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Chow rings of matroids as permutation representations

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Abstract. Given a matroid and a group of its matroid automorphisms, we study the induced group action on the Chow ring of the matroid. This turns out to always be a permutation action. Work of Adiprasito, Huh and Katz showed that the Chow ring satisfies Poincaré duality and the Hard Lefschetz theorem. We lift these to statements about this permutation action, and suggest further conjectures in this vein.

Keywords: matroid, Chow ring, Koszul, log-concave, unimodal, Kahler package, Burnside ring, equivariant, Polya freqency, real-rooted

1 Introduction

A *matroid* $\mathcal{M}$ is a combinatorial abstraction of lists of vectors $v_1, v_2, \ldots, v_n$ in a vector space, recording only the information about which subsets of the vectors are linearly independent or dependent, forgetting their coordinates. In groundbreaking work, Adiprasito, Huh and Katz [1] affirmed long-standing conjectures of Rota–Heron–Welsh and Mason about vectors and matroids via a new methodology. Their work employed a certain graded $\mathbb{Z}$-algebra $A(\mathcal{M})$ called the *Chow ring* of $\mathcal{M}$, introduced by Feichtner and Yuzvinsky [4] as a generalization of the Chow ring of DeConcini and Procesi's *wonderful compactifications* for hyperplane arrangement complements. A remarkable integral Gröbner basis result proven by Feichtner and Yuzvinsky [4, Thm. 2] shows that for a matroid $\mathcal{M}$ of rank $r+1$ with Chow ring $A(\mathcal{M}) = \bigoplus_{k=0}^{r} A^k$, each homogeneous component is free abelian: $A^k \cong \mathbb{Z}^{a_k}$ for some Hilbert function $(a_0, a_1, \ldots, a_r)$. A key step in the work of Adiprasito, Huh and Katz shows not only *symmetry* and *unimodality* for the Hilbert function

$$a_k = a_{r-k} \text{ for } r \leq k/2 \tag{1.1}$$

$$a_0 \leq a_1 \leq \cdots \leq a_{\lfloor \frac{r}{2} \rfloor} = a_{\lceil \frac{r}{2} \rceil} \geq \cdots \geq a_{r-1} \geq a_r, \tag{1.2}$$

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but in fact proves that $A(\mathcal{M})$ enjoys a trio of properties referred to as the *Kähler package*, reviewed in Section 2.2 below. The first of these properties is *Poincaré duality*, proving (1.1) via a natural $\mathbb{Z}$-module isomorphism $A^{r-k} \cong \mathrm{Hom}_{\mathbb{Z}}(A^k, \mathbb{Z})$. The second property, called the *Hard Lefschetz Theorem*, shows that after tensoring over $\mathbb{Z}$ with $\mathbb{R}$ to obtain $A(\mathcal{M})_{\mathbb{R}} = \bigoplus_{k=0} A^k_{\mathbb{R}}$, one can find *Lefschetz elements* ω in $A^1_{\mathbb{R}}$ such that multiplication by ω^{r-2k} give $\mathbb{R}$-linear isomorphisms $A^k_{\mathbb{R}} \to A^{r-k}_{\mathbb{R}}$ for $k \leq \frac{r}{2}$. In particular, multiplication by ω mapping $A^k_{\mathbb{R}} \to A^{k+1}_{\mathbb{R}}$ is *injective* for $k < \frac{r}{2}$, strengthening the unimodality (1.2).

We are interested in how these *Poincaré duality* and *Hard Lefschetz* properties interact with the group $G := \mathrm{Aut}(\mathcal{M})$ of symmetries of the matroid $\mathcal{M}$. It is not hard to check that G acts via graded $\mathbb{Z}$-algebra automorphisms on $A(\mathcal{M})$, giving $\mathbb{Z}G$-module structures on each A^k, and $\mathbb{R}G$-module structures on each $A^k_{\mathbb{R}}$. One can also check (see the proof of Corollary 6 below) that $A^r \cong \mathbb{Z}$ with trivial G-action. From this, the Poincaré duality pairing immediately gives rise to a $\mathbb{Z}G$-module isomorphism

$$A^{r-k} \cong \mathrm{Hom}_{\mathbb{Z}}(A^k, \mathbb{Z}) \tag{1.3}$$

where g in G acts on φ in $\mathrm{Hom}_{\mathbb{Z}}(A^k, \mathbb{Z})$ via $\varphi \mapsto \varphi \circ g^{-1}$; similarly $A^{r-k} \cong \mathrm{Hom}_{\mathbb{R}}(A^k, \mathbb{R})$ as $\mathbb{R}G$-modules. Furthermore, it is not hard to check (see Corollary 6 below) that one can pick an explicit Lefschetz element ω as in [1] which is G-fixed, giving $\mathbb{R}G$-module isomorphisms and injections

$$\begin{array}{ccc} A^k_{\mathbb{R}} \xrightarrow{\sim} A^{r-k}_{\mathbb{R}} \quad \text{for } r \leq \dfrac{k}{2} & \qquad & A^k_{\mathbb{R}} \hookrightarrow A^{k+1}_{\mathbb{R}} \quad \text{for } r < \dfrac{k}{2} \\ a \longmapsto a \cdot \omega^{r-2k} & & a \longmapsto a \cdot \omega. \end{array} \tag{1.4}$$

Our goal is to use Feichtner and Yuzvinsky's Gröbner basis result to prove a combinatorial strengthening of the isomorphisms and injections (1.3), (1.4). To this end, recall that the matroid $\mathcal{M}$ can be specified by its family $\mathbf{F}$ of *flats*; then the *Chow ring* $A(\mathcal{M})$ is presented as a quotient of the polynomial ring $S := \mathbb{Z}[x_F]$ having one variable x_F for each nonempty flat F in $\mathbf{F} \setminus \{\varnothing\}$. The presentation takes the form $A(\mathcal{M}) := S/(I+J)$ where I, J are certain ideals of S defined more precisely in Definition 1 below.

Feichtner and Yuzvinsky exhibited a Gröbner basis for $I + J$ that leads to the following standard monomial $\mathbb{Z}$-basis for $A(\mathcal{M})$, which we call the *FY-monomials* of $\mathcal{M}$:

$$\mathrm{FY} := \{x_{F_1}^{m_1} x_{F_2}^{m_2} \cdots x_{F_\ell}^{m_\ell} : (\varnothing =: F_0) \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_\ell, \text{ and } m_i \leq \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1\}.$$

Here $\mathrm{rk}(F)$ denotes the matroid rank of the flat F. The subset FY^k of FY-monomials $x_{F_1}^{m_1} \cdots x_{F_\ell}^{m_\ell}$ of total degree $m_1 + \cdots + m_\ell = k$ then gives a $\mathbb{Z}$-basis for A^k. One can readily check (see Corollary 4 below) that the group $G = \mathrm{Aut}(\mathcal{M})$ permutes the $\mathbb{Z}$-basis FY^k for A^k, endowing A^k with the structure of a *permutation representation*, or *G-set*. Our main result is this strengthening of the isomorphisms and injections seen in (1.3), (1.4).

Theorem 1. *For every matroid $\mathcal{M}$ of rank $r+1$, there exist*

(i) G-equivariant bijections $\pi : \mathrm{FY}^k \xrightarrow{\sim} \mathrm{FY}^{r-k}$ for $k \leq \frac{r}{2}$, and

(ii) G-equivariant injections $\lambda : \mathrm{FY}^k \hookrightarrow \mathrm{FY}^{k+1}$ for $k < \frac{r}{2}$.

2 Background

Among the many definitions of a matroid $\mathcal{M}$ on ground set E, the most useful here specifies its collection of *flats* $\mathbf{F} \subsetneq 2^E$, satisfying certain axioms. When ordered by inclusion the collection of flats $(\mathbf{F}, \subseteq)$ forms a geometric lattice; in an abuse of notation, we will use $\mathbf{F}$ to refer to both the lattice and the set. This lattice is ranked, with rank function denoted $\mathrm{rk}(F)$. The *rank* of the matroid $\mathcal{M}$ itself is defined to be $\mathrm{rk}(E)$, and we assume throughout that $\mathrm{rk}(E) = r + 1$. An *automorphism* of the matroid $\mathcal{M}$ is any permutation $g : E \to E$ of the ground set E that carries flats to flats: for all F in $\mathbf{F}$ one has $g(F)$ in $\mathbf{F}$. Let $G = \mathrm{Aut}(\mathcal{M})$ denote the group of all automorphisms of $\mathcal{M}$. Since such automorphisms respect the partial order via inclusion on $\mathbf{F}$, they also preserve the rank function: $\mathrm{rk}(g(F)) = \mathrm{rk}(F)$ for all g in G and F in $\mathbf{F}$.

2.1 Chow Rings

As defined in the Introduction, Feichtner and Yuzvinsky [4] introduced the Chow ring $A(\mathcal{M})$ of a matroid $\mathcal{M}$.

Definition 1. The *Chow ring* $A(\mathcal{M})$ of a matroid $\mathcal{M}$ is the quotient Z-algebra

$$A(\mathcal{M}) := S/(I + J)$$

where $S = \mathbb{Z}[x_F]$ is a polynomial ring having one variable x_F for each nonempty flat $F \in \mathbf{F} \setminus \{\varnothing\}$, and where I, J are the following ideals of S:

- I is generated by products $x_F x_{F'}$ where F, F' are incomparable flats,
- J is generated by the linear elements $\sum_{a \in F \in \mathbf{F}} x_F$ for each atom a in the lattice $\mathbf{F}$.

The presentation of the Chow ring $A(\mathcal{M})$ only uses the information about the partial order on the lattice of flats $\mathbf{F}$ has some consequences. For one, the Chow ring depends only upon the associated *simple matroid* of $\mathcal{M}$ (one without loops and parallel edges); hence, we assume all matroids to be such. Another consequence is that any element g in $G = \mathrm{Aut}(\mathcal{M})$ will send the generators of the ideals I, J to other such generators. Thus $I + J$ is a G-stable ideal, and G acts on $A(\mathcal{M})$.

Note if one considers $S = \mathbb{Z}[x_F]$ as a graded $\mathbb{Z}$-algebra, then the ideals I, J are generated by homogeneous elements. Hence the quotient $A(\mathcal{M}) = S/(I+J)$ inherits the structure of a graded $\mathbb{Z}$-algebra $A(\mathcal{M}) = \bigoplus_{k=0}^{\infty} A^k$. Since the action of $G = \mathrm{Aut}(\mathcal{M})$ on the Chow ring preserves rank and hence degree, both $A(\mathcal{M})$ and each homogeneous component A^k become $\mathbb{Z}G$-modules.

The following crucial result appears as [4, Thm. 2]. To state it, define an *FY-monomial order* on $S = \mathbb{Z}[x_F]_{\varnothing \neq F \in \mathbf{F}}$ to be any monomial order based on a linear order of the variables with $x_F > x_{F'}$ if $F \subsetneq F'$.

Theorem 2. *Given a matroid $\mathcal{M}$ and any FY-monomial order on $S = \mathbb{Z}[x_F]_{\varnothing \neq F \in \mathbf{F}}$, the ideal $I + J$ presenting $A(\mathcal{M}) = S/(I+J)$ has a monic Gröbner basis $\{g_{F,F'}\}$ indexed by $F \neq F'$ in $\mathbf{F}$, with $g_{F,F'}$ and their initial terms $\mathrm{in}_{\prec}(g_{F,F'})$ as shown here:*

condition on $F \neq F'$ in $\mathbf{F}$	$g_{F,F'}$	$\mathrm{in}_{\prec}(g_{F,F'})$
F, F' non-nested	$x_F x_{F'}$	$x_F x_{F'}$
$\varnothing \neq F \subsetneq F'$	$x_F \left(\sum_{\substack{F'' \in \mathbf{F}: \\ F'' \supseteq F'}} x_{F''} \right)^{\mathrm{rk}(F') - \mathrm{rk}(F)}$	$x_F \cdot x_{F'}^{\mathrm{rk}(F') - \mathrm{rk}(F)}$
$\varnothing = F \subsetneq F'$	$\left(\sum_{\substack{F'' \in \mathbf{F}: \\ F'' \supseteq F'}} x_{F''} \right)^{\mathrm{rk}(F')}$	$x_{F'}^{\mathrm{rk}(F')}$

Corollary 3. *([4, Cor. 1]) For a matroid $\mathcal{M}$ of rank $r+1$, the Chow ring $A(\mathcal{M})$ has these properties:*

(i) *$A(\mathcal{M})$ is free as a $\mathbb{Z}$-module, with $\mathbb{Z}$-basis given by the set of what we call* FY-monomials

$$\mathrm{FY} := \{ x_{F_1}^{m_1} x_{F_2}^{m_2} \cdots x_{F_\ell}^{m_\ell} : F_1 \subsetneq \cdots \subsetneq F_\ell \in \mathbf{F}, \text{ and } m_i \leq \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1 \}. \quad (2.1)$$

(ii) *$A(\mathcal{M})$ vanishes in degrees strictly above r, that is, $A(\mathcal{M}) = \bigoplus_{k=0}^{r} A^k$.*

(iii) *A^r has $\mathbb{Z}$-basis $\{x_E^r\}$, and so a $\mathbb{Z}$-module isomorphism $\deg : A^r \longrightarrow \mathbb{Z}$ sending $x_E^r \longmapsto 1$.*

Assertions (ii) and (iii) follow immediately from the first. To see this, note that the typical FY-monomial $x_{F_1}^{m_1} x_{F_2}^{m_2} \cdots x_{F_\ell}^{m_\ell}$, has total degree

$$\sum_{i=1}^{\ell} m_i \leq \sum_{i=1}^{\ell} (\mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1) = \mathrm{rk}(F_\ell) - \ell \leq (r+1) - 1 = r + 1.$$

Equality here occurs only if $\ell = 1$ and $F_\ell = E$, in which case the FY-monomial is x_E^r.

For any matroid automorphism g, the fact that $\mathrm{rk}(g(F)) = \mathrm{rk}(F)$ for every flat F in $\mathbf{F}$ implies that g sends any FY-monomial to another FY-monomial:

$$x_{F_1}^{m_1} x_{F_2}^{m_2} \cdots x_{F_\ell}^{m_\ell} \overset{g}{\longmapsto} x_{g(F_1)}^{m_1} x_{g(F_2)}^{m_2} \cdots x_{g(F_\ell)}^{m_\ell}.$$

This has a corollary, inspired by work of H.-C. Liao on Boolean matroids [8, Thm. 2.5].

Corollary 4. *For any matroid $\mathcal{M}$, the group $G = \mathrm{Aut}(\mathcal{M})$ permutes the set* FY*, as well as its subset of degree k monomials* $\mathrm{FY}^k \subset \mathrm{FY}$*. Consequently, the $\mathbb{Z}G$-modules on the Chow ring $A(\mathcal{M})$ and each of its homogeneous components A^k lift to G-permutation representations on* FY *and each* FY^k*.*

Example 1. Let $\mathcal{M} = U_{4,5}$ be the uniform matroid of rank 4 on $E = \{1, 2, 3, 4, 5\}$, associated to a list of 5 *generic* vectors v_1, v_2, v_3, v_4, v_5 in a 4-dimensional vector space, so that any quadruple v_i, v_j, v_k, v_ℓ is linearly independent. One has these flats of various ranks:

rank	flats $F \in \mathbf{F}$
0	$\varnothing$
1	$1, 2, 3, 4, 5$
2	$12, 13, 14, 15, 23, 24, 25, 34, 35, 45$
3	$123, 124, 125, 134, 135, 145, 234, 235, 245, 345$
4	$E = 12345$

The Chow ring $A(\mathcal{M}) = S/(I + J)$, where $S = \mathbb{Z}[x_i, x_{jk}, x_{\ell mn}, x_E]$ with $\{i\}, \{j,k\}, \{\ell, m, n\}$ running through all one, two and three-element subsets of $E = \{1, 2, 3, 4, 5\}$, and

$$I = \Big(x_F x_{F'} \Big)_{F \not\subset F', F' \not\subset F}, \qquad J = \Bigg(x_i + \sum_{\substack{1 \leq j < k \leq 5 \\ i \in \{j,k\}}} x_{jk} + \sum_{\substack{1 \leq \ell < m < n \leq 5 \\ i \in \{\ell, m, n\}}} x_{\ell mn} + x_E \Bigg)_{i = 1,2,3,4,5}.$$

The FY-monomial bases for A^0, A^1, A^2, A^3 are shown here, together with the G-equivariant maps λ:

$$\begin{array}{ccccccc}
\mathbf{FY^0} & & \mathbf{FY^1} & & \mathbf{FY^2} & & \mathbf{FY^3} \\
1 & \overset{\lambda}{\longmapsto} & x_E & \overset{\lambda}{\longmapsto} & x_E^2 & & x_E^3 \\
 & & x_{ijk} & \overset{\lambda}{\longmapsto} & x_{ijk}^2 & & \\
 & & x_{ij} & \overset{\lambda}{\longmapsto} & x_{ij} \cdot x_E & &
\end{array}$$

Thus in this case, the ranks of the free $\mathbb{Z}$-modules (A^0, A^1, A^2, A^3) form the symmetric, unimodal sequence $(a_0, a_1, a_2, a_3) = (1, 21, 21, 1)$. Here the bijection $\pi : \mathrm{FY}^0 \to \mathrm{FY}^3$ necessarily maps $1 \longmapsto x_E^3$, and the bijection $\pi : \mathrm{FY}^1 \to \mathrm{FY}^2$ coincides with the map $\lambda : \mathrm{FY}^1 \to \mathrm{FY}^2$ above.

2.2 The Kähler package

The following theorem on the Kähler package for $A(\mathcal{M})$ compiles some of the main results of the work of Adiprasito, Huh and Katz [1].

Theorem 5. *For a matroid $\mathcal{M}$ of rank $r+1$, the Chow ring $A(\mathcal{M})$ satisfies the Kähler package:*

- *(Poincaré duality) For every $k \leq \frac{r}{2}$, one has a perfect $\mathbb{Z}$-bilinear pairing*

$$\begin{aligned} A^k \times A^{r-k} &\longrightarrow \mathbb{Z} \\ (a, b) &\longmapsto \deg(a \cdot b) \end{aligned}$$

- *(Hard Lefschetz) Tensoring over $\mathbb{Z}$ with $\mathbb{R}$, the (real) Chow ring $A_{\mathbb{R}}(\mathcal{M}) = \sum_{k=0}^{r} A_{\mathbb{R}}^k$ contains **Lefschetz elements** ω in $A_{\mathbb{R}}^1$, meaning that $a \mapsto a \cdot \omega^{r-2k}$ is an $\mathbb{R}$-linear isomorphism $A_{\mathbb{R}}^k \to A_{\mathbb{R}}^{r-k}$ for $k \leq \frac{r}{2}$. In particular, multiplication by ω is an injection $A_{\mathbb{R}}^k \to A_{\mathbb{R}}^{k+1}$ for $k < \frac{r}{2}$.*
- *(Hodge-Riemann-Minkowski inequalities) The Lefschetz elements ω define quadratic forms $a \longmapsto (-1)^k \deg(a \cdot \omega^{r-2k} \cdot a)$ on $A_{\mathbb{R}}^k$ that become positive definite upon restriction to the kernel of the map $A_{\mathbb{R}}^k \longrightarrow A_{\mathbb{R}}^{r-k+1}$ that sends $a \longmapsto a \cdot \omega^{r-2k+1}$.*

In fact, they show that one obtains a Lefschetz element ω whenever $\omega = \sum_{\varnothing \neq F \in \mathbf{F}} c_F x_F$ has coefficients c_F coming from restricting to $\mathbf{F}$ any function $A \mapsto c_A$ that maps $2^E \to \mathbb{R}$ and satisfies these two properties:

(1) the *strict submodular inequality* $c_A + c_B > c_{A \cap B} + c_{A \cup B}$ for all $A \neq B$, and

(2) $c_\varnothing = c_E = 0$.

This has consequences for G acting on $A(\mathcal{M})$ and each A^k. Properties (1) and (2) above are refined by Theorem 1's parts (i) and (ii) respectively.

Corollary 6. *For any matroid $\mathcal{M}$, one has an isomorphism of $\mathbb{Z}G$-modules $A^{r-k} \to A^k$ for each $k \leq \frac{r}{2}$ and $\mathbb{R}G$-module maps $A_{\mathbb{R}}^k \to A_{\mathbb{R}}^{k+1}$ which are injective for $k < \frac{r}{2}$.*

3 Results

We recall the statement of the theorem, involving the FY-monomial $\mathbb{Z}$-basis for $A(\mathcal{M})$ in Corollary 3:

$$\mathrm{FY} := \{x_{F_1}^{m_1} x_{F_2}^{m_2} \cdots x_{F_\ell}^{m_\ell} : \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_\ell \text{ in } \mathbf{F}, \text{ and } m_i \leq \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1\}$$

This also means that the FY-monomials FY^k of degree k form a $\mathbb{Z}$-basis for A^k for each $k = 0, 1, 2, \ldots, r$.

Theorem 1 *For every matroid $\mathcal{M}$ of rank $r+1$, there exist*

(i) G-equivariant bijections $\pi : \mathrm{FY}^k \xrightarrow{\sim} \mathrm{FY}^{r-k}$ for $k \leq \frac{r}{2}$, and

(ii) G-equivariant injections $\lambda : \mathrm{FY}^k \hookrightarrow \mathrm{FY}^{k+1}$ for $k < \frac{r}{2}$.

The prove this, we organize monomials according to the fibers of the following map.

Definition 2. Define the *extended support* $\mathrm{supp}_+(a) \subset \mathbf{F}$ of an FY-monomial $a = x_{F_1}^{m_1} \cdots x_{F_\ell}^{m_\ell}$ by

$$\mathrm{supp}_+(a) := \{F_1, \ldots, F_\ell\} \cup \{E\} = \begin{cases} \{F_1, \ldots, F_\ell\} \cup \{E\} & \text{if } F_\ell \subsetneq E, \\ \{F_1, \ldots, F_\ell\} & \text{if } F_\ell = E. \end{cases}$$

Define a partial order $<_+$ on the FY-monomials in which $a <_+ b$ if a divides b and $\mathrm{supp}_+(a) = \mathrm{supp}_+(b)$.

For integers $p < q$, let $[p, q]$ denote the integer linear order inclusively from p to q. Given a sequence of such pairs $p_i < q_i$ for $i = 1, 2, \ldots, m$, let

$$\prod_{i=1}^{n} [p_i, q_i] = [p_1, q_1] \times [p_2, q_2] \times \cdots \times [p_m, q_m] \tag{3.1}$$

denote their Cartesian product, partially ordered componentwise.

Proposition 7. *For any nested flag $\{F_1 \subsetneq \cdots \subsetneq F_\ell \subsetneq E\}$ in $\mathbf{F}$ containing E, with conventions $F_0 := \varnothing$ and $F_{\ell+1} := E$, the fiber $\mathrm{supp}_+^{-1}\{F_1, \ldots, F_\ell, E\}$ is the set of monomials $\{x_{F_1}^{m_1} x_{F_2}^{m_2} \cdots x_{F_\ell}^{m_\ell} x_E^{m_{\ell+1}}\}$ satisfying these inequalities:*

$$\begin{aligned} &1 \leq m_i \leq \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1 \text{ for } i \leq \ell, \\ &0 \leq m_{\ell+1} \leq \mathrm{rk}(E) - \mathrm{rk}(F_\ell) - 1 = r - \mathrm{rk}(F_\ell). \end{aligned}$$

Consequently, the minimum and maximum degree of monomials in $\mathrm{supp}_+^{-1}\{F_1, \ldots, F_\ell, E\}$ are ℓ and $r - \ell$, and one has a poset isomorphism

$$\begin{aligned} (\mathrm{supp}_+^{-1}\{F_1, \ldots, F_\ell, E\}, <_+) \quad &\longrightarrow \quad \prod_{i=1}^{\ell} [1, \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1] \ \times \ [0, r - \mathrm{rk}(F_\ell)] \\ x_{F_1}^{m_1} x_{F_2}^{m_2} \cdots x_{F_\ell}^{m_\ell} x_E^{m_{\ell+1}} \quad &\longmapsto \quad (m_1, m_2, \ldots, m_\ell, m_{\ell+1}). \end{aligned}$$

Most assertions of the proposition are immediate from the definition of the order on FY-monomials $<_+$ and the map supp_+. The minimum and maximum degrees of monomials in $\mathrm{supp}_+^{-1}\{F_1,\ldots,F_\ell,E\}$) are achieved by

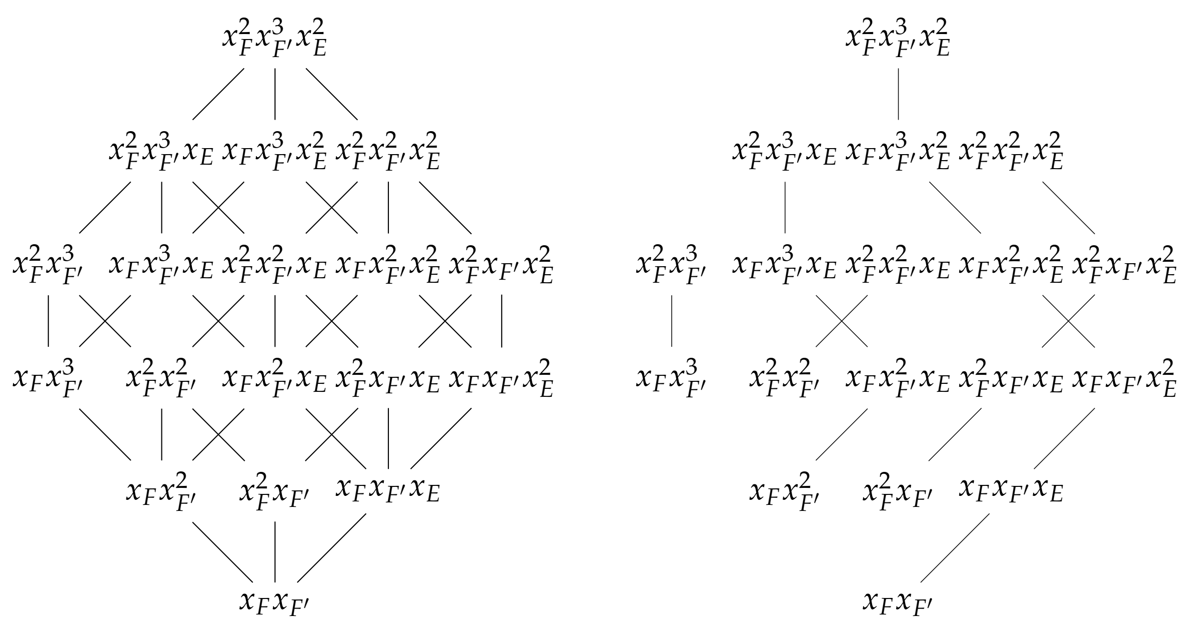
$$\begin{aligned}\deg(x_{F_1}^1 x_{F_2}^1 \cdots x_{F_\ell}^1 x_E^0) &= \ell \quad \text{and}\\ \deg\left(\prod_{i=1}^{\ell} x_{F_i}^{\mathrm{rk}(F_i)-\mathrm{rk}(F_i)-1} \cdot x_E^{\mathrm{rk}(E)-\mathrm{rk}(F_\ell)-1)}\right) &= \sum_{i=1}^{\ell+1} \big(\mathrm{rk}(F_i)-\mathrm{rk}(F_{i-1})-1\big)\\ &= \mathrm{rk}(E)-(\ell+1)\\ &= r-\ell.\end{aligned}$$

The idea behind the proof of Theorem 1 stems from the observation that all products of chains, as in (3.1), have *symmetric chain decompositions*, which can then be pulled back to each fiber $\mathrm{supp}_+^{-1}\{F_1,\ldots,F_\ell,E\}$.

Definition 3. A *symmetric chain decomposition (SCD)* of a finite ranked poset P of rank r is a disjoint decomposition $P=\bigsqcup_{i=1}^t C_i$ in which each C_i is a totally ordered subset containing one element of each rank $\{\rho_i,\rho_i+1,\ldots,r-\rho_i-1,r-\rho_i\}$ for some $\rho_i \le \lfloor \frac{r}{2} \rfloor$.

It is not hard to check that when posets P_1, P_2 each have an SCD, then so does their Cartesian product. In particular, all products of chains have an SCD. *Fix one such SCD for each product poset in* (3.1), *once and for all,* and use the isomorphisms from Proposition 7 to induce an SCD on each fiber $\mathrm{supp}_+^{-1}\{F_1,\ldots,F_\ell,E\}$.

Example 2. Assume $\mathcal{M}$ has $\mathrm{rk}(E)=10=r+1$ with $r=9$, and one has a pair of nested flats $F\subset F'$ with $\mathrm{rk}(F)=3, \mathrm{rk}(F')=7$. Then the poset $\mathrm{supp}_+^{-1}\{F,F',E\}$ and one choice of SCD for it look as follows:



4 Further questions and conjectures

So far, we have mentioned that the unimodality statement (1.2), asserting for $k < \frac{r}{2}$ that one has $a_k \leq a_{k+1}$, is weaker than the statement in Corollary 6 asserting that there are injective $\mathbb{R}G$-module maps $A^k_{\mathbb{R}} \to A^{k+1}_{\mathbb{R}}$, which is weaker than Theorem 1(ii) asserting that there are injective G-equivariant maps of the G-sets $\mathrm{FY}^k \hookrightarrow \mathrm{FY}^{k+1}$. Here, we wish to consider not only unimodality for $(a_0, a_1, \ldots, a_r)$, but other properties like *log-concavity*, *the Pólya frequency property*, and how to similarly lift them to statements regarding $\mathbb{R}G$-modules and G-permutation representations. In phrasing this, it helps to consider the character and Burnside rings.

Definition 4. For a finite group G, its *virtual (complex) character ring* $R_{\mathbb{C}}(G)$ is the free $\mathbb{Z}$-submodule of the ring of (conjugacy) class functions $\{f : G \to \mathbb{C}\}$, having as a $\mathbb{Z}$-basis the irreducible complex characters of G. If a character χ can be written as a positive linear combination of irreducible characters of G, we say that χ is a *genuine character*, and write $\chi \geq_{R_{\mathbb{C}}(G)} 0$.

Similarly, one can define its *Burnside ring* $B(G)$ by now having as basis the isomorphism classes $[X]$ of finite G-sets X. Then $B(G)$ is the $\mathbb{Z}$-module that mods out by the span of all elements $[X \sqcup Y] - ([X] + [Y])$ and if $b \in B(G)$ can be written as a positive linear combination of isomorphism classes, then b is a *genuine permutation representation*, and $b \geq_{B(G)} 0$.

4.1 PF sequences and log-concavity

For a sequence of *positive* real numbers $(a_0, a_1, \ldots, a_r)$, the property of *unimodality* lies at the bottom of a hierarchy of concepts

$$\begin{array}{ccccccccc} \text{unimodal} & \Leftarrow & PF_2 & \Leftarrow & PF_3 & \Leftarrow & \cdots & \Leftarrow & PF_\infty \\ & & \| & & & & & & \| \\ & & \text{(strongly) log-concave} & & & & & & PF \end{array} \tag{4.1}$$

which we next review, along with their equivariant and Burnside ring extensions.

Definition 5. Say a sequence of positive reals $(a_0, a_1, \ldots, a_r)$ is *strongly log-concave* (or PF_2) if $0 \leq i \leq j \leq k \leq \ell \leq r$ and $i + \ell = j + k$ implies

$$a_i a_\ell \leq a_j a_k, \text{ or equivalently, } \det \begin{bmatrix} a_j & a_\ell \\ a_i & a_k \end{bmatrix} \geq 0.$$

For $\ell = 2, 3, 4, \ldots$, say that the sequence is PF_ℓ if the associated (infinite) *Toeplitz matrix*

$$T(a_0, \ldots, a_r) := \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{r-1} & a_r & 0 & 0 & \cdots \\ 0 & a_0 & a_1 & \cdots & a_{r-2} & a_{r-1} & a_r & 0 & \cdots \\ 0 & 0 & a_0 & \cdots & a_{r-3} & a_{r-2} & a_{r-1} & a_r & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

has all *nonnegative* square minor subdeterminants of size $m \times m$ for $1 \leq m \leq \ell$. Say that the sequence is a *Pólya frequency sequence* (or PF_∞, or just PF) if it is PF_ℓ for all $\ell = 2, 3, \ldots$.

Definition 6. For a finite group G and (genuine, nonzero) $\mathbb{C}G$-modules $(A^0, A^1, \ldots, A^r)$, define the analogous notions of *equivariant unimodality*, *equivariant strong log-concavity*, *equivariant PF_r or PF_∞* by replacing the numerical inequalities in Definition 5 by inequalities in $\mathbb{R}_\mathbb{C}(G)$, or, similarly, one can define all these concepts to be Burnside if these inequalities are in the Burnside ring $B(G)$.

We've seen for Chow rings $A(\mathcal{M})$ of rank $r+1$ matroids $\mathcal{M}$, and $G = \mathrm{Aut}(\mathcal{M})$, the sequence $(a_0, a_1, \ldots, a_r)$ with $a_k := \mathrm{rk}_\mathbb{Z} A_k$ is *unimodal*; after tensoring with $\mathbb{C}$, the sequence of $\mathbb{C}G$-modules $(A^0_\mathbb{C}, A^1_\mathbb{C}, \ldots, A^r_\mathbb{C})$ is *equivariantly unimodal*; and the sequence of G-sets $(\mathrm{FY}^0, \mathrm{FY}^1, \ldots, \mathrm{FY}^r)$ is *Burnside unimodal*.

Conjecture 1. *In the Chow ring of a rank $r+1$ matroid $\mathcal{M}$, one has that*

(i) (Ferroni-Schröter [6, Conj. 10.19]) $(a_0, \ldots, a_r)$ is PF_∞.

(ii) $(A^0_\mathbb{C}, \ldots, A^r_\mathbb{C})$ is equivariantly PF_∞.

(iii) $(\mathrm{FY}^0, \ldots, \mathrm{FY}^r)$ is Burnside PF_2 (Burnside log-concave).

Of course, in Conjecture 1, assertion (ii) implies assertion (i). However the same is not true of assertion (iii): it would only imply the weaker PF_2 part of the conjectural assertion (ii), and only imply the PF_2 part of Ferroni and Schröter's assertion (i), but not their PF_∞ assertions. We have some evidence for the following two further conjectures.

Conjecture 2. *For a Boolean matroid $\mathcal{M}$ of rank n and $i \leq j \leq k \leq \ell$ with $i + \ell = j + k$, not only is the element $[\mathrm{FY}^j][\mathrm{FY}^k] - [\mathrm{FY}^i][\mathrm{FY}^\ell] \geq_{B(\mathfrak{S}_n)} 0$, so that it is a genuine permutation representation, but furthermore one whose orbit-stabilizers are all Young subgroups $\mathfrak{S}_\lambda$.*

Conjecture 3. *For a matroid $\mathcal{M}$ of rank $r+1$ with Chow ring $A(\mathcal{M})$, and any composition $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ with $m := \sum_i \alpha \leq r$, the analogous Toeplitz minors of G-sets have*

$$\det \begin{bmatrix} [\mathrm{FY}^{\alpha_1}] & [\mathrm{FY}^{\alpha_1+\alpha_2}] & [\mathrm{FY}^{\alpha_1+\alpha_2+\alpha_3}] & \cdots & [\mathrm{FY}^{m}] \\ [\mathrm{FY}^{0}] & [\mathrm{FY}^{\alpha_2}] & [\mathrm{FY}^{\alpha_2+\alpha_3}] & \cdots & [\mathrm{FY}^{m-\alpha_1}] \\ 0 & [\mathrm{FY}^{0}] & [\mathrm{FY}^{\alpha_3}] & \cdots & [\mathrm{FY}^{m-(\alpha_1+\alpha_2)}] \\ 0 & 0 & & & \vdots \\ \vdots & \vdots & & & [\mathrm{FY}^{\alpha_{\ell-1}+\alpha_\ell}] \\ 0 & 0 & \cdots & [\mathrm{FY}^{0}] & [\mathrm{FY}^{\alpha_\ell}] \end{bmatrix} \geq_{B(G)} 0.$$

4.2 Further Questions

So far, we have focused on the Chow ring of a matroid $\mathcal{M}$ using its *maximal* building set.

One relevant example of such a building set is the *minimal* building set, which is stable under the full automorphism group $\mathrm{Aut}(\mathcal{M})$, and which arises, for example, in the study of the moduli space $\overline{M}_{0,n}$ of genus 0 curves with n marked points; see, e.g., Dotsenko [3], Gibney and Maclagan [7].

Question 4. *Does the analogue of Theorem 1 hold for the Chow ring of a matroid $\mathcal{M}$ with respect to any G-stable building set? In particular, what about the minimal building set?*

In [9, Lem. 3.1], Stembridge provides a generating function for the symmetric group representations on each graded component of the Chow ring for all Boolean matroids; see also Liao [8]. Furthermore, Stembridge's expression exhibits them as *permutation representations*, whose orbit-stabilizers are all *Young subgroups* in the symmetric group.

Question 5. *Can one provide such explicit expressions as permutation representations for other families of matroids with symmetry?*

Hilbert functions $(a_0, a_1, \ldots, a_r)$ for Chow rings of rank $r+1$ matroids are not only symmetric and unimodal, but satisfy the stronger condition of *γ-positivity*, as shown by : one has *nonnegativity* for all coefficients $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor \frac{r}{2} \rfloor})$ appearing in the unique expansion

$$\sum_{i=0}^{r} a_i t^i = \sum_{i=0}^{\lfloor \frac{r}{2} \rfloor} \gamma_i \; t^i (1+t)^{r-2i}.$$

This has been shown, independently by Ferroni, Matherne, Stevens and Vecchi [5, Thm. 3.25] and by Wang (see [5, p. 29]), that the γ-positivity for Hilbert series of Chow rings of matroids follows from results of [2].

One also has the notion of *equivariant γ-positivity* for a sequence of G-representations $(A_0, A_1, \ldots, A_r)$: upon replacing each a_i with the element $[A_i]$ of $R_{\mathbb{C}}(G)$, one asks that the uniquely defined coefficients γ_i in $R_{\mathbb{C}}(G)$ have $\gamma_i \geq_{R_{\mathbb{C}}(G)} 0$.

Conjecture 6. *For any matroid $\mathcal{M}$ of rank $r+1$ and its Chow ring $A(\mathcal{M}) = \bigoplus_i A^i$, the sequence of G-representations $(A^0_{\mathbb{C}}, A^1_{\mathbb{C}}, \ldots, A^r_{\mathbb{C}})$ is equivariantly γ-positive.*

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