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Cluster algebras and tilings for the $m = 4$ amplituhedron

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Abstract. The *amplituhedron* $\mathcal{A}^Z_{n,k,m}$ is the image of the positive Grassmannian $\mathrm{Gr}^{\geq 0}_{k,n}$ under the map $\tilde{Z} : \mathrm{Gr}^{\geq 0}_{k,n} \to \mathrm{Gr}_{k,k+m}$ induced by a positive linear map $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$. It was originally introduced in physics in order to give a geometric interpretation of scattering amplitudes. More specifically, one can compute scattering amplitudes in $\mathcal{N} = 4$ SYM by 'tiling' the $m = 4$ amplituhedron $\mathcal{A}^Z_{n,k,4}$ — that is, decomposing $\mathcal{A}^Z_{n,k,4}$ into 'tiles' (closures of images of $4k$-dimensional cells of $\mathrm{Gr}^{\geq 0}_{k,n}$ on which $\tilde{Z}$ is injective). In this article we deepen both our understanding of tiles and tilings of the $m = 4$ amplituhedron and the connection with cluster algebras. Firstly, we prove the *cluster adjacency conjecture* for BCFW tiles of $\mathcal{A}^Z_{n,k,4}$, which says that facets of tiles are cut out by collections of compatible cluster variables for $\mathrm{Gr}_{4,n}$. Secondly, we describe each BCFW tile as the semialgebraic set in $\mathrm{Gr}_{k,k+4}$ where certain cluster variables have particular signs. Finally, we prove the *BCFW tiling conjecture*, which says that any way of iterating the BCFW recurrence gives rise to a tiling of the amplituhedron $\mathcal{A}^Z_{n,k,4}$. Along the way, we introduce a method to construct seeds for $\mathrm{Gr}_{4,n}$ comprised of high-degree cluster variables, which may be of independent interest in the study of cluster algebras.

Keywords: positroid, amplituhedron, cluster algebras, tile, tiling, BCFW

1 Introduction

The (tree) *amplituhedron* $\mathcal{A}^Z_{n,k,m}$ is the image of the positive Grassmannian $\mathrm{Gr}^{\geq 0}_{k,n}$ under the *amplituhedron map* $\tilde{Z} : \mathrm{Gr}^{\geq 0}_{k,n} \to \mathrm{Gr}_{k,k+m}$. It was introduced by Arkani-Hamed and Trnka [4] in order to give a geometric interpretation of *scattering amplitudes* in $\mathcal{N} = 4$ super Yang Mills theory (SYM): in particular, one can compute $\mathcal{N} = 4$ SYM scattering amplitudes

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by 'tiling' the $m = 4$ amplituhedron $\mathcal{A}^Z_{n,k,4}$ — that is, by decomposing the amplituhedron into smaller 'tiles' — and summing the 'volumes' of the tiles. While the case $m = 4$ is most important for physics, the amplituhedron is defined for any positive n, k, m with $k + m \leq n$, and has a very rich geometric and combinatorial structure. It generalizes cyclic polytopes (when $k = 1$), cyclic hyperplane arrangements [19] (when $m = 1$), and the positive Grassmannian (when $k = n - m$), and it is connected to the hypersimplex and the positive tropical Grassmanian [23, 26] (when $m = 2$). The amplituhedron is also an example of a *Grassmann polytope* ('Grasstope') and conjectured to be a *positive geometry* [1, 21]. The followings are two of the guiding problems about the amplituhedron.

The first is the *cluster adjacency conjecture*, which says that facets of tiles are cut out by collections of compatible cluster variables. This was motivated by physics where cluster algebras were shown to describe singularities of scattering amplitudes in $\mathcal{N} = 4$ SYM [16]. In particular, [7, 8] conjectured that the terms in tree-level amplitudes coming from the BCFW recursions are rational functions whose poles correspond to compatible cluster variables of the Grassmannian $\mathrm{Gr}_{4,n}$, see also [25]. The cluster adjacency conjecture, formulated for the $m = 2$ and $m = 4$ amplituhedron in [22] and [17], was proved for all tiles of the $m = 2$ amplituhedron in [26].

The second is the *BCFW tiling conjecture*, which says that any way of iterating the BCFW recurrence gives rise to a collection of cells whose images tile the $m = 4$ amplituhedron $\mathcal{A}^Z_{n,k,4}$. This arose alongside the definition of the amplituhedron [4] in order to give a geometric interpretation of the recurrence Britto–Cachazo–Feng–Witten [6] introduced to compute scattering amplitudes. BCFW-like tilings of the $m = 1$ and $m = 2$ amplituhedron were proved in [19] and [5], building on [3] and [20]. Finally, extending the work of [20], it was proved in [10] that the 'standard' way of performing the BCFW recursion gives a tiling for the $m = 4$ amplituhedron.

Main results. In this paper we build on [26] and [10] to give a very complete picture of the $m = 4$ amplituhedron. We show that arbitrary *BCFW cells* give tiles (Theorem 3.5) and that they satisfy the cluster adjacency conjecture (Theorem 3.15). We strengthen the connection with cluster algebras by associating to each BCFW tile a collection of compatible cluster variables for $\mathrm{Gr}_{4,n}$ (Definition 3.11), which we use to describe the tile as a semialgebraic set in $\mathrm{Gr}_{k,k+4}$ (Theorem 3.13). For 'standard' BCFW tiles, one can also give a non-recursive description of these cluster variables and the underlying quiver, and define an associated cluster algebra [9, Sections 8, 9]. Finally, we use these results to prove the BCFW tiling conjecture for the $m = 4$ amplituhedron (Theorem 3.17).

Further motivation. From the point of view of cluster algebras, the study of tiles for the amplituhedron $\mathcal{A}_{n,k,m}$ is useful because it is closely related to the cluster structure on the Grassmannian $\mathrm{Gr}_{m,n}$, as was shown for $m = 2$ in [26] and as this paper demonstrates for $m = 4$. In particular, for $m = 4$, the *BCFW product* (Definition 3.2) used to recursively build tiles (Definition 3.3) has a cluster quasi-homomorphism counterpart called *product promotion* (Definition 3.6), that can be used to recursively construct cluster variables and seeds in $\mathrm{Gr}_{4,n}$ (Theorem 3.7).

In the closely related field of *total positivity*, one prototypical problem is to give an efficient characterization of the 'positive part' of a space as the subset where a certain minimal collection of functions take on positive values [13] ('positivity test'). For example, for any cluster $\mathbf{x}$ for $\mathrm{Gr}_{k,n}$ [28], the *positive Grassmannian* $\mathrm{Gr}_{k,n}^{>0}$ can be described as the region in $\mathrm{Gr}_{k,n}$ where all the cluster variables of $\mathbf{x}$ are positive.

We think of Theorem 3.13 as a 'positivity test' for membership in a BCFW tile of the amplituhedron. See [26, Theorem 6.8] for an analogous result for $m = 2$, and [9, Conjecture 7.17] for some conjectures for general m.

From the point of view of discrete geometry, it is interesting to study tiles and more generally Grasstopes because one can think of them as a generalization of polytopes in the Grassmannian. In particular, the positivity tests for the positive Grassmannian and BCFW tiles can be thought of as analogues of the hyperplane description of polytopes. Finally, it would be interesting to show that tiles are positive geometries.

2 Background

2.1 The (positive) Grassmannian

The *Grassmannian* $\mathrm{Gr}_{k,n}(\mathbb{F})$ is the space of all k-dimensional subspaces of an n-dimensional vector space $\mathbb{F}^n$. Let $[n]$ denote $\{1,\dots,n\}$, and $\binom{[n]}{k}$ denote the set of all k-element subsets of $[n]$. We can represent a point $V \in \mathrm{Gr}_{k,n}(\mathbb{F})$ as the row-span of a full-rank $k \times n$ matrix C with entries in $\mathbb{F}$. Then for $I = \{i_1 < \cdots < i_k\} \in \binom{[n]}{k}$, we let $\langle I \rangle_V = \langle i_1\, i_2\, \dots\, i_k \rangle_V$ be the $k \times k$ minor of C using the columns I. The $\langle I \rangle_V$ are called the *Plücker coordinates* of V, and are independent of the choice of matrix representative C (up to common rescaling). The *Plücker embedding* $V \mapsto \{\langle I \rangle_V\}_{I \in \binom{[n]}{k}}$ embeds $\mathrm{Gr}_{k,n}(\mathbb{F})$ into projective space[1] . If C has columns $v_1,\dots,v_n$, we may also identify $\langle i_1\, i_2\, \dots\, i_k \rangle$ with $v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k}$, hence e.g. $\langle i_1\, i_2\, \dots\, i_k \rangle = -\langle i_2\, i_1\, \dots\, i_k \rangle$. In this paper we will often be working with the *real* Grassmannian $\mathrm{Gr}_{k,n} = \mathrm{Gr}_{k,n}(\mathbb{R})$. We will also denote by $\mathrm{Gr}_{k,N}$ the Grassmannians of k-planes in a vector space with basis indexed by a set $N \subset [n]$.

Definition 2.1 (Positive Grassmannian)**.** [24, 27] We say that $V \in \mathrm{Gr}_{k,n}$ is *totally nonnegative* if (up to a global change of sign) $\langle I \rangle_V \geq 0$ for all $I \in \binom{[n]}{k}$. Similarly, V is *totally positive* if $\langle I \rangle_V > 0$ for all $I \in \binom{[n]}{k}$. We let $\mathrm{Gr}_{k,n}^{\geq 0}$ and $\mathrm{Gr}_{k,n}^{>0}$ denote the set of totally nonnegative and totally positive elements of $\mathrm{Gr}_{k,n}$, respectively. $\mathrm{Gr}_{k,n}^{\geq 0}$ is called the *totally nonnegative Grassmannian*, or sometimes just the *positive Grassmannian*.

[1] We will sometimes abuse notation and identify C with its row-span; we will also drop the subscript V on Plücker coordinates when it does not cause confusion.

If we partition $\mathrm{Gr}_{k,n}^{\geq 0}$ into strata based on which Plücker coordinates are strictly positive and which are 0, we obtain a cell decomposition of $\mathrm{Gr}_{k,n}^{\geq 0}$ into *positroid cells* [27]. Each positroid cell S gives rise to a matroid $\mathcal{M}$, whose bases are precisely the k-element subsets I such that the Plücker coordinate $\langle I \rangle$ does not vanish on S; $\mathcal{M}$ is called a *positroid*.

There are many ways to index positroid cells in $\mathrm{Gr}_{k,n}^{\geq 0}$ [27], such as *plabic graphs*:

Definition 2.2. Let G be a *plabic graph*, i.e. a planar bipartite graph[2] embedded in a disk, with black vertices $1, 2, \ldots, n$ on the boundary of the disk. An *almost perfect matching M* of G is a collection of edges which covers each internal vertex of G exactly once. The *boundary* of M, denoted ∂M, is the set of boundary vertices covered by M. The positroid associated to G is the collection $\mathcal{M} = \mathcal{M}(G) := \{\partial M : M \text{ an almost perfect matching of } G\}$.

Both $\mathrm{Gr}_{k,n}$ and $\mathrm{Gr}_{k,n}^{\geq 0}$ admit the following set of operations, which will be useful to us.

Definition 2.3 (Operations on the Grassmannian)**.** We define the following maps on $\mathrm{Mat}_{k,n}$, which descends to maps on $\mathrm{Gr}_{k,n}$ and $\mathrm{Gr}_{k,n}^{\geq 0}$, which we denote in the same way:

- (cyclic shift) We define the *cyclic shift* as the map $\mathrm{cyc} : \mathrm{Mat}_{k,n} \to \mathrm{Mat}_{k,n}$ which sends $v_1 \mapsto (-1)^{k-1} v_n$ and $v_i \mapsto v_{i-1}, 2 \leq i \leq n$, and in terms of Plückers: $\langle I \rangle \mapsto \langle I - 1 \rangle$.
- (reflection) We define *reflection* as the map $\mathrm{refl} : \mathrm{Mat}_{k,n} \to \mathrm{Mat}_{k,n}$ which sends $v_i \mapsto v_{n+1-i}$ and rescales a row by $(-1)^{\binom{k}{2}}$, and in terms of Plückers: $\langle I \rangle \mapsto \langle n + 1 - I \rangle$.
- (zero column) We define the map $\mathrm{pre}_i : \mathrm{Mat}_{k,[n]\setminus\{i\}} \to \mathrm{Mat}_{k,n}$ which adds a zero column at i, and in terms of Plückers: $\langle I \rangle \mapsto \langle I \rangle$.

Here, $I - 1$ is obtained from $I \in \binom{[n]}{k}$ by subtracting 1 (mod n) from each element of I and $n + 1 - I$ is obtained from I by subtracting each element of I from $n + 1$.

2.2 The amplituhedron

Building on [2, 18], Arkani-Hamed and Trnka [4] introduced the *(tree) amplituhedron*, which they defined as the image of the positive Grassmannian under a positive linear map. Let $\mathrm{Mat}_{n,p}^{>0}$ denote the set of $n \times p$ matrices whose maximal minors are positive.

Definition 2.4 (Amplituhedron)**.** Let $Z \in \mathrm{Mat}_{n,k+m}^{>0}$, where $k + m \leq n$. The *amplituhedron map* $\tilde{Z} : \mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,k+m}$ is defined by $\tilde{Z}(C) := CZ$, where C is a $k \times n$ matrix representing an element of $\mathrm{Gr}_{k,n}^{\geq 0}$, and CZ is a $k \times (k + m)$ matrix representing an element of $\mathrm{Gr}_{k,k+m}$. The *amplituhedron* $\mathcal{A}_{n,k,m}^{Z} \subset \mathrm{Gr}_{k,k+m}$ is the image $\tilde{Z}(\mathrm{Gr}_{k,n}^{\geq 0})$.

In this article we will be concerned with the case where $m = 4$.

[2] We will always assume that plabic graphs are *reduced* [27, Definition 12.5].

Definition 2.5 (Tiles)**.** Fix k, n, m with $k + m \leq n$ and choose $Z \in \mathrm{Mat}^{>0}_{n,k+m}$. Given a positroid cell S of $\mathrm{Gr}^{\geq 0}_{k,n}$, we let $Z^\circ_S := \tilde{Z}(S)$ and $Z_S := \overline{\tilde{Z}(S)} = \tilde{Z}(\overline{S})$. We call Z_S and Z°_S a *tile* and an *open tile* for $\mathcal{A}^Z_{n,k,m}$ if $\dim(S) = km$ and $\tilde{Z}$ is injective on S.

Definition 2.6 (Tilings)**.** A *tiling* of $\mathcal{A}^Z_{n,k,m}$ is a collection $\{Z_S \mid S \in \mathcal{C}\}$ of tiles, such that their union equals $\mathcal{A}^Z_{n,k,m}$ and the open tiles $Z^\circ_S, Z^\circ_{S'}$ are pairwise disjoint.

There is a natural notion of *facet* of a tile, generalizing the notion of facet of a polytope.

Definition 2.7 (Facet of a cell and a tile)**.** Given two positroid cells S' and S, we say that S' is a *facet* of S if $S' \subset \partial S$ and S' has codimension 1 in $\overline{S}$. If S' is a facet of S and Z_S is a tile of $\mathcal{A}^Z_{n,k,m}$, we say that $Z_{S'}$ is a *facet* of Z_S if $Z_{S'} \subset \partial Z_S$ and has codimension 1 in Z_S.

Definition 2.8 (Twistor coordinates)**.** Fix $Z \in \mathrm{Mat}^{>0}_{n,k+m}$ with rows $Z_1, \ldots, Z_n \in \mathbb{R}^{k+m}$. Given $Y \in \mathrm{Gr}_{k,k+m}$ with rows $y_1, \ldots, y_k$, and $\{i_1, \ldots, i_m\} \subset [n]$, we define the *twistor coordinate* $\langle\langle i_1 i_2 \cdots i_m \rangle\rangle$ to be the determinant of the matrix with rows $y_1, \ldots, y_k, Z_{i_1}, \ldots, Z_{i_m}$.

Note that the twistor coordinates are defined only up to a common scalar multiple. An element of $\mathrm{Gr}_{k,k+m}$ is uniquely determined by its twistor coordinates [19]. Moreover, $\mathrm{Gr}_{k,k+m}$ can be embedded into $\mathrm{Gr}_{m,n}$ so that the twistor coordinate $\langle\langle i_1 \ldots i_m \rangle\rangle$ is the pullback of the Plücker coordinate $\langle i_1, \ldots, i_m \rangle$ in $\mathrm{Gr}_{m,n}$.

Definition 2.9. We refer to a homogeneous polynomial in twistor coordinates as a *functionary*. For $S \subseteq \mathrm{Gr}^{\geq 0}_{k,n}$, we say a functionary F has a definite sign $s \in \{\pm 1\}$ (or vanishes) on Z°_S if for all $Z \in \mathrm{Mat}^{>0}_{n,k+4}$ and for all $Y \in Z^\circ_S$, $F(Y)$ has sign s (or 0, respectively).

Functionaries will be crucial to describe tiles of the amplituhedron, to prove the main theorems about cluster adjacency and BCFW tilings, and to connect with cluster algebras.

2.3 Cluster Algebras

Cluster algebras were introduced by Fomin and Zelevinsky in [14], motivated by the study of total positivity; see [12] for an introduction. We give a quick definition of cluster algebras from quivers. All cluster algebras here will be of *geometric type*.

A *quiver* Q is an oriented graph given by a finite set of vertices. For a quiver without oriented cycles of length 1 and 2, one can define a *quiver mutation* $\mu_k(Q)$ at each vertex k of Q. This operation, described in [14], is an involution: $\mu_k^2(Q) = Q$.

Definition 2.10. Choose $s \geq r$ positive integers. Let $\mathcal{F}$ be an *ambient field* of rational functions in r independent variables over $\mathbb{C}(x_{r+1}, \ldots, x_s)$. A *labeled seed* in $\mathcal{F}$ is a pair $(\mathbf{x}, Q)$, where $\mathbf{x} = (x_1, \ldots, x_s)$ forms a free generating set for $\mathcal{F}$ and Q is a quiver with vertices $1, 2, \ldots, r$ called *mutable*, and vertices $r + 1, \ldots, s$ called *frozen*.

We call $\mathbf{x}$ a *cluster* and its elements $\{x_1, \ldots, x_s\}$ *cluster variables*. The variables $x_1, \ldots, x_r$ are called *mutable*, and the variables $c = \{x_{r+1}, \ldots, x_s\}$ are called *frozen*.

$$
\begin{tikzcd}[column sep=small, row sep=small]
\langle 4\,5\,6\,7\rangle \arrow[dr] & & & & \\
& \langle 3\,5\,6\,7\rangle \arrow[r] \arrow[d] & \langle 3\,4\,6\,7\rangle \arrow[r] \arrow[d] & \langle 3\,4\,5\,7\rangle \arrow[r] \arrow[d] & \langle 3\,4\,5\,6\rangle \\
& \langle 2\,5\,6\,7\rangle \arrow[r] \arrow[d] & \langle 2\,3\,6\,7\rangle \arrow[r] \arrow[d] \arrow[ul] & \langle 2\,3\,4\,7\rangle \arrow[r] \arrow[d] \arrow[ul] & \langle 2\,3\,4\,5\rangle \arrow[ul] \\
& \langle 1\,5\,6\,7\rangle & \langle 1\,2\,6\,7\rangle \arrow[ul] & \langle 1\,2\,3\,7\rangle \arrow[ul] & \langle 1\,2\,3\,4\rangle \arrow[ul]
\end{tikzcd}
$$

Figure 1: The rectangle seed $\Sigma_{4,7}$. Mutable variables are in the colored box.

Definition 2.11 (*Seed mutations*)**.** Let $(\mathbf{x}, Q)$ be a labeled seed in $\mathcal{F}$, and let $k \in \{1, \ldots, r\}$. The *seed mutation* μ_k in direction k transforms $(\mathbf{x}, Q)$ into the labeled seed $\mu_k(\mathbf{x}, Q) = (\mathbf{x}', \mu_k(Q))$, where the cluster $\mathbf{x}' = (x'_1, \ldots, x'_s)$ is defined as follows: $x'_j = x_j$ for $j \neq k$, whereas $x'_k \in \mathcal{F}$ is determined by the *exchange relation*

$$x'_k\, x_k = \prod_{i:\, i \to k} x_i + \prod_{i:\, i \leftarrow k} x_i\,. \tag{2.1}$$

Where $i \to k$ (or $i \leftarrow k$) denotes an edge oriented from vertex i to k (or k to i). Note that one omits arrows between two frozen vertices as they do not affect seed mutation.

Definition 2.12. Let $\mathbb{T}_r$ be an *r-regular tree* whose edges are labeled by $1, \ldots, r$, so that edges emanating from each vertex receive different labels. A *cluster pattern* is an assignment of a labeled seed $\Sigma_t = (\mathbf{x}_t, Q_t)$ to every vertex $t \in \mathbb{T}_n$, such that the seed assigned to the endpoint of an edge k emanated from t is obtained by mutating Σ_t in direction k.

Definition 2.13 (*Cluster algebra*)**.** Given a cluster pattern, we denote as $\mathcal{X}$ the union of all mutable variables of all the seeds in the pattern. Let $\mathbb{C}[c^{\pm 1}]$ be the *ground ring* consisting of Laurent polynomials in the frozen variables. The *cluster algebra* $\mathcal{A}$ associated with a given pattern is the $\mathbb{C}[c^{\pm 1}]$-subalgebra of the ambient field $\mathcal{F}$ generated by all mutable variables, with coefficients which are Laurent polynomials in the frozen variables: $\mathcal{A} = \mathbb{C}[c^{\pm 1}][\mathcal{X}]$. We denote $\mathcal{A} = \mathcal{A}(\mathbf{x}, Q)$, where $(\mathbf{x}, Q)$ is any seed in the underlying cluster pattern. We say that $\mathcal{A}$ has *rank* r because each cluster contains r mutable variables. Cluster variables that belong to a common cluster are said to be *compatible*.

The Grassmannian $\mathrm{Gr}_{k,n}(\mathbb{C})$ has a cluster structure [28], defined starting from particularly nice seeds called *rectangles seed* $\Sigma_{k,n}$, see Figure 1 and the exposition of [11].

Theorem 2.14 ([28])**.** *Let* $\mathrm{Gr}^{\circ}_{k,n}$ *be the open subset of the Grassmannian where the frozen variables don't vanish. Then the coordinate ring* $\mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{k,n}]$ *of the affine cone over* $\mathrm{Gr}^{\circ}_{k,n}$ *is the cluster algebra* $\mathcal{A}(\Sigma_{k,n})$.

Moreover, the operations on the Grassmannian $\mathrm{cyc}, \mathrm{refl}, \mathrm{pre}$ in Definition 2.3 induce maps on $\mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{k,N}]$ which are compatible with the cluster structure of Theorem 2.14:

Proposition 2.15. *The maps* $\mathrm{cyc}, \mathrm{refl} : \mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{k,n}] \to \mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{k,n}], \mathrm{pre}_i : \mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{k,[n]\setminus\{i\}}] \to \mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{k,n}]$ *take cluster variables to cluster variables and preserve compatibility and exchange relations.*

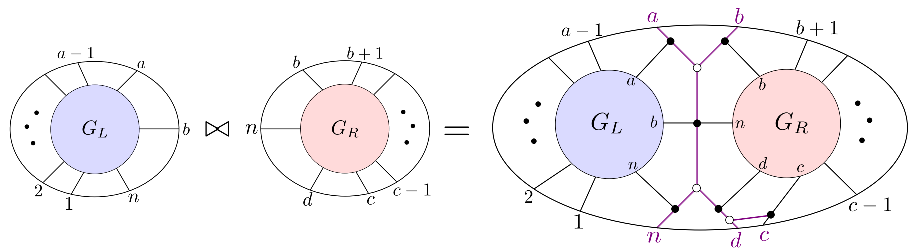

Figure 2: The BCFW product $S_L \bowtie S_R$ of S_L and S_R in terms of their plabic graphs.

3 Results

Our first main result is proving that a class of cells called *BCFW cells* give tiles for $\mathcal{A}^Z_{n,k,4}$. We will build BCFW cells recursively using the *BCFW product*. Let us first introduce some notation we will use throughout this section.

Notation 3.1. Choose integers $1 \le a < b < c < d < n$ with a, b and c, d, n consecutive. Let[3] $N_L = \{n, 1, 2, \dots, a, b\}, N_R = \{b, \dots, c, d, n\}$ and $D = (a, b, c, d, n)$. Also fix $k \le n$ and two nonnegative integers $k_L \le |N_L|$ and $k_R \le |N_R|$ such that $k_L + k_R + 1 = k$. Note that, for any set of indices $N \subset [n]$, our results hold with N instead of $[n]$, by replacing 1 and n in the definition with the smallest and largest elements of N, respectively.

Definition 3.2 (BCFW product)**.** Let $S_L \subseteq \mathrm{Gr}_{k_L,N_L}, S_R \subseteq \mathrm{Gr}^{\ge 0}_{k_R,N_R}$ as in Notation 3.1 and G_L, G_R be the respective plabic graphs. The *BCFW product* of S_L and S_R is the positroid cell $S_L \bowtie S_R \subseteq \mathrm{Gr}_{k,n}$ corresponding the plabic graph in the right-hand side of Figure 2. When it is not clear from the context, we will say $\bowtie$ is performed 'with indices D'.

We now introduce the family of *BCFW cells* to be the set of positroid cells which is closed under the operations in Definitions 2.3 and 3.2:

Definition 3.3 (BCFW cells)**.** The set of *BCFW cells* is defined recursively. For $k = 0$, let the trivial cell $\mathrm{Gr}^{>0}_{0,n}$ be a BCFW cell. If S is a BCFW cell, so is the cell obtained by applying cyc, refl, pre to S. If S_L, S_R are BCFW cells, so is their BCFW product $S_L \bowtie S_R$.

Example 3.4. For $k = 1$, the BCFW cells in $\mathrm{Gr}^{\ge 0}_{1,n}$ are as in Figure 3 (left). They have Plücker coordinates $\langle a \rangle, \langle b \rangle, \langle c \rangle, \langle d \rangle, \langle e \rangle > 0$ and all others zero. In Figure 3 (right), $S_{ex} \subset \mathrm{Gr}^{\ge 0}_{2,7}$ is obtained as $S_L \bowtie S_R$, with S_L, S_R BCFW cells in $\mathrm{Gr}^{\ge 0}_{1,N_L}, \mathrm{Gr}^{\ge 0}_{0,N_R}$ respectively, with $N_L = \{7, 1, 2, 3, 4\}, N_R = \{4, 5, 6, 7\}$ and $D = (3, 4, 5, 6, 7)$.

Theorem 3.5 (BCFW tiles)**.** *The amplituhedron map is injective on each BCFW cell. That is, the closure $Z_S := \overline{\tilde{Z}(S)}$ of the image of a BCFW cell S is a* tile*, which we refer to as a* BCFW tile.

[3]Note that we will overload the notation and let n index an element of a vector space basis for different vector spaces; however, in what follows, the meaning should be clear from context.

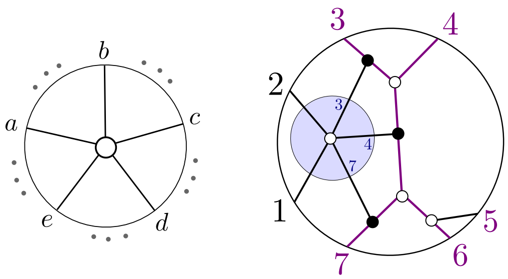

Figure 3: Plabic graphs of: a BCFW cell in $\mathrm{Gr}_{1,n}^{\geq 0}$ (left); a BCFW cell $S_{ex} \subset \mathrm{Gr}_{2,7}^{\geq 0}$ (right).

A key ingredient to prove Theorem 3.5 is inverting the amplituhedron map on BCFW tiles [9, Theorem 7.7] by using *product promotion* – an operation which interacts nicely both with the cluster structure on the Grassmannian and with the BCFW product.

Definition 3.6. Using Notation 3.1, *product promotion* is the homomorphism

$$\Psi_D = \Psi : \mathbb{C}(\widehat{\mathrm{Gr}}_{4,N_L}) \times \mathbb{C}(\widehat{\mathrm{Gr}}_{4,N_R}) \to \mathbb{C}(\widehat{\mathrm{Gr}}_{4,n}),$$

induced by the following substitution:

$$\text{on } \widehat{\mathrm{Gr}}_{4,N_L}\colon\, b \;\mapsto\; \frac{(ba)\cap(cdn)}{\langle a\,c\,d\,n\rangle}, \quad \text{on } \widehat{\mathrm{Gr}}_{4,N_R}\colon\, n \;\mapsto\; \frac{(ba)\cap(cdn)}{\langle a\,b\,c\,d\rangle},\, d \;\mapsto\; \frac{(dc)\cap(abn)}{\langle a\,b\,c\,n\rangle}.$$

The vector $(ij)\cap(rsq) := v_i\langle j\,r\,s\,q\rangle - v_j\langle i\,r\,s\,q\rangle = -v_r\langle i\,j\,s\,q\rangle + v_s\langle i\,j\,r\,q\rangle - v_q\langle i\,j\,r\,s\rangle$ is in the intersection of the 2-plane and the 3-plane spanned by v_i, v_j and v_r, v_s, v_q, respectively.

We show[4] that Ψ is in fact a *quasi-homomorphism* (see [15]) from the cluster algebra[5] $\mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{4,N_L}] \times \mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{4,N_R}]$ to a sub-cluster algebra of $\mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{4,n}]$. See [15, Definition 3.1, Proposition 3.2] for the precise definition of a quasi-homomorphism.

Theorem 3.7. *Product promotion Ψ is a quasi-homomorphism of cluster algebras. In particular, Ψ maps a cluster variable (respectively, cluster) of $\mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{4,N_L}] \times \mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{4,N_R}]$, to a cluster variable (respectively, sub-cluster) of $\mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{4,n}]$, up to multiplication by Laurent monomials in $\mathcal{T}' := \{\langle a\,b\,c\,n\rangle, \langle a\,b\,c\,d\rangle, \langle b\,c\,d\,n\rangle, \langle a\,c\,d\,n\rangle\}$.*

Remark 3.8. Definition 3.6 and Theorem 3.7 extend also to the degenerate cases, e.g. for $a = 1$ (*upper promotion*), where $\Psi : \mathbb{C}(\widehat{\mathrm{Gr}}_{4,N_R}) \to \mathbb{C}(\widehat{\mathrm{Gr}}_{4,n})$, see [9, Section 4.3].

Definition 3.9. Let x be a cluster variable of $\mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{4,N_L}]$ or $\mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{4,N_R}]$. We define the *rescaled product promotion* $\overline{\Psi}(x)$ of x to be the cluster variable of $\mathrm{Gr}_{4,n}$ obtained from $\Psi(x)$ by removing[6] the Laurent monomial in $\mathcal{T}'$ (c.f. Theorem 3.7).

[4] We will sometime omit the dependence on the indices $D = \{a, b, c, d, n\}$ in Ψ (and $\overline{\Psi}$) for brevity.
[5] $\mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{4,N_L}] \times \mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{4,N_R}]$ is a cluster algebra where each seed is the disjoint union of a seed of each factor.
[6] If $x = \langle bcdn\rangle$, then $\overline{\Psi}(x) = \Psi(x) = x$.

The fact that product promotion is a cluster quasi-homomorphism may be of independent interest in the study of the cluster structure on $\mathrm{Gr}_{4,n}$. Much of the work thus far on the cluster structure of the Grassmannian has focused on cluster variables which are polynomials in Plücker coordinates with low degree; by constrast, the cluster variables we obtain can have arbitrarily high degree in Plücker coordinates (e.g. see the *chain polynomials* in [9, Theorem 8.3]). We introduce the following notation:

$$\langle a\,b\,c\,|\,d\,e\,|\,f\,g\,h\rangle := \langle a,b,c,(d\,e)\cap(f\,g\,h)\rangle = \langle a\,b\,c\,d\rangle\,\langle e\,f\,g\,h\rangle - \langle a\,b\,c\,e\rangle\,\langle d\,f\,g\,h\rangle. \quad (3.1)$$

Example 3.10. For N_L and N_R as in Example 3.4, the only Plücker which changes is: $\Psi(\langle 1\,2\,4\,7\rangle) = \langle 1\,2\,7|3\,4|5\,6\,7\rangle/\langle 3\,4\,6\,7\rangle$, and $\overline{\Psi}(\langle 1\,2\,4\,7\rangle) = \langle 1\,2\,7|3\,4|5\,6\,7\rangle$ which is a quadratic cluster variable in $\mathrm{Gr}_{4,7}$, e.g. obtained by mutating $\langle 2367\rangle$ in $\Sigma_{4,7}$ of Figure 1.

Using rescaled product promotion and the operations in Proposition 2.15, we associate to each BCFW tile Z_S a collection of compatible cluster variables $\mathbf{x}(S)$ for $\mathrm{Gr}_{4,n}$.

Definition 3.11 (Cluster variables for BCFW tiles)**.** Let $S \subset \mathrm{Gr}_{k,n}^{\geq 0}$ be a BCFW cell. We define the set of *coordinate cluster variables* $\mathbf{x}(S)$ for S recursively as follows:

- If $S = S_L \bowtie S_R$ with indices $D_k = (a_k, b_k, c_k, d_k, n_k)$, then
$$\mathbf{x}(S) = \overline{\Psi}_{D_k}(\mathbf{x}(S_L) \cup \mathbf{x}(S_R)) \cup \{\langle I\rangle, I \in \binom{D_k}{4}\}, \quad (3.2)$$
- If $S = \begin{cases} \mathrm{pre}_i(S') \\ \mathrm{cyc}(S') \\ \mathrm{refl}(S') \end{cases}$ then $\mathbf{x}(S) = \begin{cases} \mathbf{x}(S') \\ \mathrm{cyc}^{-1}(\mathbf{x}(S')) \\ \mathrm{refl}(\mathbf{x}(S')) \end{cases}$,

and for the base case $k = 0$, we set $\mathbf{x}(S) = \emptyset$. Here, $\mathrm{cyc}^{-1} = \mathrm{cyc}^{n-1}$.

For a BCFW cell S, $\mathbf{x}(S)$ depends on the sequence of operations in Definition 3.3 used to build S, but we will drop this dependence for brevity.

Note that $\mathbf{x}(S)$ is a collection of compatible cluster variables for $\mathrm{Gr}_{4,n}$ [9, Lemma 7.6].

Example 3.12. From Example 3.4, $S_{ex} = S_L \bowtie S_R$ and $\mathbf{x}(S_L) = \{\langle I\rangle, I \in \binom{D_L}{4}\}, \mathbf{x}(S_R) = \emptyset$, where $D_L = \{1,2,3,4,7\}$. Then by Example 3.10 the coordinate cluster variables $\mathbf{x}(S_{ex})$ are: $\overline{\Psi}(\mathbf{x}(S_L)) = \mathbf{x}(S_L) \setminus \{\langle 1247\rangle\} \cup \{\langle 127|34|567\rangle\}$ together with $\{\langle I\rangle, I \in \binom{D}{4}\}$.

Given a cluster variable x in $\mathrm{Gr}_{4,n}$, we will denote as $x(Y)$ the functionary on $\mathrm{Gr}_{k,k+4}$ (cf. Definition 2.9) obtained by identifying Plücker coordinates $\langle I\rangle$ in $\mathrm{Gr}_{4,n}$ with twistor coordinates $\langle\langle I\rangle\rangle$ in $\mathrm{Gr}_{k,k+4}$ (cf. Definition 2.8). Then interpreting each cluster variable as a functionary, we describe each BCFW tile as the semialgebraic subset of $\mathrm{Gr}_{k,k+4}$ where the coordinate cluster variables take on particular signs.

Theorem 3.13 (Sign description of BCFW tiles)**.** *Let Z_S be a BCFW tile. For each element x of $\mathbf{x}(S)$, the functionary $x(Y)$ has a definite sign s_x on Z_S° and*

$$Z_S^\circ = \{Y \in \mathrm{Gr}_{k,k+4} : s_x\, x(Y) > 0 \textit{ for all } x \in \mathbf{x}(S)\}.$$

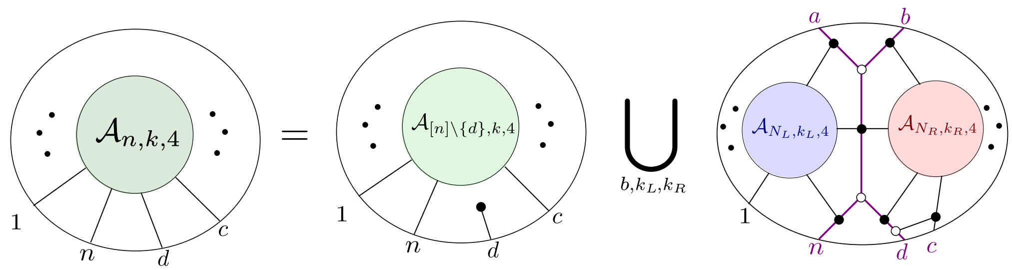

Figure 4: BCFW tiling for $\mathcal{A}_{n,k,4}$. On the right: the first term is obtained by tiling $\mathcal{A}_{[n]\setminus\{d\},k,4}$ (from $\mathcal{T}_{pre}$); the second term is the union over b, k_L, k_R as in Definition 3.16 of the collections of tiles obtained by tiling $\mathcal{A}_{N_L,k_L,4}$ and $\mathcal{A}_{N_R,k_R,4}$ (from $\mathcal{T}_{k_L,k_R,b,n}$).

Example 3.14. The open tile $Z^{\circ}_{ex} := \tilde{Z}(S_{ex})$, with S_{ex} from Example 3.4, is the semialgebraic set in $\mathrm{Gr}_{2,6}$ where the functionaries $x(Y)$, with $x \in \mathbf{x}(S_{ex})$ of Example 3.12 are positive, except when $x \in \{\langle 3567\rangle, \langle 3457\rangle, \langle 2347\rangle, \langle 3567\rangle\}$, for which $x(Y)$ are negative.

One can study facets of tiles (see Definition 2.7) by describing associated functionaries which vanish on them. Given a functionary $F(\langle\langle I\rangle\rangle)$ on $\mathrm{Gr}_{k,k+4}$, we can obtain a polynomial $F(\langle I\rangle)$ in the Plücker coordinates in $\mathrm{Gr}_{4,n}$. Then the coordinate cluster variables in Definition 3.11 are a key tool in the proof of cluster adjacency conjecture for BCFW tiles:

Theorem 3.15 (Cluster adjacency for BCFW tiles)**.** *Let Z_S be a BCFW tile of $\mathcal{A}^Z_{n,k,4}$. Each facet $Z_{S'}$ of Z_S lies on a hypersurface cut out by a functionary $F_{S'}(\langle\langle I\rangle\rangle)$ such that $F_{S'}(\langle I\rangle)$ is in* $\mathbf{x}(S)$. *Thus $\{F_{S'}(\langle I\rangle) : Z_{S'}$ a facet of $Z_S\}$ is a collection of compatible cluster variables of* $\mathrm{Gr}_{4,n}$.

Finally, we show how to use BCFW tiles to tile $\mathcal{A}^Z_{n,k,4}$ (Definition 2.6). Theorems 3.5 and 3.13 are important ingredients to prove our last main result Theorem 3.17. We use Notation 3.1, fix $n \geq k+4$, and define $b_{min} := 2$ if $k_L = 0$ and otherwise $b_{min} := k_L + 3$.

Definition 3.16 (BCFW collections)**.** We say that a collection $\mathcal{T}$ of $4k$-dimensional BCFW cells in $\mathrm{Gr}^{\geq 0}_{k,n}$ is a *BCFW collection of cells* for $\mathcal{A}_{n,k,4}$ if it has the following recursive form:

- If $k = 0$ or $k = n-4$, $\mathcal{T}$ is the single BCFW cell $\mathrm{Gr}^{>0}_{0,n}$ or $\mathrm{Gr}^{>0}_{n-4,n}$, respectively.
- If $\mathcal{T} = \{S\}$ is a BCFW collection of cells, so is $\{\mathrm{refl}\, S\}_{S\in\mathcal{T}}$ and $\{\mathrm{cyc}\, S\}_{S\in\mathcal{T}}$.
- Otherwise $\mathcal{T} = \mathcal{T}_{pre} \bigsqcup_{b,k_L,k_R} \mathcal{T}_{k_L,k_R,b,n}$, where
  - b ranges from b_{min} to $n-3-k_R$, and k_L, k_R as in Notation 3.1;
  - $\mathcal{T}_{pre} = \{\mathrm{pre}_d(S)\}_{S\in\mathcal{C}}$, where $\mathcal{C}$ is a BCFW collection of cells for $\mathcal{A}_{[n]\setminus\{d\},k,4}$;
  - $\mathcal{T}_{k_L,k_R,b,n} = \{S_L \bowtie S_R\}_{(S_L,S_R)\in\mathcal{C}_L\times\mathcal{C}_R}$ where $\mathcal{C}_L$ and $\mathcal{C}_R$ are BCFW collections of cells for $\mathcal{A}_{N_L,k_L,4}$ and $\mathcal{A}_{N_R,k_R,4}$.

Theorem 3.17 (BCFW tilings)**.** *Every BCFW collection of cells $\mathcal{T} = \{S\}$ as in Definition 3.16 gives rise to a tiling $\{Z_S\}_{S\in\mathcal{T}}$ of the amplituhedron $\mathcal{A}^Z_{n,k,4}$, which we refer to as a* BCFW tiling.

See Figure 4 for an illustration. This generalizes the main result of [10], which proved the same result for the *standard* BCFW cells, and proves the main conjecture of [4].

Non-BCFW tiles are also expected to satisfy cluster adjacency, have a sign description in terms of cluster variables, and appear in tilings of $\mathcal{A}^Z_{n,k,4}$, see [9, Section 12.2].

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