. The m-painted n-trees already appeared in [5, Sect. 3.1], inspired from the case $m = 1$ studied in [17, 8, 2]. They are mixtures (in the sense of [5]) between the permutations of $[m]$ and the binary trees with n nodes. The m-lighted n-shades are introduced in this paper, inspired from the case $m = 1$ studied in [1, 14, 4, 6, 11]. Here again, the central character is a natural surjective map from the former to the latter. Namely, the shadow map sends an m-painted n-tree to the m-lighted n-shade obtained by collecting the arity sequence along the right branch. In other words, this map records the shadow projected on the right of the tree when the sun sets on the left of the tree.

We first use this map for lattice purposes. It was proved in [5] that the right rotation digraph on binary m-painted n-trees (a mixture of the simple transposition digraph on permutations and the right rotation digraph on binary trees) defines a lattice. We consider here the right rotation digraph on unary m-lighted n-shades. We prove that it defines as well a lattice by showing that the shadow map is a meet semilattice morphism (but not a lattice morphism). When $m = 0$, this gives an unusual meet semilattice morphism from the Tamari lattice to the Boolean lattice (distinct from the usual lattice morphism given by the canopy map). When $m = 1$, this gives a connection, reminiscent of [14], between the painted tree rotation lattice and the Hochschild lattice [4, 6, 11].

We then use the shadow map for polytopal purposes. The refinement poset on all m-painted n-trees is isomorphic to the face lattice of the (m, n)-multiplihedron $\mathbb{M}\mathrm{ul}(m, n)$. This polytope is a deformed permutahedron (*a.k.a.* polymatroid [7], or generalized permutahedron [15]) obtained as the shuffle product [5] of an m-permutahedron with an n-associahedron of [10]. Oriented in a suitable direction, the skeleton of the (m, n)-multiplihedron is isomorphic to the right rotation digraph on binary m-painted n-trees [5]. Similarly, we show that the refinement poset on all m-lighted n-shades is isomorphic to the face lattice of the (m, n)-Hochschild polytope $\mathbb{H}\mathrm{och}(m, n)$. We obtain this polytope by deleting some inequalities in the facet description of the (m, n)-multiplihedron. We also work out the vertex description of the (m, n)-Hochschild polytope. We obtain a deformed permutahedron whose oriented skeleton is isomorphic to the right rotation digraph on unary m-lighted n-shades. When $m = 0$, the $(0, n)$-multiplihedron is the n-associahedron and the $(0, n)$-Hochschild polytope is a skew cube (which is not a parallelotope). When $m = 1$, the $(1, n)$-multiplihedron is the classical multiplihedron [17, 8, 2], and the $(1, n)$-Hochschild polytope is a deformed permutahedron realizing the Hochschild lattice [4, 6, 11], answering an open question of F. Chapoton.

We refer to [12] for many details and all proofs omitted in this extended abstract due to space limitations. The interested reader will in particular find enumerative formulas and cubic coordinates for multiplihedra and Hochschild polytopes.

1 Painted trees, lighted shades, and the shadow map

1.1 m-painted n-trees

We start with the combinatorics of m-painted n-trees already studied in detail in [5, Sect. 3.1]. It was inspired from the case $m = 1$ studied in [17, 8, 2].

An *n-tree* is a rooted plane tree with $n + 1$ leaves. As usual, we orient such a tree towards its root and label its vertices in inorder. Namely, each node with ℓ children is labeled by an $(\ell - 1)$-subset $\{x_1, \ldots, x_{\ell-1}\}$ of $[n]$ such that all labels in its ith subtree are larger than x_{i-1} and smaller than x_i (where by convention $x_0 = 0$ and $x_\ell = n + 1$). Note in particular that unary nodes receive an empty label. A *cut* of an n-tree T is a subset c of nodes of T containing precisely one node along the path from the root to any leaf of T. A cut c is *below* a cut c' if the unique node of c is after the unique node of c' along any path from the root to a leaf of T (note that we draw trees growing downward).

Definition 1 ([5, Def. 105])**.** *An m-painted n-tree $\mathbb{T} := (T, C, \mu)$ is an n-tree T together with a sequence $C := (c_1, \ldots, c_k)$ of k cuts of T and an ordered partition μ of $[m]$ into k parts for some $k \in [m]$, such that*
- *c_i is below c_{i+1} for all $i \in [k-1]$,*
- *$\bigcup C := c_1 \cup \cdots \cup c_k$ contains all unary nodes of T.*

We represent an m-painted n-tree $\mathbb{T} := (T, C, \mu)$ as a downward growing tree T, where the cuts of C are red horizontal lines, labeled by the corresponding parts of μ. As there is no ambiguity, we write 12 for the set $\{1, 2\}$. See Figures 1 to 3 for illustrations.

We now associate to each m-painted n-tree a preposet (*i.e.* a reflexive and transitive binary relation) on $[m + n]$. See Figure 1.

Definition 2. *Consider an m-painted n-tree $\mathbb{T} := (T, C, \mu)$. Orient T towards its root, label each node x of T by the union of the part in μ corresponding to the cut of C passing through x (empty set if x is in no cut of C) and the inorder label of x in T shifted by m, and merge all nodes contained in each cut. Define $\preccurlyeq_{\mathbb{T}}$ as the preposet on $[m + n]$ where $i \preccurlyeq_{\mathbb{T}} j$ if there is a (possibly empty) oriented path from the node containing i to the node containing j in the resulting oriented graph.*

We now use these preposets to define the refinement poset on m-painted n-trees.

Definition 3 ([5, Def. 108])**.** *The m-painted n-tree refinement poset is the poset on m-painted n-trees ordered by refinement of their corresponding preposets, that is, $\mathbb{T} \le \mathbb{T}'$ if $\preccurlyeq_{\mathbb{T}} \supseteq \preccurlyeq_{\mathbb{T}'}$.*

In the following statement, we denote by $|T|$ the number of nodes of a tree T (including unary nodes), and define $|C| := k$ and $|\bigcup C| := |c_1 \cup \cdots \cup c_k|$ for $C = (c_1, \ldots, c_k)$.

Proposition 4 ([5, Props. 107 & 116])**.** *The m-painted n-tree refinement poset is a meet semilattice ranked by $m + n - |T| - |C| + |\bigcup C|$.*

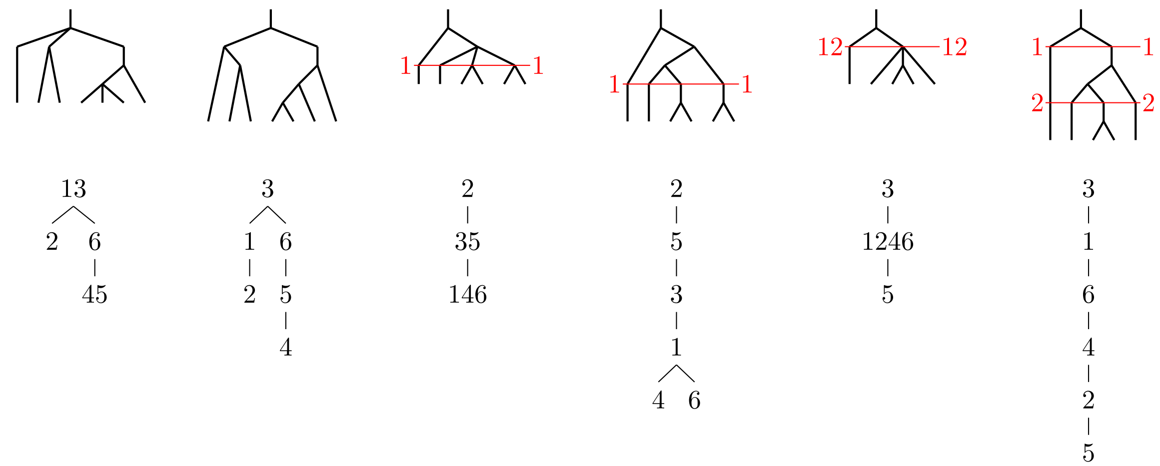

Figure 1: Some m-painted n-trees (top) and their preposets (bottom). Here $m + n = 6$.

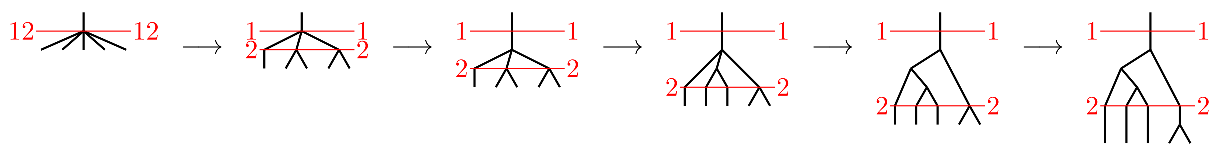

Figure 2: Refinements of some 2-painted 4-trees.

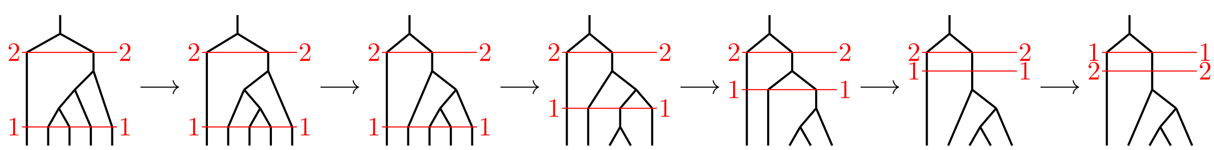

Figure 3: Rotations of some binary 2-painted 4-trees.

We now define another lattice, but on rank 0 m-painted n-trees. See Figures 3 and 7.

Definition 5 ([5, Def. 112]). *An m-painted n-tree $\mathbb{T} := (T, C, \mu)$ is binary if it has rank 0, meaning that all nodes in $\bigcup C$ are unary, while all nodes not in $\bigcup C$ are binary. The binary m-painted n-tree right rotation digraph is the directed graph on binary m-painted n-tree with an edge $(\mathbb{T}, \mathbb{T}')$ if and only if there exists $1 \le i < j \le m + n$ such that $\preccurlyeq_{\mathbb{T}} \smallsetminus \{(i, j)\} = \preccurlyeq_{\mathbb{T}'} \smallsetminus \{(j, i)\}$.*

Proposition 6 ([5, Def. 119]). *The binary m-painted n-tree right rotation digraph is the Hasse diagram of a lattice.*

Example 7. *When $m = 0$, the 0-painted n-tree rotation lattice is the Tamari lattice [18]. When $m = 1$, the 1-painted n-tree rotation lattice is the multiplihedron lattice introduced in [5].*

Remark 8. *Note that the m-painted n-tree rotation lattice is meet semidistributive, but not join semidistributive when $m \ge 1$.*

Let us finally mention that m-painted n-trees have interesting enumerative properties. See [12, Sect. 1.1] for formulas for some m-painted n-trees generating functions.

1.2 m-lighted n-shades

We now introduce the main new characters of this paper, which will later appear as certain shadows of m-painted n-trees.

Definition 9. *An n-shade is a sequence of (possibly empty) tuples of integers, whose total sum is n. An m-lighted n-shade* $\mathbb{S} := (S, C, \mu)$ *is an n-shade S together with a set C of k distinguished positions in S, containing all positions of empty tuples of S, and an ordered partition* μ *of* $[m]$ *into k parts for some* $k \in [m]$.

We represent an m-lighted n-shade $\mathbb{S} := (S, C, \mu)$ as a vertical line, with the tuples of the sequence S in black on the left, and the cuts of C in red on the right, all from top to bottom. As there is no ambiguity, we write 12 for the tuple $(1, 2)$ or the set $\{1, 2\}$. See Figures 4 to 6 for illustrations.

We now associate to each m-lighted n-shade a preposet on $[m + n]$. See Figure 4

Definition 10. *Consider an m-lighted n-shade* $\mathbb{S} := (S, C, \mu)$*. The preceeding sum* $ps(x)$ *of an entry x in a tuple of S is m plus the sum of all entries that appear weakly before x in S (meaning either the entries in a strictly earlier tuple of S, or the weakly earlier entries in the same tuple as x). Define* $\preccurlyeq_{\mathbb{S}}$ *as the preposet on* $[m + n]$ *given by the relations*

- $i \preccurlyeq_{\mathbb{S}} j$ *if* $i, j \in [m]$ *and i appears weakly after j in* μ,
- $k \preccurlyeq_{\mathbb{S}} ps(y)$ *if x and y are elements of tuples of S such that the tuple of x appears weakly after the tuple of y, and* $ps(x) - x < k \le ps(x)$,
- $i \preccurlyeq_{\mathbb{S}} ps(x)$ *if* $i \in [m]$ *and x is an element of a tuple of S which appears weakly before the cut containing i,*
- $k \preccurlyeq_{\mathbb{S}} i$ *if* $i \in [m]$ *and* $ps(x) - x < k \le ps(x)$ *for some element x of a tuple of S which appears weakly after the cut containing i.*

We now use these preposets to define the refinement poset on m-lighted n-shades.

Definition 11. *The m-lighted n-shade refinement poset is the poset on m-lighted n-shades defined by refinement of their corresponding preposets, that is,* $\mathbb{S} \le \mathbb{S}'$ *if* $\preccurlyeq_{\mathbb{S}} \supseteq \preccurlyeq_{\mathbb{S}'}$.

For a sequence $S := (s_1, \dots, s_\ell)$ of tuples, we define $|S| := \ell$ and $\|S\| := \sum_{i \in [\ell]} |s_i|$, where $|s_i|$ is the length of the tuple s_i.

Proposition 12. *The m-lighted n-shade refinement poset is a meet semilattice ranked by* $m - |S| + \|S\|$.

We now define another lattice, but on rank 0 m-lighted n-shades. See Figures 6 and 7.

Definition 13. *An m-lighted n-shade* $\mathbb{S} := (S, C, \mu)$ *is unary if it has rank 0, meaning that all tuples in* $\bigcup C$ *are empty tuples, while all tuples not in* $\bigcup C$ *are singletons. The unary m-lighted n-shade right rotation digraph is the directed graph on unary m-lighted n-shades with an edge* $(\mathbb{S}, \mathbb{S}')$ *if and only if there exists* $1 \le i < j \le m + n$ *such that* $\preccurlyeq_{\mathbb{S}} \smallsetminus \{(i, j)\} = \preccurlyeq_{\mathbb{S}'} \smallsetminus \{(j, i)\}$.

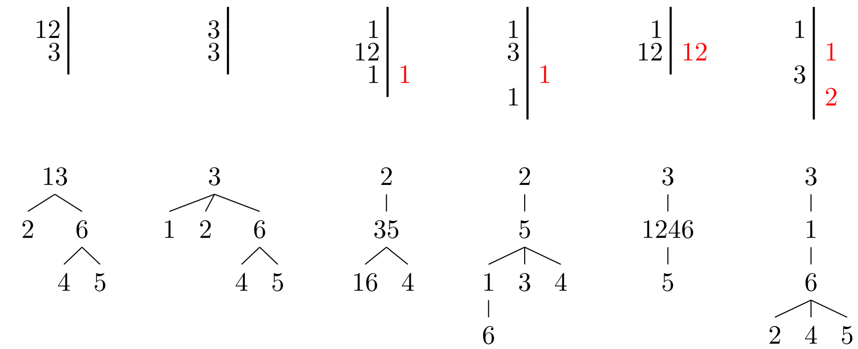

Figure 4: Some m-lighted n-shades (top) and their preposets (bottom). Here $m + n = 6$.

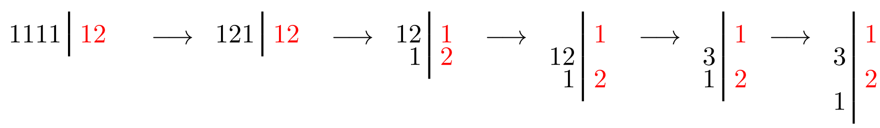

Figure 5: Refinements of some 2-lighted 4-shades.

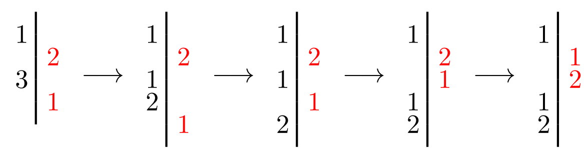

Figure 6: Rotations of some unary 2-lighted 4-shades.

Remark 14. *We observe that any unary m-lighted n-shade $\mathbb{S}$ with singleton tuples $s_1, \dots, s_k$ admits $m + k - 1 + \sum_{i \in [k]}(s_i - 1) = m + n - 1$ (left or right) rotations. In other words, the (undirected) rotation graph is regular of degree $m + n - 1$.*

Proposition 15. *The unary m-lighted n-shade right rotation digraph is the Hasse diagram of a lattice.*

Example 16. *When $m = 0$, the 0-lighted n-shade rotation lattice is boolean. When $m = 1$, the 1-lighted n-shade rotation lattice is the Hochschild lattice studied in [4, 6, 11].*

Remark 17. *Computational experiments indicate that the m-lighted n-shade rotation lattice is constructible by interval doubling (hence semidistributive and congruence uniform). However, in contrast to the case when $m \leq 1$, it is not extremal (see [11] for context), and its Coxeter polynomial is not a product of cyclotomic polynomials (see [3] and [6, Appendix] for context). Nevertheless, its subposet induced by unary m-lighted n-shades where the labels of the lights are ordered seems to enjoy all these nice properties.*

Let us finally mention that m-lighted n-shades have interesting enumerative properties. See [12, Sect. 1.1] for formulas for some m-lighted n-shades generating functions.

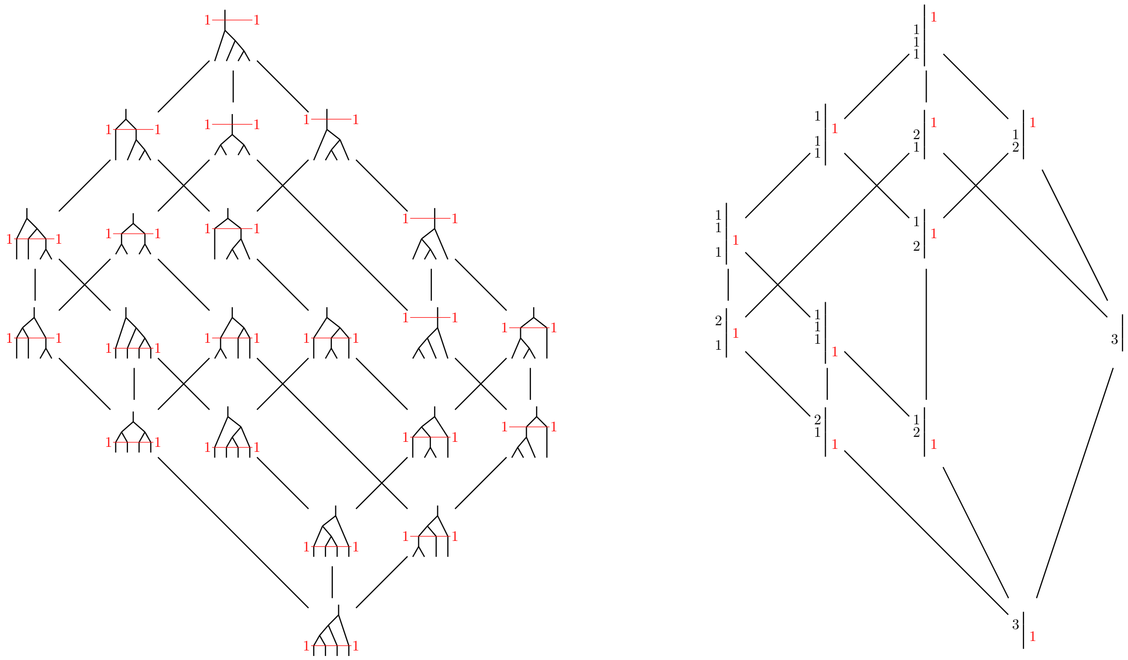

Figure 7: The 1-painted 3-tree (left) and 1-lighted 3-shade (right) rotation lattices.

1.3 Shadow map

We now describe the shadow map sending an m-painted n-tree to an m-lighted n-shade. Intuitively, the shadow is what you see on the right of the tree when the sun sets on its left. For instance, the m-painted n-trees of Figure 1 are sent to the m-lighted n-shade of Figure 4. We call *right branch* of a tree T the path from the root to the rightmost leaf of T.

Definition 18. *The shadow of an n-tree T is the n-shade $\mathrm{Sh}(T)$ obtained by*

- *contracting all edges joining a child to a parent which does not lie on the right branch of T,*
- *replacing each node on the right branch of T by the tuple of the arities of its children except its rightmost.*

The shadow of a cut c in T is the position $\mathrm{Sh}(c)$ in $\mathrm{Sh}(T)$ of the unique node of the right branch of T contained in c. For a sequence $C = (c_1, \dots, c_k)$, define $\mathrm{Sh}(C) := (\mathrm{Sh}(c_1), \dots, \mathrm{Sh}(c_k))$. The shadow of an m-painted n-tree $\mathbb{T} := (T, C, \mu)$ is the m-lighted n-shade $\mathrm{Sh}(\mathbb{T}) := (\mathrm{Sh}(S), \mathrm{Sh}(C), \mu)$.

Given two meet semilattices $(M, \wedge)$ and $(M', \wedge')$, a map $f : M \to M'$ is a *meet semilattice morphism* if $f(x \wedge y) = f(x) \wedge' f(y)$ for all $x, y \in M$.

Theorem 19. *The shadow map is a surjective meet semilattice morphism from the binary m-painted n-tree rotation lattice to the unary m-lighted n-shade rotation lattice. See Figure 7.*

Remark 20. *Note that the shadow map is not a join semilattice morphism. For instance,*

2 Multiplihedra and Hochschild polytopes

2.1 Multiplihedra

We now consider the (m, n)-multiplihedron which realizes the m-painted n-tree refinement lattice. It is illustrated for $m = 1$ and $n = 3$ in Figure 8. Although they were previously constructed when $m = 1$ in [17, 8, 2], we use here the construction of [5, Sect. 3]. This construction is just an example of the shuffle product on deformed permutahedra, introduced in [5, Sect. 2]. However, we do not need the generality of this operation and define the (m, n)-multiplihedron using its vertex and facet descriptions.

Definition 21. *Consider a binary m-painted n-tree $\mathbb{T} := (T, C, \mu)$. We associate to $\mathbb{T}$ a point $\boldsymbol{a}(\mathbb{T})$ whose pth coordinate is*

- *if $p \le m$, the number of binary nodes and cuts weakly below the cut labeled by p,*
- *if $p \ge m + 1$, the number of cuts below plus the product of the numbers of leaves in the left and right subtrees of the node of T labeled by $p - m$ in inorder.*

See Figure 9 for some examples.

Definition 22. *Consider the hyperplane $\mathbb{H}_{m+n}$ of $\mathbb{R}^{m+n}$ defined by $\langle\, \boldsymbol{x} \mid \mathbf{1}_{[m+n]} \,\rangle = \binom{m+n+1}{2}$. Moreover, for each rank $m + n - 2$ m-painted n-tree $\mathbb{T} := (T, C, \mu)$, consider the halfspace $\boldsymbol{H}(\mathbb{T})$ of $\mathbb{R}^{m+n}$ defined by $\langle\, \boldsymbol{x} \mid \mathbf{1}_{A \cup B} \,\rangle \ge \binom{|A|+1}{2} + \binom{|B_1|+1}{2} + \cdots + \binom{|B_k|+1}{2} + |A| \cdot |B|$, where*

- *A denotes the set of elements of $[m]$ which label the cut of C not containing the root of T (hence, $A = \varnothing$ if C has only one cut, which contains the root of T),*
- *$B := B_1 \cup \cdots \cup B_k$ where $B_1, \ldots, B_k$ are the inorder labels shifted by m of the non-unary nodes of T distinct from the root of T.*

See Figure 9 for some examples.

Theorem 23 ([5, Props. 116, 122, 123])**.** *The m-painted n-tree refinement lattice is anti-isomorphic to the face lattice of the (m, n)-multiplihedron $\mathbb{M}\mathrm{ul}(m, n)$, defined equivalently as*

(i) the convex hull of the vertices $\boldsymbol{a}(\mathbb{T})$ for all binary m-painted n-trees $\mathbb{T}$,

(ii) the intersection of the hyperplane $\mathbb{H}_{m+n}$ with the halfspaces $\boldsymbol{H}(\mathbb{T})$ for all rank $m + n - 2$ m-painted n-trees $\mathbb{T}$.

Proposition 24 ([5, Prop. 118])**.** *The normal fan of the (m, n)-multiplihedron $\mathbb{M}\mathrm{ul}(m, n)$ is the fan whose cones are the preposet cones of the preposets $\preccurlyeq_{\mathbb{T}}$ of all m-painted n-trees $\mathbb{T}$.*

Proposition 25 ([5, Prop. 119])**.** *The skeleton of the (m, n)-multiplihedron $\mathbb{M}\mathrm{ul}(m, n)$ oriented in the direction $\boldsymbol{\omega}_{m+n} := (m + n, \ldots, 1) - (1, \ldots, m + n)$ is isomorphic to the right rotation digraph on binary m-painted n-trees.*

Example 26. *When $m = 0$, the $(0, n)$-multiplihedron is Loday's associahedron [10]. When $m = 1$, the $(1, n)$-multiplihedron is the classical multiplihedron alternatively constructed in [8, 2].*

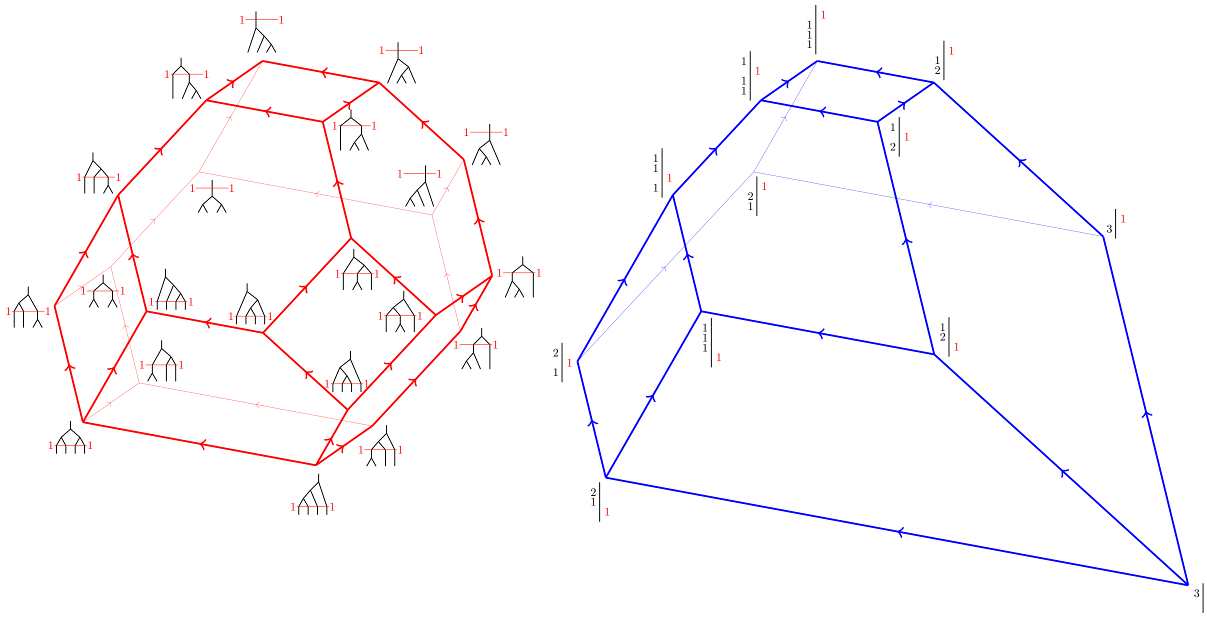

Figure 8: Multiplihedron $\mathbb{M}\mathrm{ul}(1,3)$ (left) and Hochschild polytope $\mathbb{H}\mathrm{och}(1,3)$ (right).

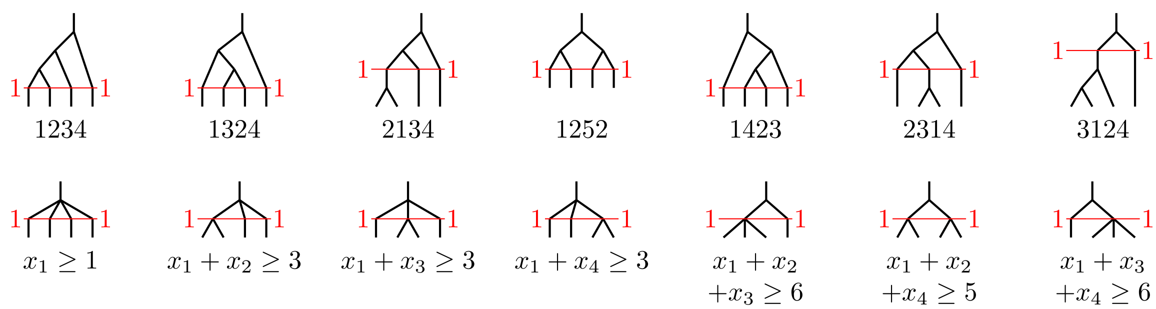

Figure 9: Some vertices (top) and facet defining inequalities (bottom) of $\mathbb{M}\mathrm{ul}(1,3)$.

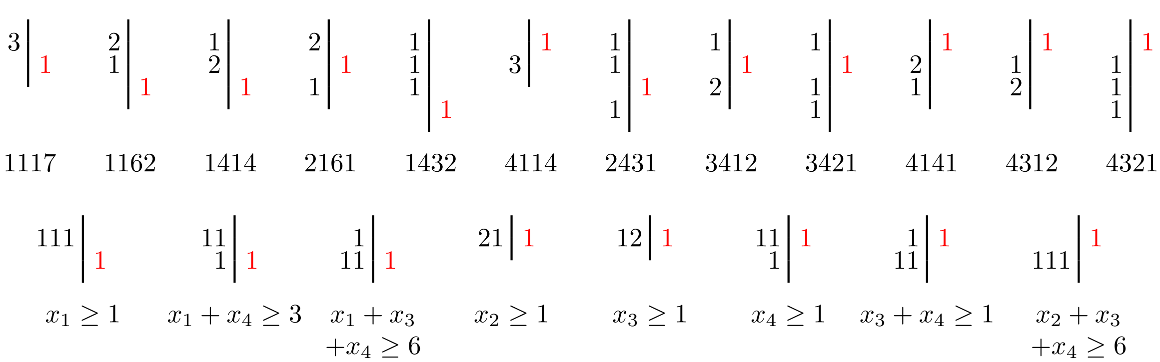

Figure 10: Vertices (top) and facet defining inequalities (bottom) of $\mathbb{H}\mathrm{och}(1,3)$.

2.2 Hochschild polytopes

We now construct the (m, n)-Hochschild polytope which realizes the m-lighted n-shade refinement lattice. It is illustrated for $m = 1$ and $n = 3$ in Figure 8. Recall that we denote by $ps(x)$ the preceeding sum of an entry x in an m-lighted n-shade (see Definition 10).

Definition 27. *Consider a unary m-lighted n-shade $\mathbb{S} := (S, C, \mu)$ and denote by $s_1, s_2, \dots, s_k$ the values of the singleton tuples of S. We associate to $\mathbb{S}$ a point $\boldsymbol{a}(\mathbb{S})$ whose pth coordinate is*
- *if $p \le m$, then the number of cuts plus the sum of the entries s_i which are weakly below the cut labeled p,*
- *if there is $j \in [k]$ such that $p = ps(s_j)$, then $1 + s_j\big(m + n - p + c_p\big) + \binom{s_j}{2}$ where c_p is the number of cuts below s_j,*
- *1 otherwise.*

See Figure 10 for some examples.

Definition 28. *Consider the hyperplane $\mathbb{H}_{m+n}$ of $\mathbb{R}^{m+n}$ defined by $\langle\, \boldsymbol{x} \mid \mathbf{1}_{[m+n]} \,\rangle = \binom{m+n+1}{2}$. Moreover, for each rank $m+n-2$ m-lighted n-shade $\mathbb{S} := (S, C, \mu)$, consider the halfspace $\boldsymbol{H}(\mathbb{S})$ of $\mathbb{R}^{m+n}$ defined by $\langle\, \boldsymbol{x} \mid \mathbf{1}_{A \cup B} \,\rangle \ge \binom{|A|+|B|+1}{2}$, where*
- *A denotes the set of elements of $[m]$ which label the cut of C not containing the first tuple of S (hence, $A = \varnothing$ if C has only one cut, which contains the first tuple of S),*
- *$B = \{m+q\}$ if S is a single tuple with 2 in position q, and $B = \{m+q+1, \dots, m+n\}$ if $S = (s_1, s_2)$ is a pair of tuples with $|s_1| = q$.*

See Figure 10 for some examples.

Theorem 29. *The m-lighted n-shade refinement lattice is anti-isomorphic to the face lattice of the (m, n)-Hochschild polytope $\mathbb{H}\mathrm{och}(m, n)$, defined equivalently as*
1. *(i) the convex hull of the vertices $\boldsymbol{a}(\mathbb{S})$ for all unary m-lighted n-shades $\mathbb{S}$,*
2. *(ii) the intersection of the hyperplane $\mathbb{H}_{m+n}$ with the halfspaces $\boldsymbol{H}(\mathbb{S})$ for all rank $m+n-2$ m-lighted n-shades $\mathbb{S}$.*

Proposition 30. *The normal fan of the (m, n)-Hochschild polytope $\mathbb{H}\mathrm{och}(m, n)$ is the fan whose cones are the preposet cones of the preposets $\preccurlyeq_{\mathbb{S}}$ of all m-lighted n-shades $\mathbb{S}$.*

Proposition 31. *The skeleton of the (m, n)-Hochschild polytope $\mathbb{H}\mathrm{och}(m, n)$ oriented in the direction $\boldsymbol{\omega}_{m+n} := (m+n, \dots, 1) - (1, \dots, m+n)$ is isomorphic to the right rotation digraph on unary m-lighted n-shades.*

Remark 32. *It follows from Remark 14 that the (m, n)-Hochschild polytope is simple and the m-lighted n-shade fan is simplicial.*

Remark 33. *As mentioned in the introduction, there are deep similarities between the behaviors of*
- *the permutahedron $\mathbb{P}\mathrm{erm}(d)$ and the associahedron $\mathbb{A}\mathrm{sso}(d)$,*
- *the multiplihedron $\mathbb{M}\mathrm{ul}(m, n)$ and the Hochschild polytope $\mathbb{H}\mathrm{och}(m, n)$.*

Some comments on the behavior of the latter for the reader familiar with the behavior of the former:

- *The* (m,n)*-Hochschild polytope* $\mathbb{H}\mathrm{och}(m,n)$ *can be obtained by deleting inequalities in the facet description of the* (m,n)*-multiplihedron* $\mathbb{M}\mathrm{ul}(m,n)$*.*
- *The common facet defining inequalities of* $\mathbb{M}\mathrm{ul}(m,n)$ *and* $\mathbb{H}\mathrm{och}(m,n)$ *are precisely those that contain a common vertex of* $\mathbb{M}\mathrm{ul}(m,n)$ *and* $\mathbb{H}\mathrm{och}(m,n)$*.*
- *In contrast, the vertex barycenters of the* (m,n)*-multiplihedron* $\mathbb{M}\mathrm{ul}(m,n)$ *and of the* (m,n)*-Hochschild polytope* $\mathbb{H}\mathrm{och}(m,n)$ *do not coincide.*
- *When* $m=0$*, the* $(0,n)$*-Hochschild polytope* $\mathbb{H}\mathrm{och}(0,n)$ *is a skew cube distinct from the parallelepiped obtained by considering the canopy congruence on binary trees.*

Example 34. *When* $m=0$*, the* $(0,n)$*-Hochschild polytope is a skew cube, distinct from the parallelotope* $\sum_{i\in[n-1]}[\boldsymbol{e}_i,\boldsymbol{e}_{i+1}]$*. When* $m=1$*, the* $(1,n)$*-Hochschild polytope gives a realization of the Hochschild lattice [4, 6, 11]. Note that the unoriented rotation graph on* 1*-lighted* n*-shades was already known to be isomorphic to the unoriented skeleton of a deformed permutahedron called freehedron and obtained as a truncation of the standard simplex [16], or more precisely as the Minkowski sum* $\sum_{i\in[n]}\triangle_{\{1,\dots,i\}}+\sum_{i\in[n]}\triangle_{\{i,\dots,n\}}$ *of the faces of the standard simplex corresponding to initial and final intervals, see Figure 11. However, orienting the skeleton of the freehedron in direction* $\boldsymbol{\omega}_{m+n}$*, we obtain a poset different from the Hochschild lattice, and which is not even a lattice. Indeed, in Figure 11 (left) the two blue vertices have no join while the two red vertices have no meet. In fact, the Hasse diagram of the Hochschild lattice cannot be obtained as a Morse orientation given by a linear functional on the freehedron. Finally, observe that the freehedron cannot be obtained by removing inequalities in the facet description of the permutahedron or of the multiplihedron. See Figure 11 where the removahedra have the wrong combinatorics.*

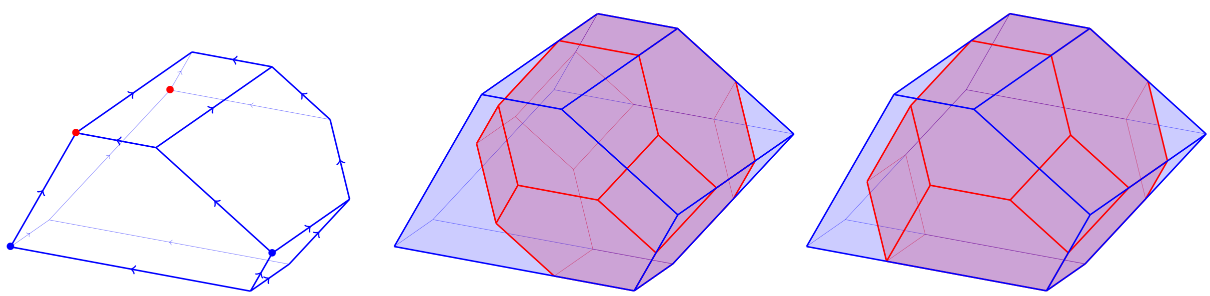

Figure 11: The freehedron obtained as Minkowski sum of the faces of the standard simplex corresponding to initial or final intervals (left), and failed attempts to obtain it as a removahedron of the permutahedron (middle) or of the multiplihedron (right).

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