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Acyclonestohedra

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Abstract. Given a building set $\mathcal{B}$ and an oriented matroid $\mathcal{M}$ on the same ground set, we define the acyclic nested complex as the simplicial complex of nested sets on $\mathcal{B}$ which are in some sense acyclic with respect to $\mathcal{M}$. We prove that this complex is always the face lattice of an oriented matroid, obtained as a stellar subdivision of the positive tope of the oriented matroid $\mathcal{M}$. When the oriented matroid $\mathcal{M}$ is the oriented matroid of a vector configuration $\boldsymbol{A}$, we moreover prove that this complex is the boundary complex of an acyclonestohedron, a polytope obtained as the section of a nestohedron for $\mathcal{B}$ by the evaluation space of $\boldsymbol{A}$. Our work specializes to explicit polytopal realizations of the poset associahedra and affine poset cyclohedra of Galashin.

Résumé. Étant donné un ensemble de construction $\mathcal{B}$ et un matroïde orienté $\mathcal{M}$ sur le même ensemble, nous définissons le complexe imbriqué acyclique comme le complexe simplicial des ensembles imbriqués de $\mathcal{B}$ qui sont acycliques pour $\mathcal{M}$ en un certain sens. Nous montrons que ce complexe est toujours le treillis des faces d'un matroïde orienté, obtenu par subdivisions stellaires du tope positif du matroïde orienté $\mathcal{M}$. Quand le matroïde orienté $\mathcal{M}$ est le matroïde orienté d'une configuration de vecteurs $\boldsymbol{A}$, nous montrons que ce complexe est le complexe de bord d'un acyclonestoèdre, un polytope obtenu comme la section du nestoèdre de $\mathcal{B}$ par l'espace vectoriel des évaluations linéaires sur $\boldsymbol{A}$. Notre travail se spécialise à des réalisations polytopales explicites des associaèdres d'ordres et des cycloèdres d'ordres affines de Galashin.

Keywords: building sets, nested complexes, oriented matroids, poset associahedra

Introduction

Motivated by the recent work of Galashin on poset associahedra and affine poset cyclohedra [8], we introduce the acyclic nested complexes and the acyclonestohedra, some simplicial complexes and polytopes at the interface between nestohedra [10, 4, 6, 12] (Section 1.1) and oriented matroids [1] (Section 1.2).

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The input data is an oriented building set $(\mathcal{B}, \mathcal{M})$ (Section 2.1), that is, a building set $\mathcal{B}$ and an oriented matroid $\mathcal{M}$ on the same ground set so that any circuit of $\mathcal{M}$ is a block of $\mathcal{B}$. The acyclic nested complex $\mathfrak{A}(\mathcal{B}, \mathcal{M})$ is the simplicial complex of nested sets on $\mathcal{B}$ which are in some sense acyclic with respect to $\mathcal{M}$ (Section 2.2).

Prototypical examples are graphical oriented building sets. The graphical oriented building set of a directed graph D is formed by the graphical building set of the line graph $L(D)$ together with the graphical oriented matroid of D. The graphical acyclic nested complex is then given by all tubings T on $L(D)$ such that for each tube $t \in T$, the contraction in the restriction $D_{|t}$ of all arcs contained in some tube $s \in T$ with $s \subsetneq t$ yields an acyclic directed graph. It is not difficult to see that this definition actually only depends upon the transitive closure of D and coincides with the poset associahedron of [8]. A similar (but slightly more intricate) construction shows that the affine poset cyclohedra of [8] are also acyclic nested complexes of specific oriented building sets.

Our main results are geometric realizations of acyclic nested complexes (Sections 2.3 and 2.4). We show that the acyclic nested complex of an oriented building set $(\mathcal{B}, \mathcal{M})$ is

(i) the face lattice of an oriented matroid obtained by stellar subdivisions of $\mathcal{M}$,

(ii) the boundary complex of a convex polytope, obtained by stellar subdivisions of the positive tope of $\mathcal{M}$ when the latter is realizable,

(iii) the boundary complex of the polar of the acyclonestohedron, a polytope obtained as the section of a nestohedron for $\mathcal{B}$ with the evaluation space of $\boldsymbol{A}$, when $\mathcal{M}$ is realized by the vector configuration $\boldsymbol{A}$.

Note that (i) is valid for all oriented matroids (realizable or not), while (ii) and (iii) only apply to realizable oriented matroids. The advantage of (iii) over (ii) is that it leads to explicit realizations with controlled integer coordinates. For poset associahedra and affine poset cyclohedra, (ii) recovers the construction of [8] using stellar subdivisions of order polytopes, and (iii) answers a question left open in [8], and independently settled in [11].

In fact, the oriented building sets and their acyclic nested complexes are closely related to the lattice building sets and their lattice nested complexes of [4, 6]. Namely, we show that the building sets on the Las Vergnas face lattice of $\mathcal{M}$ are obtained from the oriented building sets $(\mathcal{B}, \mathcal{M})$ by keeping only the blocks of $\mathcal{B}$ which are faces of $\mathcal{M}$, and that the two notions of nested complexes coincide (Section 3). We exploit this correspondence in both directions: we recover our results on stellar subdivisions as reformulations of [5, 4], and we use our acyclonestohedra to get explicit polytopal realizations with integer coordinates for the all nested complexes over face lattices of realizable matroids.

Finally, Galashin's main motivation for poset associahedra was that they model compactifications of the space of order preserving maps $P \to \mathbb{R}$, which can be identified with the interior of an order polytope. We observe that results of [7] imply that acyclonestohedra are associated to nice compactifications of interiors of polytopes (Section 4).

Many details and all proofs are omitted in this extended abstract due to space limitations. We refer to the long version of this work which should soon become available.

1 Preliminaries

1.1 Building sets, nested complexes, and nestohedra

We start with the classical definitions of building sets, nested sets, nested complexes, and nestohedra from [10, 4, 6, 12] and their specializations to the graphical case [2].

Definition 1.1 ([10, 4, 6]). A *building set* on S is a set $\mathcal{B}$ of non-empty subsets of S such that

- $\mathcal{B}$ contains all singletons $\{s\}$ for $s \in S$, and
- if $B, B' \in \mathcal{B}$ and $B \cap B' \neq \varnothing$, then $B \cup B' \in \mathcal{B}$.

We denote by $\kappa(\mathcal{B})$ its set of *$\mathcal{B}$-connected components*, *i.e.*, its inclusion maximal elements.

Example 1.2 ([2]). Consider a graph G on S. A *tube* of G is a non-empty subset of S which induces a connected subgraph of G. The set $\mathcal{B}(G)$ of all tubes of G is a *graphical building set*. Moreover, the blocks of $\kappa(\mathcal{B}(G))$ are the connected components of G.

Remark 1.3 ([3]). More generally, a hypergraph H on S defines a building set $\mathcal{B}(H)$ on S given by all non-empty subsets of S which induce connected subhypergraphs of H. Any building set $\mathcal{B}$ on S is the building set of a hypergraph, but not always of a graph.

Definition 1.4 ([10, 4, 6]). A *nested set* is a subset $\mathcal{N}$ of $\mathcal{B}$ containing $\kappa(\mathcal{B})$ such that

- for any $B, B' \in \mathcal{N}$, either $B \subseteq B'$ or $B' \subseteq B$ or $B \cap B' = \varnothing$,
- for any $k \geq 2$ pairwise disjoint $B_1, \dots, B_k \in \mathcal{N}$, the union $B_1 \cup \dots \cup B_k$ is not in $\mathcal{B}$.

The *nested complex* of $\mathcal{B}$ is the simplicial complex $\mathfrak{N}(\mathcal{B})$ whose faces are $\mathcal{N} \smallsetminus \kappa(\mathcal{B})$ for all nested sets $\mathcal{N}$ on $\mathcal{B}$.

Example 1.5 ([2]). Consider a graph G on S. Two tubes t, t' of G are *compatible* if they are either nested (*i.e.*, $t \subseteq t'$ or $t' \subseteq t$), or disjoint and non-adjacent (*i.e.*, $t \cup t' \notin \mathcal{B}(G)$). A *tubing* on G is a set T of pairwise compatible tubes of G containing all connected components $\kappa(G)$. Tubings are precisely the nested sets of the graphical building set $\mathcal{B}(G)$. The nested complex $\mathfrak{N}(\mathcal{B}(G))$ is a *graphical nested complex*.

We now introduce restrictions and contractions of building sets. These operations are used to describe links of nested complexes [12, Prop. 3.2], and will be crucial here to define acyclic nested complexes.

Definition 1.6. For any $R \subseteq S$, define

- the *restriction* of $\mathcal{B}$ to R as the building set $\mathcal{B}_{|R} := \{B \in \mathcal{B} \mid B \subseteq R\}$ on R,
- the *contraction* of R in $\mathcal{B}$ as the building set $\mathcal{B}_{/R} := \{B \smallsetminus R \mid B \in \mathcal{B},\ B \not\subseteq R\}$ on $S \smallsetminus R$.

Example 1.7. For a graph G on S and $R \subseteq S$,

- $\mathcal{B}(G)_{|R} = \mathcal{B}(G_{|R})$ where $G_{|R}$ is the subgraph of G induced by R,
- $\mathcal{B}(G)_{/R} = \mathcal{B}(G_{/R})$ where $G_{/R}$ is the *reconnected complement* of R in G, *i.e.*, the graph on $S \smallsetminus R$ with an edge $\{r, s\}$ if there is a path between r and s in G with vertices in $R \cup \{r, s\}$, see [2].

Finally, we recall the definition of the nestohedron which realizes the nested complex. See for instance Figure 1. We denote by $(\boldsymbol{e}_s)_{s\in S}$ the standard basis of $\mathbb{R}^S$. For a building set $\mathcal{B}$, we denote by $\mathbb{R}^{\mathcal{B}}_{+} := \{\boldsymbol{\lambda} \in \mathbb{R}^{\mathcal{B}} \mid \lambda_B > 0 \text{ for all } B \in \mathcal{B} \text{ with } |B| \geq 2\}$.

Definition 1.8 ([10, 6])**.** For a building set $\mathcal{B}$ and a positive vector $\boldsymbol{\lambda} = (\lambda_B)_{B\in\mathcal{B}} \in \mathbb{R}^{\mathcal{B}}_{+}$, the *nestohedron* $\mathsf{Nest}(\mathcal{B}, \boldsymbol{\lambda})$ is the Minkowski sum $\sum_{B\in\mathcal{B}} \lambda_B \triangle_B$, where $\triangle_B := \operatorname{conv}\{\boldsymbol{e}_b \mid b \in B\}$ denotes the face of the standard simplex $\triangle_S$ corresponding to B.

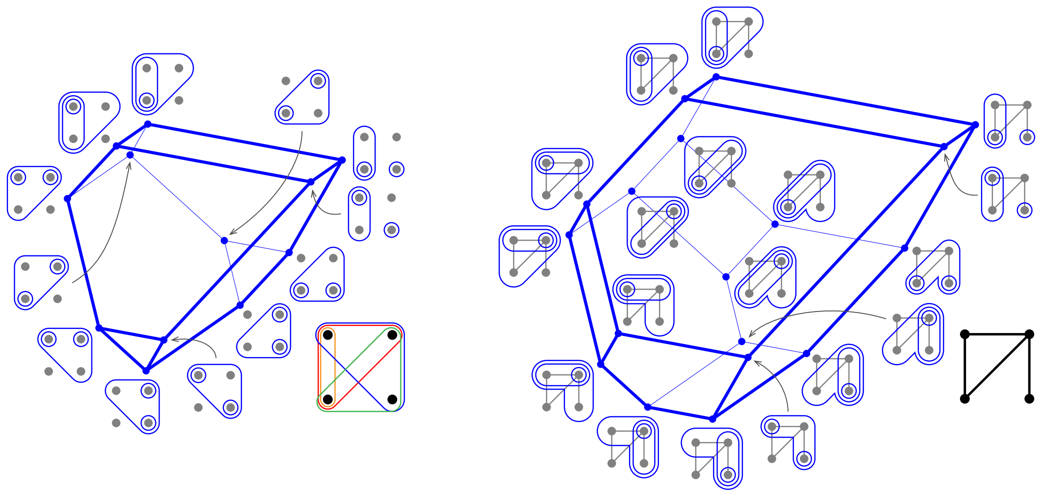

Figure 1: A nestohedron whose vertices are labeled by the corresponding maximal nested sets (left), and a graph associahedron whose vertices are labeled by the corresponding maximal tubings (right). The maximal block or tubing is always omitted.

Theorem 1.9 ([10, 6, 12])**.** *For a building set $\mathcal{B}$ and any $\boldsymbol{\lambda} \in \mathbb{R}^{\mathcal{B}}_{+}$, the nested complex $\mathfrak{N}(\mathcal{B})$ is isomorphic to the boundary complex of the polar of the nestohedron* $\mathsf{Nest}(\mathcal{B}, \boldsymbol{\lambda})$*.*

Proposition 1.10. *For $\boldsymbol{\lambda} \in \mathbb{R}^{\mathcal{B}}_{+}$, the vertex of the nestohedron* $\mathsf{Nest}(\mathcal{B}, \boldsymbol{\lambda})$ *corresponding to a maximal nested set $\mathcal{N}$ is*

$$\boldsymbol{v}(\mathcal{N}, \boldsymbol{\lambda}) = \sum_{s\in S} \sum_{B\in\mathcal{B},\, s\in B\subseteq B(v,\mathcal{N})} \lambda_B \boldsymbol{e}_s,$$

where $B(s, \mathcal{N})$ denotes the inclusion minimal block of $\mathcal{N}$ containing s.

Proposition 1.11. *For $\boldsymbol{\lambda} \in \mathbb{R}^{\mathcal{B}}_{+}$, the nestohedron* $\mathsf{Nest}(\mathcal{B}, \boldsymbol{\lambda})$ *is given by the equalities $g_B(\boldsymbol{x}) = 0$ for all $B \in \kappa(\mathcal{B})$ and the inequalities $g_B(\boldsymbol{x}) \geq 0$ for all $B \in \mathcal{B}$, where*

$$g_B(\boldsymbol{x}) := \Big\langle \sum_{b\in B} \boldsymbol{e}_b \;\Big|\; \boldsymbol{x} \Big\rangle - \sum_{B'\in\mathcal{B},\, B'\subseteq B} \lambda_{B'}.$$

Example 1.12. For a graph G on S, the nestohedra of $\mathcal{B}(G)$ are the *graph associahedra* of G, introduced in [2]. For instance, the associahedron of the complete graph is a permutahedron, the associahedron of a path graph is an associahedron, and the associahedron of a cycle graph is a cyclohedron.

1.2 Vector configurations and oriented matroids

We now recall some aspects of oriented matroids. We only give the precise definition for those associated to vector configurations and refer to [1] for the general definition.

Definition 1.13. For a finite vector configuration $\boldsymbol{A} := (\boldsymbol{a}_s)_{s\in S} \in (\mathbb{R}^d)^S$, we denote by

- $\mathcal{D}(\boldsymbol{A}) := \{\boldsymbol{\delta} \in \mathbb{R}^S \mid \sum_{s\in S} \delta_s \boldsymbol{a}_s = \boldsymbol{0}\}$ the space of linear *dependences* on $\boldsymbol{A}$,
- $\mathcal{D}^*(\boldsymbol{A}) := \{(f(\boldsymbol{a}_s))_{s\in S} \in \mathbb{R}^S \mid f \in (\mathbb{R}^d)^*\}$ the space of *evaluations* of linear forms on $\boldsymbol{A}$.

Note that, $\mathcal{D}^*(\boldsymbol{A})$ and $\mathcal{D}(\boldsymbol{A})$ are orthogonal spaces whose dimensions are the *rank* $\mathrm{rk}(\boldsymbol{A})$ and the he *corank* $\mathrm{rk}^*(\boldsymbol{A}) := |S| - \mathrm{rk}(\boldsymbol{A})$ of $\boldsymbol{A}$ respectively.

Notation 1.14. Define $\sigma(S) := \{(x_+, x_-) \mid x_+, x_- \subseteq S \text{ and } x_+ \cap x_- = \varnothing\}$. The *signature* of $\boldsymbol{\delta} \in \mathbb{R}^S$ is $\sigma(\boldsymbol{\delta}) := (\{s \in S \mid \delta_s > 0\}, \{s \in S \mid \delta_s < 0\})$ in $\sigma(S)$. For $x = (x_+, x_-) \in \sigma(S)$, we define the *support* of x by $\underline{x} := x_+ \cup x_-$, and the *opposite* of x by $-x := (x_-, x_+)$.

Definition 1.15. The *oriented matroid* $\mathcal{M}(\boldsymbol{A})$ of a finite vector configuration $\boldsymbol{A} \subset \mathbb{R}^d$ is the combinatorial data given equivalently by

- the *vectors* $\mathcal{V}(\boldsymbol{A})$ of $\boldsymbol{A}$, *i.e.,* signatures of linear dependences of $\boldsymbol{A}$,
- the *covectors* $\mathcal{V}^*(\boldsymbol{A})$ of $\boldsymbol{A}$, *i.e.,* signatures of linear evaluations on $\boldsymbol{A}$,
- the *circuits* $\mathcal{C}(\boldsymbol{A})$ of $\boldsymbol{A}$, *i.e.,* support minimal signatures of linear dependences of $\boldsymbol{A}$,
- the *cocircuits* $\mathcal{C}^*(\boldsymbol{A})$ of $\boldsymbol{A}$, *i.e.,* support minimal signatures of linear evaluations on $\boldsymbol{A}$.

Example 1.16. Consider a directed graph D with vertex set V and arc set S (loops and multiple arcs are allowed). Let $(\boldsymbol{b}_v)_{v\in V}$ denote the standard basis of $\mathbb{R}^V$. The *incidence configuration* $\boldsymbol{A}_D$ of D has a vector $\boldsymbol{a}_{(u,v)} := \boldsymbol{b}_u - \boldsymbol{b}_v \in \mathbb{R}^V$ for each arc (u, v) of D. Its oriented matroid, whose ground set is the set S of arcs of D, is the *graphical oriented matroid* $\mathcal{M}(D)$ of D. See [9, Prop. 1.1.7 & Chap. 5] and [1, Sect. 1.1].

In this paper, we will consider abstract oriented matroids, which are combinatorial abstractions for the dependences and evaluations of vector configurations considered in Definitions 1.13 and 1.15. We rest on [1] to avoid the detailed axioms.

Definition 1.17. An *oriented matroid* on S is the combinatorial data $\mathcal{M}$ given by four subsets of $\sigma(S)$, the *vectors* $\mathcal{V}(\mathcal{M})$, *covectors* $\mathcal{V}^*(\mathcal{M})$, *circuits* $\mathcal{C}(\mathcal{M})$ and *cocircuits* $\mathcal{C}^*(\mathcal{M})$, which satisfy the axioms of [1, Sect. 3].

Definition 1.18. An oriented matroid $\mathcal{M}$ is *realizable* if there exists a vector configuration $\boldsymbol{A} := (\boldsymbol{a}_s)_{s\in S} \in (\mathbb{R}^d)^S$ such that $\mathcal{V}(\mathcal{M}) = \mathcal{V}(\boldsymbol{A})$, $\mathcal{V}^*(\mathcal{M}) = \mathcal{V}^*(\boldsymbol{A})$, $\mathcal{C}(\mathcal{M}) = \mathcal{C}(\boldsymbol{A})$, and $\mathcal{C}^*(\mathcal{M}) = \mathcal{C}^*(\boldsymbol{A})$ (these four conditions are actually equivalent).

Definition 1.19. An oriented matroid $\mathcal{M}$ is acyclic if it has no positive circuit.

Example 1.20. A realizable oriented matroid $\mathcal{M}(\boldsymbol{A})$ is acyclic if and only if $\boldsymbol{A}$ has no positive dependence, *i.e.,* if and only if $\boldsymbol{A}$ is contained in a positive linear half-space of $\mathbb{R}^d$. A graphical oriented matroid $\mathcal{M}(D)$ is acyclic if and only if D is acyclic (no directed cycle).

Definition 1.21. Let $\mathcal{M}$ be an acyclic oriented matroid. A set $F \subseteq S$ is a *face* of $\mathcal{M}$ if it is the complement of a non-negative covector, *i.e.*, $(S \smallsetminus F, \varnothing) \in \mathcal{V}^*(\mathcal{M})$. The *Las Vergnas face lattice* $\mathcal{F}(\mathcal{M})$ is the poset of faces of $\mathcal{M}$ ordered by inclusion.

We conclude with restrictions and contractions in oriented matroids.

Definition 1.22. For any $R \subseteq S$, define

- the *restriction* $\mathcal{M}_{|R}$ as the oriented matroid on R with circuits $\{c \in \mathcal{C}(\mathcal{M}) \mid \underline{c} \subseteq R\}$,
- the *contraction* $\mathcal{M}_{/R}$ as the oriented matroid on $S \smallsetminus R$ with vectors $\{v \smallsetminus R \mid v \in \mathcal{V}(\mathcal{M})\}$, where $v \smallsetminus R := (v_+ \smallsetminus R, v_- \smallsetminus R)$.

Example 1.23. For a vector configuration $\boldsymbol{A} := (\boldsymbol{a}_s)_s \in S$ and $R \subseteq S$,

- $\mathcal{M}(\boldsymbol{A})_{|R} = \mathcal{M}(\boldsymbol{A}_{|R})$ where $\boldsymbol{A}_{|R}$ is the vector subconfiguration $(\boldsymbol{a}_r)_{r \in R}$,
- $\mathcal{M}(\boldsymbol{A})_{/R} = \mathcal{M}(\boldsymbol{A}_{/R})$ where $\boldsymbol{A}_{/R}$ is the vector configuration obtained by projecting the vectors $\boldsymbol{a}_s$ with $s \notin R$ on the space orthogonal to all vectors $\boldsymbol{a}_r$ with $r \in R$.

For a directed graph D and a subset R of arcs of D,

- $\mathcal{M}(D)_{|R} = \mathcal{M}(D_{|R})$ where $D_{|R}$ is the subgraph of D formed by the arcs in R,
- $\mathcal{M}(D)_{/R} = \mathcal{M}(D_{/R})$ where $D_{/R}$ is the contraction of the arcs of R in D.

2 Acyclic nested complexes and acyclonestohedra

2.1 Oriented building sets

Definition 2.1. An *oriented building set* is a pair $(\mathcal{B}, \mathcal{M})$ where $\mathcal{B}$ is a building set and $\mathcal{M}$ is an oriented matroid on the same ground set S such that $\underline{c} \in \mathcal{B}$ for any $c \in \mathcal{C}(\mathcal{M})$. We say that $(\mathcal{B}, \mathcal{M})$ is *realizable* if $\mathcal{M}$ is realizable.

Example 2.2. Consider a directed graph D with vertex set V and arc set S. The *line graph* of D is the graph $L(D)$ on S with an edge between two arcs of D if and only if they share an endpoint. The *graphical oriented building set* of D is the pair $(\mathcal{B}(L(D)), \mathcal{M}(D))$. Note that it is indeed an oriented building set: S is the ground set of both $\mathcal{B}(L(D))$ and $\mathcal{M}_D$, and the circuits in $\mathcal{M}(D)$ are cycles in D, hence of $L(D)$, thus belong to $\mathcal{B}(L(D))$.

Lemma 2.3. *If $(\mathcal{B}, \mathcal{M})$ is an oriented building set on S and $R \subseteq S$, then both $(\mathcal{B}_{|R}, \mathcal{M}_{|R})$ and $(\mathcal{B}_{/R}, \mathcal{M}_{/R})$ are oriented building sets on R and $S \smallsetminus R$ respectively.*

Definition 2.4. Given an oriented building set $(\mathcal{B}, \mathcal{M})$, a nested set $\mathcal{N}$ on $\mathcal{B}$ and $B \in \mathcal{N}$, we consider the oriented buiding set $(\mathcal{B}, \mathcal{M})_{B \in \mathcal{N}} := (\mathcal{B}_{B \in \mathcal{N}}, \mathcal{M}_{B \in \mathcal{N}})$ on $S_{B \in \mathcal{N}} := B \smallsetminus R$ defined by $\mathcal{B}_{B \in \mathcal{N}} := (\mathcal{B}_{|B})_{/R}$ and $\mathcal{M}_{B \in \mathcal{N}} := (\mathcal{M}_{|B})_{/R}$, where $R = R_{B \in \mathcal{N}} := \bigcup_{B' \in \mathcal{N},\, B' \subsetneq B} C$.

Example 2.5. Consider the graphical oriented building set of a directed graph D of Example 2.2, and a tube t in a tubing T of $L(D)$. The oriented building set $(\mathcal{B}(L(D)), \mathcal{M}(D))_{t \in T}$ is the graphical oriented building set of the directed graph obtained as the contraction in the restriction $D_{|t}$ of all arcs contained in some tube $s \in T$ with $s \subsetneq t$.

2.2 Acyclic nested complexes

Definition 2.6. The *acyclic nested complex* of an oriented building set $(\mathcal{B}, \mathcal{M})$ is the simplicial complex $\mathfrak{A}(\mathcal{B}, \mathcal{M})$ whose faces are $\mathcal{N} \smallsetminus \kappa(\mathcal{B})$ for all nested sets $\mathcal{N}$ of $\mathcal{B}$ such that $\mathcal{M}_{B \in \mathcal{N}}$ is acyclic for every $B \in \mathcal{N}$.

Remark 2.7. Observe that:

- For any building set $\mathcal{B}$ on S, the nested complex $\mathfrak{N}(\mathcal{B})$ is the acyclic nested complex $\mathfrak{A}(\mathcal{B}, \mathcal{I})$, where $\mathcal{I}$ is the independent (*i.e.*, no circuit) oriented matroid on S.
- If $\mathcal{M}$ is not acyclic, then the acyclic nested complex $\mathfrak{A}(\mathcal{B}, \mathcal{M})$ is empty.
- If $\mathcal{M}$ contains a circuit $c = (c_+, c_-)$ with $|c_-| = 1$, then $\mathfrak{A}(\mathcal{B}, \mathcal{M})$ is isomorphic to $\mathfrak{A}(\mathcal{B}_{|S \smallsetminus \{s\}}, \mathcal{M}_{|S \smallsetminus \{s\}})$.

Example 2.8. From Example 2.2, consider a directed graph D and its graphical oriented building set $(\mathcal{B}(L(D)), \mathcal{M}(D))$. The *graphical acyclic nested complex* $\mathfrak{A}(\mathcal{B}(L(D)), \mathcal{M}(D))$ is then given by all tubings T on $L(D)$ such that for each tube $t \in T$, the contraction in the restriction $D_{|t}$ of all arcs contained in some tube $s \in T$ with $s \subsetneq t$ yields an acyclic directed graph. Figure 2 illustrates two graphical acyclic nested complexes. Note that these two directed graphs have the same line graph, but distinct graphical oriented matroids, and thus distinct graphical acyclic nested complexes.

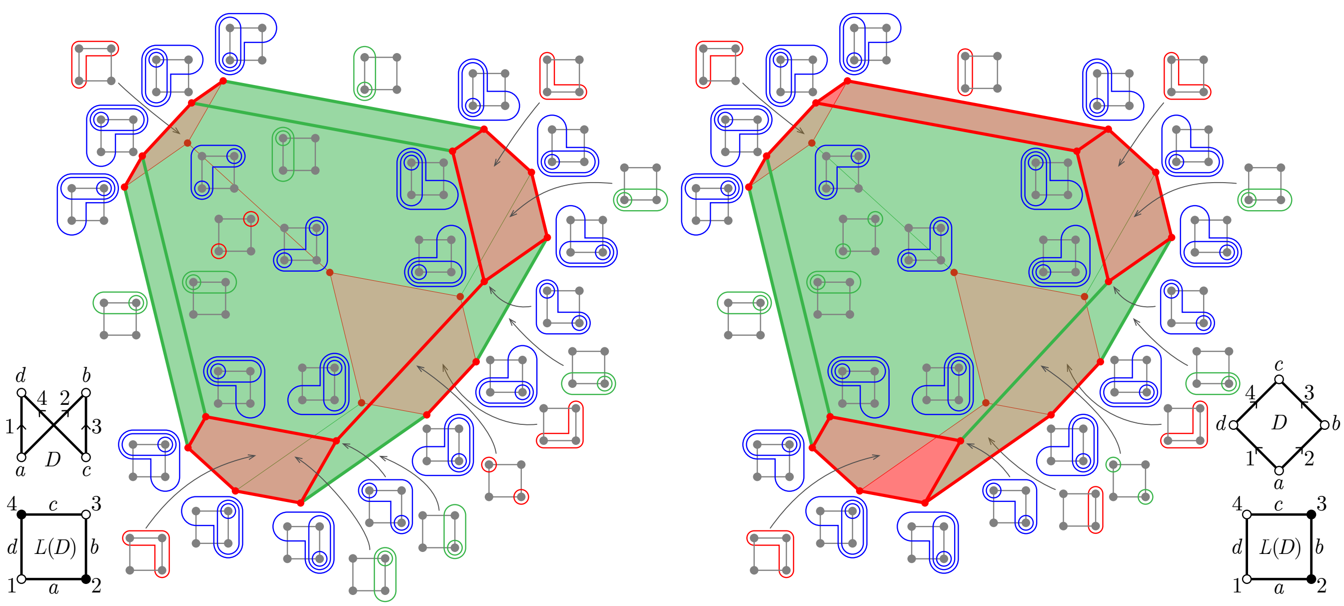

Figure 2: Two graphical acyclic nested complexes. For each one, we have drawn the directed graph D, its line graph $L(D)$ with vertices colored black and white according to the sign of the corresponding arcs in the only circuit of D, and all tubings of $L(D)$ labeling the faces of the graph associahedron, colored green if acyclic and red if cyclic.

Remark 2.9. It follows from Remark 2.7 that the graphical acyclic nested complex of D is

- isomorphic to the classical nested complex of the line graph $L(D)$ when D is an oriented forest (for instance, it is isomorphic to the simplicial permutahedron if D is a star, and to the simplicial associahedron if D is a path),
- empty if D is cyclic (*i.e.,* has an oriented cycle),
- isomorphic to the graphical acyclic nested complex of the Hasse diagram of the transitive closure of D if D is acyclic.

Hence, graphical acyclic nested complexes are in fact intrinsically associated to posets. The graphical case of Examples 2.2 and 2.8 actually motivated Definitions 2.1 and 2.6, and was inspired from the poset associahedra defined in [8]. The following statement can serve as definition of poset associahedra, which we omit here for space reason.

Proposition 2.10. *The poset associahedron of a finite poset P defined in [8] is isomorphic to the graphical acyclic nested set of the Hasse diagram of P.*

We note that affine poset associahedra of [8] are also acyclic nested complexes of certain specific oriented building sets, although their definition is slightly more intricate.

2.3 Stellar subdivisions

We now show that the acyclic nested complex of any oriented building set (realizable or not) is always the face lattice of an oriented matroid, hence a topological sphere [1, Thm. 4.3.5]. The main tool here is that of stellar subdivisions.

Definition 2.11. For a cell σ in a regular cell complex Δ, the *stellar subdivision* $\mathrm{sd}(\Delta, \sigma)$ is the cell complex obtained by gluing the cone $s * (\overline{\mathrm{star}}(\sigma, \Delta) \smallsetminus \mathrm{star}(\sigma, \Delta))$ to $\Delta \smallsetminus \mathrm{star}(\sigma, \Delta)$ along $\overline{\mathrm{star}}(\sigma, \Delta) \smallsetminus \mathrm{star}(\sigma, \Delta)$, where s is a new vertex, and $\mathrm{star}(\sigma, \Delta) := \{\tau \in \Delta \mid \sigma \subseteq \tau\}$ is the star of σ and $\overline{\mathrm{star}}(\sigma, \Delta) := \{\rho \in \Delta \mid \rho \subseteq \tau \text{ for some } \tau \in \mathrm{star}(\sigma, \Delta)\}$ is its closure.

Proposition 2.12 ([1, Prop. 9.2.3 & Sect. 7.2])**.** *Let $\mathcal{M}$ be an acyclic oriented matroid with ground set S, and F be one of its proper faces. Then the face lattice of the stellar subdivision $\mathrm{sd}(\Delta(\mathcal{M}), F)$ is isomorphic to the face lattice of an oriented matroid on $S \cup \{F\}$ (this oriented matroid is not unique, but its face lattice is). Moreover, this oriented matroid can be chosen to be realizable when $\mathcal{M}$ is realizable.*

Theorem 2.13. *For any oriented building set $(\mathcal{B}, \mathcal{M})$ (realizable or not), the acyclic nested complex $\mathfrak{A}(\mathcal{B}, \mathcal{M})$ is the face lattice of an oriented matroid, obtained by stellar subdivisions of $\mathcal{M}$.*

Corollary 2.14. *For any realizable oriented building set $(\mathcal{B}, \mathcal{M}(\boldsymbol{A}))$, the acyclic nested complex $\mathfrak{A}(\mathcal{B}, \mathcal{M}(\boldsymbol{A}))$ is isomorphic to the boundary complex of a convex polytope, obtained by stellar subdivisions of the positive tope of $\boldsymbol{A}$.*

Example 2.15. In the graphical situation discussed in Examples 2.2 and 2.8, Remark 2.9, and Proposition 2.10, we obtain that the poset associahedron of a poset P can be realized as a stellar subdivision of the order polytope of P, thus recovering the construction of [8].

2.4 Acyclonestohedra

We now consider a realizable oriented building set $(\mathcal{B}, \mathcal{M}(\boldsymbol{A}))$. From Corollary 2.14, we know that the acyclic nested complex $\mathfrak{A}(\mathcal{B}, \mathcal{M}(\boldsymbol{A}))$ is realizable as a polytope by stellar subdivisions of the positive tope of $\boldsymbol{A}$. However, this non-explicit approach does not allow any control on the coordinates of the realizations. In this section, we obtain explicit polytopal realizations with controlled integer coordinates, using sections of nestohedra.

Definition 2.16. As each $c \in \mathcal{C}(\boldsymbol{A})$ is the signature of a unique (up to rescaling) linear dependence $\delta \in \mathcal{D}(\boldsymbol{A})$, we define $r_c := \max \delta^{\neq 0} / \min \delta^{\neq 0}$ where $\delta^{\neq 0} := \{|\delta_s| \mid s \in S\} \smallsetminus \{0\}$ and $R := |\mathcal{B}| \cdot \max_{c \in \mathcal{C}(\boldsymbol{A})} r_c$. We then define $\boldsymbol{\rho} := (\rho_B)_{B \in \mathcal{B}} \in \mathbb{R}_+^{\mathcal{B}}$ by $\rho_B := 0$ if $|B| = 1$ and $\rho_B := R^{|B|}$ if $|B| \geq 2$.

We use these coefficients $\boldsymbol{\rho} \in \mathbb{R}_+^{\mathcal{B}}$ to define two polytopes $\mathsf{Acyc}(\mathcal{B}, \boldsymbol{A})$ and $\mathsf{A}\overline{\mathsf{cyc}}(\mathcal{B}, \boldsymbol{A})$ that we both call *acyclonestohedra*. While these two polytopes are affinely equivalent, the first is more natural for our construction, but the second has the advantage to live in the right dimensional space.

Definition 2.17. The *acyclonestohedron* $\mathsf{Acyc}(\mathcal{B}, \boldsymbol{A})$ is the polytope in $\mathbb{R}^S$ defined as the intersection of the nestohedron $\mathsf{Nest}(\mathcal{B}, \boldsymbol{\rho})$ with the evaluation space $\mathcal{D}^*(\boldsymbol{A})$ of $\boldsymbol{A}$.

Definition 2.18. The *acyclonestohedron* $\mathsf{A}\overline{\mathsf{cyc}}(\mathcal{B}, \boldsymbol{A})$ is the polytope of $\mathbb{R}\boldsymbol{A}$ defined by the equalities $\overline{g}_B(\boldsymbol{y}) = 0$ for all $B \in \kappa(\mathcal{B})$ and the inequalities $\overline{g}_B(\boldsymbol{y}) \geq 0$ for all $B \in \mathcal{B}$, where

$$\overline{g}_B(\boldsymbol{y}) := \Big\langle \sum_{b \in B} \boldsymbol{a}_b \;\Big|\; \boldsymbol{y} \Big\rangle - \sum_{B' \in \mathcal{B},\, B' \subseteq B} \rho_{B'}.$$

Proposition 2.19. *The acyclonestohedron* $\mathsf{Acyc}(\mathcal{B}, \boldsymbol{A}) \subset \mathbb{R}^S$ *of Definition 2.17 and the acyclonestohedron* $\mathsf{A}\overline{\mathsf{cyc}}(\mathcal{B}, \boldsymbol{A}) \subset \mathbb{R}\boldsymbol{A}$ *of Definition 2.18 are affinely equivalent.*

Theorem 2.20. *For any realizable oriented building set* $(\mathcal{B}, \mathcal{M}(\boldsymbol{A}))$*, the acyclic nested complex* $\mathfrak{A}(\mathcal{B}, \mathcal{M}(\boldsymbol{A}))$ *is isomorphic to the boundary complex of the polar of the acyclonestohedron* $\mathsf{Acyc}(\mathcal{B}, \boldsymbol{A})$ *(or equivalently of* $\mathsf{A}\overline{\mathsf{cyc}}(\mathcal{B}, \boldsymbol{A})$*).*

Remark 2.21. Following Remark 2.7, note that if $\boldsymbol{A}$ is linearly independent, then its evaluation space $\mathcal{D}^*(\boldsymbol{A})$ is $\mathbb{R}^S$, and the acyclonestohedra $\mathsf{Acyc}(\mathcal{B}, \boldsymbol{A})$ and $\mathsf{A}\overline{\mathsf{cyc}}(\mathcal{B}, \boldsymbol{A})$ both coincide with the classical nestohedron $\mathsf{Nest}(\mathcal{B}, \boldsymbol{\rho})$. For instance, the acyclonestohedron of the graphical oriented building set of an oriented forest D is the graph associahedron of $L(D)$ (for instance, a permutahedron if D is a star, and an associahedron if D is a path).

Example 2.22. Specializing Definitions 2.17 and 2.18 and Theorem 2.20 to the graphical situation discussed in Examples 2.2 and 2.8, Remark 2.9, and Proposition 2.10, we obtain that the poset associahedron of a poset P with Hasse diagram D is explicitly realized as

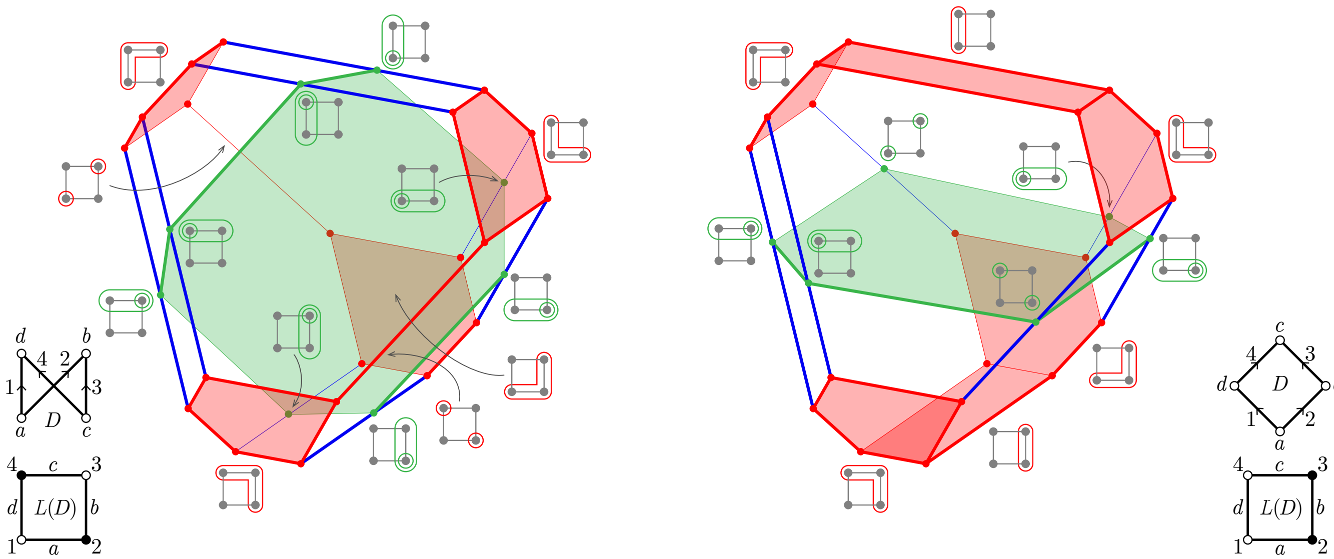

Figure 3: The graphical acyclonestohedra (green polygons) realizing the graphical acyclic nested complexes of Figure 2, obtained as the section of the line graph of $L(D)$ by the evaluation space of the graphical oriented matroid of D.

- the section of a graph associahedron of the line graph of D with the linear hyperplanes normal to $\mathbb{1}_{c_+} - \mathbb{1}_{c_-}$ for all circuits $c = (c_+, c_-)$ of D, see Figure 3,
- the polytope in $\mathbb{R}^P$ defined by the equality $\overline{g}_P(\boldsymbol{y}) = 0$ and the inequalities $\overline{g}_t(\boldsymbol{y}) \geq 0$ for all $t \in \mathcal{B}(\mathrm{P})$, where

$$\overline{g}_t(\boldsymbol{y}) := \Big\langle \sum_{\substack{p,q \in t \\ p \prec q}} \boldsymbol{b}_p - \boldsymbol{b}_q \;\Big|\; \boldsymbol{y} \Big\rangle - \sum_{\substack{t' \in \mathcal{B}(\mathrm{P}) \\ t' \subseteq t}} |\mathcal{B}(\mathrm{P})|^{|t|}.$$

This answers an open question of [8]. During the completion of this paper, we became aware that this question was independently solved in [11]. The approach of [11] is quite different but leads essentially to the same realization of poset associahedra. We actually want to acknowledge that we originally only worked with the acyclonestohedron of Definition 2.17, and that the affinely equivalent acyclonestohedron of Definition 2.18 was motivated by the approach of [11].

Remark 2.23. To conclude this section, we want to give a vague idea of the proof of Theorem 2.20. As illustrated in Figures 2 and 3, the main point is that our choice of coefficients $\boldsymbol{\rho}$ guaranties that a face of $\mathsf{Nest}(\mathcal{B}, \boldsymbol{\rho})$ intersects the evaluation space $\mathcal{D}^*(\boldsymbol{A})$ if and only if the corresponding nested set on $\mathcal{B}$ is acyclic for $\mathcal{M}(\boldsymbol{A})$. Note that the coefficients could sometimes be chosen smaller, our exponential choice is just a convenient hammer to kill all small contributions. For instance, for graphical oriented building sets, the coefficient ρ_t of a tube t can in fact be chosen of order $4^{|t|}$.

3 Nested complexes of building sets of the face lattice

Although less popular in the combinatorics community, the original definition of [4] for building sets and their nested complexes depends upon an underlying lattice.

Definition 3.1 ([4, 6])**.** A subset $\mathcal{B}$ of a finite lattice $\mathcal{L}$ is a *$\mathcal{L}$-building set* if the lower interval of any element $x \in \mathcal{L}$ is the direct product of the lower intervals of the maximal elements of $\mathcal{B}$ below x. We denote by $\kappa(\mathcal{B}) := \max(\mathcal{B})$ the set of *$\mathcal{B}$-connected components*.

Definition 3.2 ([4, 6])**.** Let $\mathcal{B}$ be an $\mathcal{L}$-building set. An *$\mathcal{L}$-nested set* $\mathcal{N}$ on $\mathcal{B}$ is a subset of $\mathcal{B}$ containing $\kappa(\mathcal{B})$ and such that for any $k \geq 2$ pairwise incomparable elements $B_1, \dots, B_k \in \mathcal{N}$, the join $B_1 \vee \dots \vee B_k$ does not belong to $\mathcal{B}$. The *$\mathcal{L}$-nested complex* of $\mathcal{B}$ is the simplicial complex $\mathfrak{N}_{\mathcal{L}}(\mathcal{B})$ whose faces are $\mathcal{N} \smallsetminus \kappa(\mathcal{B})$ for all $\mathcal{L}$-nested sets $\mathcal{N}$ on $\mathcal{B}$.

Example 3.3. If $\mathcal{L}$ is the boolean lattice on S, then the $\mathcal{L}$-building sets are the building sets on S of Definition 1.1 and the $\mathcal{L}$-nested sets are the nested sets of Definition 1.4.

We will use these definitions over the Las Vergnas face lattice $\mathcal{F}(\mathcal{M})$ of the oriented matroid $\mathcal{M}$, see Definition 1.21. We first select the facial part of an oriented building set.

Definition 3.4. The *facial building set* $\widehat{\mathcal{B}}$ of an oriented building set $(\mathcal{B}, \mathcal{M})$ is the set of blocks $B \in \mathcal{B}$ that are also faces of $\mathcal{M}$.

Theorem 3.5. *The facial building sets of $\mathcal{M}$ coincide with the $\mathcal{F}(\mathcal{M})$-building sets.*

Definition 3.6. The *facial nested complex* $\mathfrak{N}_{\mathcal{F}(\mathcal{M})}(\widehat{\mathcal{B}})$ is the $\mathcal{F}(\mathcal{M})$-nested complex of $\widehat{\mathcal{B}}$.

Theorem 3.7. *Let $\widehat{\mathcal{B}}$ be the facial building set of an oriented building set $(\mathcal{B}, \mathcal{M})$. Then the acyclic nested complex $\mathfrak{A}(\mathcal{B}, \mathcal{M})$ and the facial nested complex $\mathfrak{N}_{\mathcal{F}(\mathcal{M})}(\widehat{\mathcal{B}})$ coincide.*

Example 3.8. If $\mathcal{M}$ is independent (*i.e.,* no circuit), then its positive tope is a simplex, its Las Vergnas face lattice is boolean, so that we are in the classical situation of Example 3.3.

We conclude with a few remarks in light of Theorems 3.5 and 3.7. First, we observe that this interpretation actually recovers the results of Section 2.3. Namely,

- [5, Cor. 4.3] proved that the nested complex of a finite atomic meet-semilattice is homeomorphic to its order complex. Since the face lattices of oriented matroids encode face lattices of regular cell decompositions of spheres [1, Thm. 4.3.5], their order complexes are the face lattices of the barycentric subdivisions of these spheres.
- The stellar subdivisions of Theorem 2.13 are actually oriented matroid realizations of the combinatorial blowups of [4, Thm. 3.4] on face lattices of oriented matroids.

In turn, our acyclonestohedra of Section 2.4 provide explicit polytopal realizations with integer coordinates for the $\mathcal{F}(\mathcal{M})$-nested complexes over realizable matroids. To sum up:

Corollary 3.9. *Any $\mathcal{F}(\mathcal{M})$-nested complex of any $\mathcal{F}(\mathcal{M})$-building set over the face lattice $\mathcal{F}(\mathcal{M})$ of an acyclic oriented matroid $\mathcal{M}$ is the face lattice of an oriented matroid obtained by stellar subdivisions of the positive tope of $\mathcal{M}$. When $\mathcal{M}$ is realizable, it can be realized as a polytope either by realizing these stellar subdivisions polytopaly, or as the polar of a section of a nestohedron.*

4 Compactifications

Galashin's main motivation for defining poset associahedra was that they model compactifications of the space of order preserving maps $P \to \mathbb{R}$, which can be identified with the interior of an order polytope. In fact, the connection above reveals that all acyclonestohedra are associated to nice compactifications of interiors of polytopes, via [7].

Theorem 4.1 ([7]). *Consider a realizable oriented building set $(\mathcal{B}, \mathcal{M}(\boldsymbol{A}))$, and let P be the polytope associated to the positive tope. Then there is a compactification $P^{\mathcal{B}}$ of the interior of P that is a stratified C^∞ manifold with corners such that*

(i) except for the open dense stratum, all the strata lie in the boundary,

(ii) the codimension 1 strata are in correspondence with the facial blocs of $\widehat{\mathcal{B}}$,

(iii) the intersection of the closures of the strata indexed by a subset $\mathcal{N} \subseteq \widehat{\mathcal{B}}$ is non empty if and only if $\mathcal{N}$ is a $\mathcal{F}(\mathcal{M})$-nested set,

(iv) the strata of $P^{\mathcal{B}}$ can be indexed by the faces of the acyclic nested complex $\mathfrak{A}(\mathcal{B}, \mathcal{M}(\boldsymbol{A}))$.

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