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Smirnov words and the Delta Conjectures

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Abstract. We provide a combinatorial interpretation for the symmetric function $\Theta_{e_k}\Theta_{e_l}\nabla e_{n-k-l}|_{t=0}$ in terms of Smirnov words, which are words where adjacent letters are distinct. The motivation for this work is the study of a diagonal coinvariant ring with one set of commuting and two sets of anti-commuting variables, whose Frobenius characteristic is conjectured to be the symmetric function in question. It is intimately related to the two Delta conjectures, as our work is a step towards a unified formulation of these.

Résumé. Nous donnons une interprétation combinatoire à la fonction symétrique $\Theta_{e_k}\Theta_{e_l}\nabla e_{n-k-l}|_{t=0}$ en termes de mots de Smirnov, qui sont les mots dont les lettres adjacentes sont distinctes. La motivation de ce travail est l'étude de l'anneau des coinvariants diagonaux avec un jeu de variables commutatives et deux jeux de variables anticommutatives, dont la caractéristique de Frobenius est, conjecturalement, la fonction symétrique en question. Elles est intimement liée aux conjectures Delta, ce travail constituant un pas vers une formulation unifiée de ces dernières.

Keywords: Delta conjecture, coinvariant ring, Smirnov words

1 Introduction

This work is mainly concerned with a combinatorial expansion and its consequences. It is motivated by a circle of problems in representation theory, which we briefly survey in this introduction.

In the 1990s, Garsia and Haiman introduced the ring of *diagonal coinvariants* DR_n. The study of the structure of this $\mathfrak{S}_n$-module and its generalizations has been an important research topic in algebra and combinatorics ever since. The ring is defined as follows: consider the space $\mathbb{C}[\mathbf{x}_n, \mathbf{y}_n] := \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]$ and define an $\mathfrak{S}_n$-action as

$$\sigma \cdot f(x_1, \dots, x_n, y_1, \dots, y_n) := f(x_{\sigma(1)}, \dots, x_{\sigma(n)}, y_{\sigma(1)}, \dots, y_{\sigma(n)})$$

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for all $f \in \mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]$ and $\sigma \in \mathfrak{S}_n$. Let $I(\mathbf{x}_n, \mathbf{y}_n)$ be the ideal generated by the $\mathfrak{S}_n$-invariants with vanishing constant term. Then the ring of diagonal coinvariants is defined as

$$DR_n := \mathbb{C}[\mathbf{x}_n, \mathbf{y}_n] / I(\mathbf{x}_n, \mathbf{y}_n).$$

The space has a natural bi-grading: let $DR_n^{(i,j)}$ be the component of DR_n with homogeneous $\mathbf{x}$-degree i and homogeneous $\mathbf{y}$-degree j. This grading is preserved by the $\mathfrak{S}_n$-action. Garsia and Haiman conjectured, and Haiman later proved [9], a formula for the graded Frobenius characteristic of the diagonal harmonics:

$$\operatorname{grFrob}(DR_n; q, t) := \sum_{i,j \in \mathbb{N}} q^i t^j \operatorname{Frob}(DR_n^{(i,j)}) = \nabla e_n, \tag{1.1}$$

where e_n is the n-th elementary symmetric function and ∇ is the operator introduced in [1]. In [7], the authors gave a combinatorial formula for this graded Frobenius character ∇e_n, in terms of *labelled Dyck paths*, called the *shuffle conjecture*. It is now a theorem by Carlsson and Mellit [2].

The *Delta conjecture* is a pair of combinatorial formulas for the symmetric function $\Delta'_{e_{n-k-1}} e_n$ in terms of *decorated* labelled Dyck paths, stated in [8] –we detail the combinatorics in Section 5. Here $\Delta'_{e_{n-k-1}}$ is a certain symmetric function operator (depending on q, t). These conjectures reduce to the shuffle theorem when $k = 0$.

This extension of the combinatorial setting led Zabrocki, D'Adderio, Iraci and Vanden Wyngaerd to introduce extensions of DR_n [15, 3]. Consider the ring $\mathbb{C}[\mathbf{x}_n, \mathbf{y}_n, \boldsymbol{\theta}_n, \boldsymbol{\xi}_n]$ where the $\mathbf{x}_n, \mathbf{y}_n$ are the usual commuting (or *bosonic*) variables, while the $\boldsymbol{\theta}_n$, $\boldsymbol{\xi}_n$ are anti-commuting (or *fermionic*): $\theta_i \theta_j = -\theta_j \theta_i$ and $\xi_i \xi_j = -\xi_j \xi_i$ for all $1 \leq i, j \leq n$.

Again, consider the $\mathfrak{S}_n$-action that permutes all the variables simultaneously. If $I(\mathbf{x}_n, \mathbf{y}_n, \boldsymbol{\theta}_n, \boldsymbol{\xi}_n)$ now denotes the ideal generated by the $\mathfrak{S}_n$-invariants without constant term, define $TDR_n := \mathbb{C}[\mathbf{x}_n, \mathbf{y}_n, \boldsymbol{\theta}_n, \boldsymbol{\xi}_n] / I(\mathbf{x}_n, \mathbf{y}_n, \boldsymbol{\theta}_n, \boldsymbol{\xi}_n)$. This ring is naturally quadruply graded: let $TDR_n^{(i,j,k,l)}$ denote the component of TDR_n of homogeneous (i, j, k, l)-degrees.

In [15] Zabrocki conjectured

$$\sum_{i,j \in \mathbb{N}} q^i t^j \operatorname{Frob}(TDR_n^{(i,j,k,0)}) \stackrel{?}{=} \Delta'_{e_{n-k-1}} e_n. \tag{1.2}$$

Note that the symmetric function of the Delta conjectures occurs on the right-hand side. In [3], D'Adderio with the first and third named authors introduced operators Θ_f (depending on q, t), for any symmetric function f, and showed that $\Delta'_{n-k-1} e_n = \Theta_{e_k} \nabla e_{n-k}$. This permitted them to extend Zabrocki's conjecture as follows:

$$\sum_{i,j \in \mathbb{N}} q^i t^j \operatorname{Frob}(TDR_n^{(i,j,k,l)}) \stackrel{?}{=} \Theta_{e_l} \Theta_{e_k} \nabla e_{n-k-l}. \tag{1.3}$$

Special cases of the conjecture have been studied over the years. Let us call the "(a, b)-case" the structures linked to the diagonal coinvariant ring with a sets of bosonic variables and b sets of fermionic variables. The $(2, 1)$- and the $(2, 2)$-cases thus occur in (1.2) and (1.3) respectively, and the $(2, 0)$-case is the known case (1.1). The $(1, 0)$ and $(0, 1)$ cases are classical rings and the conjecture is known to hold in this case. The $(1, 1)$-case, or the *superspace coinvariant ring*, is still open, but Rhoades and Wilson in [12] showed that its Hilbert series agrees with the expected formula. The $(0, 2)$-case, or *fermionic Theta* case, was proved by Iraci, Rhoades, and Romero in [10].

In this abstract, we will turn our interest to the combinatorics that (conjecturally) occur in the $(1, 2)$-case. Following Conjecture (1.3), we thus are led to study the symmetric function $\Theta_{e_k}\Theta_{e_l}\nabla e_{n-k-l}|_{t=0}$.

Our combinatorial model is that of *segmented Smirnov words*. A Smirnov word is a word in the alphabet of positive integers such that adjacent letters are distinct. A segmented Smirnov word is the concatenation of Smirnov words with prescribed lengths (see Definition 2.1). The main result of this paper (Theorem 2.5) is an expansion in terms of segmented Smirnov words.

Theorem. *For any n, k, l, we have the identity between symmetric functions in $(x_i)_{i\geq 1}$*

$$\Theta_{e_k}\Theta_{e_l}\nabla e_{n-k-l}|_{t=0} = \sum_{w\in \mathsf{SW}(n,k,l)} q^{\mathsf{sminv}(w)} x_{w_1}x_{w_2}\cdots x_{w_n}.$$

Here $\mathsf{SW}(n, k, l)$ is the set of segmented Smirnov words with k descents and l ascents, while the power of q is given by a new *sminversion* statistic on these words (see Definition 2.3). This expansion can be expressed more compactly in terms of fundamental quasisymmetric functions (Proposition 2.7).

The proof of the main theorem relies on an algebraic recursion (Proposition 2.4) for the symmetric function under study. We show in Section 3 that the combinatorial expansion satisfies indeed the same recursion.

In Section 4, we focus on the special case $k + l = n - 1$ which turns out to be linked to various topics in the literature. In Section 5, we describe an explicit bijection between segmented Smirnov words and "doubly decorated labelled Dyck paths" (Theorem 5.1), motivated by a potential unified Delta conjecture.

2 Preliminaries and main result

Combinatorics. In this work $\mathbb{Z}_+$ is the set of positive integers, and we will fix $n \in \mathbb{Z}_+$. We write $\mu \vDash_0 n$ if μ is a *weak composition* of n, that is $\mu = (\mu_1, \mu_2, \dots)$ where the μ_i are nonnegative integers that sum to n. A composition $\alpha \vDash n$ is a finite sequence $\alpha = (\alpha_1, \dots, \alpha_t)$ of positive integers that sums to n.

Definition 2.1. A *Smirnov word* of length n is an element $w \in \mathbb{Z}_+^n$ such that $w_i \neq w_{i+1}$ for all $1 \leq i < n$. A *segmented Smirnov word* is a word $w \in \mathbb{Z}_+^n$ together with a composition $\alpha = (\alpha_1, \dots, \alpha_t) \vDash n$ such that if w is written as the concatenation $w^1 \cdots w^t$ where each w^i has length α_i, then each w^i is a Smirnov word.

Let $\mathsf{SW}(n)$ be the set of segmented Smirnov words of length n. We say that α is the *shape* of w. We call $w^1, \dots, w^t$ *segments* of w. We usually simply denote a segmented Smirnov word by w, and omit the shape α. In examples, we separate segments by vertical bars. Segmented Smirnov words of shape (n) are naturally identified with Smirnow words of length n.

Given $\mu \vDash_0 n$, we denote by $\mathsf{SW}(\mu)$ the set of segmented Smirnov words with content μ, that is they contain μ_1 occurrences of 1, μ_2 occurrences of 2, and so on. We clearly have $\mathsf{SW}(n) = \bigcup_{\mu \vDash_0 n} \mathsf{SW}(\mu)$. We call *segmented permutation* a segmented Smirnov word in $\mathsf{SW}(1^n)$. Note that these can be identified with pairs (σ, α) with $\sigma \in S_n$ and $\alpha \vDash n$.

Example 2.2. If $\mu = (2,1)$, then $\mathsf{SW}(\mu)$ has 8 elements: $1|1|2, 1|2|1, 2|1|1$ with shape $(1,1,1)$; $1|12, 1|21$ with shape $(1,2)$; $21|1, 12|1$ with shape $(2,1)$ and 121 with shape (3).

Given a Smirnov word w, we say that i is an *ascent* of w if $w_{i+1} > w_i$, and a *descent* otherwise. If $w \in \mathsf{SW}(n)$, we say that i is an ascent (resp. descent) of w if it is an ascent (resp. descent) of one of its segments. Let us denote by $\mathsf{SW}(n,k,l)$ the set of segmented Smirnov words with k descents and l ascents; note that these words have $n-k-l$ segments. For $\mu \vDash_0 n$, we also define $\mathsf{SW}(\mu,k,l)$ as the intersection $\mathsf{SW}(\mu) \cap \mathsf{SW}(n,k,l)$.

We can now define the main new statistic of this work. An index $i \in \{1, \dots, n\}$ is called *initial* (resp. *final*) if it corresponds to the first (resp. last) position of a segment, *i.e.* if it has the form $i = \alpha_1 + \cdots + \alpha_{m-1} + 1$ (resp. $i = \alpha_1 + \cdots + \alpha_m$) for some $t \in \{1, \dots, t\}$.

Definition 2.3 (The sminv statistic)**.** For a segmented Smirnov word w of shape $\alpha \vDash n$, we say that (i,j) with $1 \leq i < j \leq n$ is a *sminversion* if $w_i > w_j$ and one of the following holds:

1. j is initial in w;
2. $w_{j-1} > w_i$;
3. $i \neq j-1$, $w_{j-1} = w_i$, and $j-1$ is initial in w;
4. $i \neq j-1$ and $w_{j-2} > w_{j-1} = w_i$.

We let $\mathsf{sminv}(w)$ be the number of sminversions of w. The segmented Smirnov word $w = 321|2131$, has sminv equal to 4, since $(1,4)$, $(2,5)$, $(2,7)$ and $(4,7)$ are its sminversions. Finally, define

$$\mathsf{SW}_q(\mu,k,l) = \sum_{w \in \mathsf{SW}(\mu,k,l)} q^{\mathsf{sminv}(w)}.$$

In view of Example 2.2, we can compute that $\mathsf{SW}_q((2,1),0,0) = 1 + q + q^2$; $\mathsf{SW}_q((2,1),1,0) = 1 + q$; $\mathsf{SW}_q((2,1),1,1) = 1$; and $\mathsf{SW}_q((2,1),0,1) = 1 + q$.

Let us note two important cases where the statistic sminv simplifies:

- When w is a segmented permutation σ, only cases (1) and (2) occur.
- When w has shape (n), *i.e.* w is a Smirnov word, only cases (2) and (4).

Symmetric functions. We refer to [14, Ch. 7] for undefined terminology. Consider the ring Λ of symmetric functions in $(x_i)_{i\in\mathbb{Z}_+}$ with coefficients in $\mathbb{Q}(q)$. Let us define

$$\mathsf{SF}(n,k,l) := \Theta_{e_k}\Theta_{e_l}\nabla e_{n-k-l}|_{t=0} \in \Lambda \tag{2.1}$$

to simplify notations. Here $h_j^\perp$ is the operator dual to multiplication by h_j, with respect to the standard duality on Λ given by $\langle h_\lambda, m_\mu\rangle = \delta_{\lambda,\mu}$.

The following proposition is the key to the combinatorial interpretation:

Proposition 2.4. *For any n,k,l with $k+l<n$, $\mathsf{SF}(n,k,l)$ satisfies*

$$\begin{aligned} h_j^\perp \mathsf{SF}(n,k,l) = \sum_{r=0}^{j}\sum_{a=0}^{j}\sum_{i=0}^{j} q^{\binom{r-i}{2}} q^{\binom{a-i}{2}} \begin{bmatrix} n-k-l \\ j-r-a+i \end{bmatrix}_q \begin{bmatrix} n-k-l-(j-r-a)-1 \\ i \end{bmatrix}_q \\ \times \begin{bmatrix} n-k-l-(j-r-a+i) \\ r-i \end{bmatrix}_q \begin{bmatrix} n-k-l-(j-r-a+i) \\ a-i \end{bmatrix}_q \mathsf{SF}(n-j,k-r,l-a) \end{aligned}$$

for any $j\geq 1$. Moreover $\mathsf{SF}(0,k,l) = \delta_{k,0}\delta_{l,0}$ and $\mathsf{SF}(n,k,l) = 0$ if $n<0$.

We omit the proof of this proposition in this abstract: it comes from the specialization $t=0$ of [6, Theorem 8.2], with some extra elementary computations.

Main result. Define

$$\mathsf{SW}_{x;q}(n,k,l) = \sum_{\mu\vDash_0 n} \mathsf{SW}_q(\mu,k,l)x^\mu = \sum_{w\in\mathsf{SW}(n,k,l)} q^{\mathsf{sminv}(w)} x_w, \tag{2.2}$$

where $x_w = \prod_{i=1}^n x_{w_i}$.

Theorem 2.5. *For any n,k,l with $k+l<n$, we have the identity*

$$\mathsf{SF}(n,k,l) = \mathsf{SW}_{x;q}(n,k,l). \tag{2.3}$$

Expansion into fundamental quasisymmetric functions. Let w be a segmented Smirnov word. For $1\leq i\leq n$, we say that i is *thick* if i is initial or $w_{i-1} > w_i$, and *thin* otherwise.

Definition 2.6. Let σ be a segmented permutation of size n, and $i\in\{1,\ldots,n\}$. Let j be such that $\sigma_j = \sigma_{i+1}$. We say that i is *splitting* for σ if either of the following holds:

- i and j are in the same segment of σ, and $|i-j|=1$;
- i is thick and j is thin;
- i and j are both thin and $i<j$;
- i and j are both thick and $j<i$.

Let $\mathsf{Split}(\sigma) = \{1 \leq i \leq n-1 \mid i \text{ is splitting for } \sigma\}$. For any subset $S \subseteq [n-1]$, let $Q_{S,n}$ be the fundamental quasisymmetric function associated to S (see [14, Sec. 7.19]).

Proposition 2.7.

$$\mathsf{SW}_{x;q}(n,k,l) = \sum_{\sigma \in \mathsf{SW}(1^n,k,l)} q^{\mathsf{sminv}(\sigma)} Q_{\mathsf{Split}(\sigma),n}.$$

The proof relies on grouping terms in the right-hand side of (2.2) using a certain "reading order". We omit it in this abstract.

3 Proof of Theorem 2.5

The proof consists in showing that the series $\mathsf{SW}_{x;q}(n,k,l)$ satisfies the relations encoded in Proposition 2.4.

In detail, fix $\mu \vDash_0 n$ nonzero, let F_μ be the coefficient of x^μ in the power series $\mathsf{SF}(n,k,l)$, and let the last nonzero part of μ be $\mu_m = j$. Then by taking the inner product of $\mathsf{SF}(n,k,l)$ with h_μ in Proposition 2.4, we obtain a recurrence for F_μ. Theorem 2.5 then claims that $\mathsf{SW}_q(\mu,k,l)$ obeys the same recurrence. Explicitly, let μ^- be equal to μ except that $\mu^-_m = 0$, and let $s := n-k-l$ be the number of segments, then one has to show:

$$\begin{aligned}\mathsf{SW}_q(\mu,k,l) = \sum_{i=0}^{j}\sum_{r=i}^{j}\sum_{a=i}^{j} q^{\binom{r-i}{2}} \begin{bmatrix} s-(j-r-a+i) \\ r-i \end{bmatrix}_q q^{\binom{a-i}{2}} \begin{bmatrix} s-(j-r-a+i) \\ a-i \end{bmatrix}_q \\ \times \begin{bmatrix} s \\ j-r-a+i \end{bmatrix}_q \begin{bmatrix} s-j+r+a-1 \\ i \end{bmatrix}_q \mathsf{SW}_q(\mu^-,k-r,l-a). \end{aligned} \tag{3.1}$$

We will sketch a bijective proof below. Since it is quite technical, let us first give the simpler proof in the case $\mu = 1^n$, which boils down to the following proposition:

Proposition 3.1. *For any n,k,l with $k+l<n$, the polynomials $\mathsf{SW}_q(1^n,k,l)$ satisfy*

$$\begin{aligned}\mathsf{SW}_q(1^n,k,l) = [n-k-l]_q \Big(&\mathsf{SW}_q(1^{n-1},k,l) + \mathsf{SW}_q(1^{n-1},k-1,l) \\ &+ \mathsf{SW}_q(1^{n-1},k,l-1) + \mathsf{SW}_q(1^{n-1},k-1,l-1)\Big).\end{aligned}$$

Proof. Given a segmented permutation on $n-1$ elements, we want to insert n in all possible ways. It can be done in four different manners:

1. as a new singleton segment. This keeps the number of ascents and descents the same, and increases the number of segments by one;
2. at the beginning of a segment. This creates no ascent and one descent, and keeps the number of segments the same;
3. at the end of a segment. This creates one ascent and no descents, and keeps the number of segments the same;

4. as an element merging two adjacent segments $\cdots w_1|w_2\cdots \to \cdots w_1 n w_2 \cdots$. This creates an ascent and a descent, and decreases the number of segments by one.

Each of these insertions can be done in s different ways, if s is the number of segments in the final segmented permutation. Moreover, the construction is injective: if i is such that $\sigma_i = n$ for some $\sigma \in \mathsf{SW}(1^n)$, then by looking whether i is initial and/or final, one knows which of the four types of insertion was performed.

From this one sees that the proposition holds at $q = 1$. As for sminversions, one checks that inserting n does not modify the number of those involving letters in $\{1, \dots, n-1\}$. Moreover, the value n is part of a sminversion with all initial letters to its right. In each case, this increases sminv by all possible amounts between 0 and $s-1 = n-k-l-1$. The recursion of Proposition 3.1 follows.

□

Sketch of the proof of (3.1). The idea is the same as in the standard case above. Starting with a word in $w \in \mathsf{SW}(\mu^-)$, we want to insert j occurrences of the letter m (larger than all letters of w) to create a word w' in $\mathsf{SW}(\mu, k, l)$. As in the standard case, we distinguish if the occurrences of m are initial and/or final. The complication comes from inserting several occurrences of m.

Pick $i, a, r \geq 0$ such that $i \leq a \leq j$ and $i \leq r \leq j$. Then we insert *successively*:

- i is occurrences of m that are neither initial nor final (this is done by merging adjacent segments as in the standard case);
- $r-i$ occurrences of m that initial but not final;
- $a-i$ occurrences of m that are final but not initial;
- and finally $j-r-a+i$ singletons equal to m.

Note that the total number of occurrences of m is indeed j. Since we want $s = n-k-l$ segments in the end, we must have $s+i-(j-r-a+i) = s-j+r+a$ segments in w. Also, w must have $k-r$ descents and $l-a$ ascents so that the final word has k descents and l ascents.

The claim is that the number of ways to insert m is given by the coefficient of $\mathsf{SW}_q(\mu^-, k-r, l-a)$ in (3.1) at $q = 1$: each of the four binomial coefficients can be whown to correspond naturally to one of the cases above. To complete the proof, one needs to check that then number of sminversions behaves as wanted. We omit the details in this abstract. □

4 The maximal case $k + l = n - 1$

We focus in this section on various aspects of the case $k+l = n-1$ of Theorem 2.5. The combinatorial side now involves only Smirnov words. It is also conjecturally giving the graded Frobenius characteristic of the subspace of the $(1,2)$-coinvariant space of maximum total degree in the fermionic variables $\boldsymbol{\zeta}_n, \boldsymbol{\xi}_n$ (cf. (1.3)).

Chromatic symmetric function interpretation. Given a graph $G = (V, E)$, a *proper coloring* is a function $c \colon V \to \mathbb{Z}_+$ such that $\{i, j\} \in E \implies c(i) \neq c(j)$. If $V = [n]$, a *descent* of a coloring is an edge $\{i, j\} \in E$ such that $i < j$ and $c(i) > c(j)$. The chromatic quasisymmetric function of G is defined as

$$X_G(\mathbf{x}; q) = \sum_{\substack{c \colon V \to \mathbb{Z}_+ \\ c \text{ proper}}} q^{\mathsf{des}(c)} \prod_{v \in V} x_{c(v)},$$

where $\mathsf{des}(c)$ is the number of descents of c.

For the path graph $G_n = 1 - 2 - \cdots - n$, if c is a proper coloring then $c(1)c(2)\ldots c(n)$ is a Smirnov word of length n, and vice versa, if w is a Smirnov word of length n, then $c(i) = w_i$ is a proper coloring of G_n. It follows from Theorem 2.5

$$X_{G_n}(\mathbf{x}; u) = \sum_{k=0}^{n-1} u^k \Theta_{e_k} \Theta_{e_{n-k-1}} e_1 \Big|_{q=1, t=0}.$$

This suggests also the existence of an extra q-grading on the cohomology of the permutahedral toric variety $\mathcal{V}_n$: indeed the graded Frobenius characteristic of that cohomology is known to be given by $\omega X_{G_n}(\mathbf{x}; u)$, see [13].

Parallelogram polyominoes. A *parallelogram polyomino* of size $m \times n$ is a pair of north-east lattice paths on a $m \times n$ grid, such that first is always strictly above the second, except on the endpoints $(0, 0)$ and (m, n). A labelling of a parallelogram polyomino is an assignment of positive integer labels to each cell that has a north step of the first path as its left border, or an east step of the second path as its bottom border, such that columns are strictly increasing bottom to top and rows are strictly decreasing left to right. In [5] it is conjectured that $\Theta_{e_{m-1}} \Theta_{e_{n-1}} e_1$ enumerates labelled parallelogram polyominoes of size $m \times n$ with respect to two statistics, one of which is (a labelled version of) the area, and the other is unknown.

It is immediate to see that parallelogram polyominoes of size $(n - k) \times (k + 1)$ and area 0 are again in bijection with Smirnov words of length n with k descents, and proper colorings of G_n with k descents. Indeed, reading the labels of such a polyomino bottom to top, left to right, yields a Smirnov word of size n with k descents, and the correspondence is bijective. In particular, sminversions on Smirnov words define a statistic on this subfamily of parallelogram polyominoes, proving the conjectural identity and partially answering Problem 7.13 from [5] in the case when the area is 0.

The case $q = 0$. Note that in this case, it is known [10] that the symmetric function in Theorem 2.5 is the Frobenius characteristic of the $(0, 2)$-case. It was also shown that the high-degree part of this module has a basis indexed by noncrossing partitions. In particular, this means that there is a bijection between segmented permutations with one segment (that is, permutations) with zero sminv, and noncrossing partitions.

Lemma 4.1. *Permutations with zero* sminv *are exactly* 231*-avoiding permutations, that is permutations σ with no $i < j < k$ such that $\sigma_k < \sigma_i < \sigma_j$.*

Proof. Let σ be a permutation, and suppose that it has a 231 pattern, that is, that there exist indices $i < j < k$ such that $\sigma_k < \sigma_i < \sigma_j$. Let $m = \min j < a \leq k \mid \sigma_m < \sigma_i$; by definition, $i < j \leq m-1$, and $\sigma_{m-1} > \sigma_i$, so (i, m) is a sminversion of σ. It follows that permutations with zero sminv are 231-avoiding permutations. Since a sminversion in a permutation corresponds to a 231 pattern, this concludes the proof. □

Let π be a noncrossing partition, and let $\phi(\pi)$ be the permutation that, in one line notation, is written by listing the blocks of π sorted by their smallest element, with the elements of each segment sorted in decreasing order. Let us call *decreasing run* of a permutation σ a maximal subsequence of consecutive decreasing entries of σ (in one line notation): then the blocks of π correspond to the decreasing runs of $\phi(\pi)$. For instance, if $\pi = \{\{1,2,5\},\{3,4\},\{6,8,9\},\{7\}\}$, then $\phi(\pi) = 521439867$.

The map ϕ defines a classical bijection between noncrossing partitions of size n with $k+1$ blocks and 231-avoiding permutations with k descents. This recovers known numerology about the $(0,2)$-case.

Remark 4.2. More generally, standard segmented permutations with zero sminv can be characterized as 231-avoiding permutations where letters of a segment are smaller all than letters of the segments to its right. These can be easily counted, and we recover the total dimension of the $(0,2)$-coinvariant ring given by $\binom{2n+1}{n}$.

5 Connection with the Delta conjectures

Let us first note that we recover known combinatorial interpretations when setting $k = 0$ (resp. $l = 0$) in Theorem 2.5. Indeed this gives an expansion over segmented Smirnov words with no descents (resp. ascents), and these are easily identified with ordered multiset partitions [11]. In each case, the sminv statistic can moreover be seen to be distributed as the inv statistic on ordered set partitions.

The two different Delta conjectures are as follows:

$$\Delta'_{e_{n-k-1}} e_n = \Theta_{e_k} \nabla e_{n-k} = \sum_{D \in \mathsf{LD}(n)^{*k}} q^{\mathsf{dinv}(D)} t^{\mathsf{area}(D)} x^D \tag{5.1}$$

$$\stackrel{?}{=} \sum_{D \in \mathsf{LD}(n)^{\bullet k}} q^{\mathsf{dinv}(D)} t^{\mathsf{area}(D)} x^D. \tag{5.2}$$

The sets $\mathsf{LD}(n)^{*k}$ and $\mathsf{LD}(n)^{\bullet k}$ denote labelled Dyck paths of size n with k decorations on *rises* or *valleys*, respectively; and the statistics dinv and area depend on the decorations. So

(5.1) is referred to as the *rise version* and (5.2) as the *valley version* of the Delta conjecture. The rise version was recently proved in [4].

Let us make some of the combinatorics explicit. A *Dyck path* of size n is a lattice path starting at $(0,0)$, ending at (n,n), using only unit North (N) and East (E) steps, and staying weakly above the line $x=y$. A *labelled Dyck path* is a Dyck path together with a positive integer label on each of its vertical steps such that labels on consecutive vertical steps must be strictly increasing (from bottom to top).

A *rise* of a labelled Dyck path is a North step that is preceded by another North step. A *(contractible) valley* of a labelled Dyck path is a vertical step v that is preceded either by two horizontal steps, or by a horizontal step that is preceded by a vertical step whose label is strictly smaller than v's label.

A *decorated labelled Dyck path* D is a labelled Dyck path, together with a choice of rises and (contractible) valleys, which are *decorated*. Let $\mathsf{DRise}(D)$, resp. $\mathsf{DValley}(D)$, be the set of $i \in [n]$ such that the i-th vertical step of D is a decorated rise, resp. a decorated valley. We decorate rises with a $*$ and valleys with a $\bullet$. The set of decorated labelled Dyck paths with k decorated rises and l decorated valleys, is denoted by $\mathsf{LD}(n)^{*k,\bullet l}$. The sets $\mathsf{LD}(n)^{*k}$, resp. and $\mathsf{LD}(n)^{\bullet l}$, above correspond to setting l, resp. k, to 0.

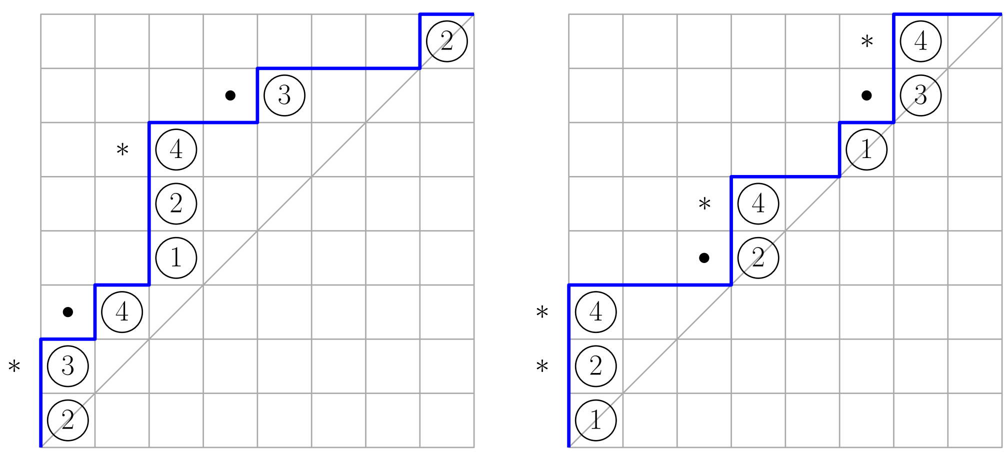

Figure 1: Elements of $\mathsf{LD}(8)^{*2,\bullet 2}$ (left) and $\mathsf{LD}_0(8)^{*4,\bullet 2}$ (right).

Given a decorated labelled Dyck path D of size n, its *area word* is the word of non-negative integers whose i-th letter equals the number of whole squares between the i-th vertical step of the path and the line $x=y$. If a is the area word of D, the *area* of D is

$$\mathsf{area}(D) := \sum_{i \in [n] \setminus \mathsf{DRise}(D)} a_i. \tag{5.3}$$

Take D to be the left path in Figure 1. We have $\mathsf{DRise}(D) = \{2,6\}$, $\mathsf{DValley} = \{3,7\}$. Its area word of D is 01112320, and so its area equals 6.

The statistic $\mathsf{dinv}(D)$ counts "diagonal inversions" minus the number of decorated valleys; we omit its precise definition in this abstract.

In [3], the authors conjectured a partial formula for a possible *unified Delta conjecture*, for which they have significant computational evidence:

$$\Theta_{e_k}\Theta_{e_l}\nabla e_{n-k-l}\big|_{q=1} \overset{?}{=} \sum_{D\in\mathsf{LD}(n)^{*k,\bullet l}} t^{\mathsf{area}(D)}x^D, \tag{5.4}$$

The goal would thus be to find a statistic $\mathsf{qstat}\colon \mathsf{LD}(n)^{*k,\bullet l} \to \mathbb{N}$ so that

$$\Theta_{e_k}\Theta_{e_l}\nabla e_{n-k-l} \overset{?}{=} \sum_{D\in\mathsf{LD}(n)^{*k,\bullet l}} q^{\mathsf{qstat}(D)}t^{\mathsf{area}(D)}x^D; \tag{5.5}$$

and such that when $k=0$ or $l=0$, the formula reduces to (5.1) or (5.2), respectively.

Let us now come back to our setting. Comparing our main Theorem 2.5 at $q=1$ with (5.4) at $t=0$, we get the conjectural existence of a bijection between labelled Dyck paths of area zero and segmented Smirnov words. This bijection exists indeed: Let $\mathsf{LD}_0(n)^{*k,\bullet l}$ be the subset of area zero Dyck paths in $\mathsf{LD}(n)^{*k,\bullet l}$.

Theorem 5.1. *For any n,k,l, there is a bijection ϕ between $\mathsf{SW}(n,k,l)$ and $\mathsf{LD}_0(n)^{*k,\bullet l}$ such that $x_w = x^{\phi(w)}$.*

Sketch of the proof. Paths in $\mathsf{LD}_0(n)^{*k,\bullet l}$ have a very specific shape: they are the concatenation of paths of the form N^iE^i, where all rises are decorated; see Figure 1, right. This precisely ensures that the area is zero, cf. Formula (5.3).

For $\mu \vDash_0 n$, and let $\mathsf{LD}_0(\mu)^{*k,\bullet l}$ be the subset of $\mathsf{LD}_0(n)^{*k,\bullet l}$ such that $x^D = x^\mu$. Using the special structure detailed above, one can then show bijectively that the cardinalities of the sets of $\mathsf{LD}_0(\mu)^{*k,\bullet l}$ decompose as $\mathsf{SW}_{q=1}(\mu,k,l)$: namely, they satisfy (3.1) at $q=1$. By matching with the bijective decomposition of $\mathsf{SW}(\mu,k,l)$ in Section 3, we can obtain a recursively defined bijection ϕ between the two sets. We omit the details in this abstract. □

What about q ? By transporting the sminv statistic through the bijection ϕ, we get a q-statistic on $\mathsf{LD}_0(n)^{*k,\bullet l}$. Now this statistic will *not* satisfy the unified Delta conjecture (5.5) at $t=0$, because it does not match the dinv-statistic coming from the rise Delta conjecture.

It is however possible to fix this –thus we do have a unified Delta conjecture at $t=0$– by recursively defining a different q-statistic on $\mathsf{SW}(n)$. Roughly put, this is done by ordering segments in *ad hoc* ways when proving the recursion for $\mathsf{SW}(\mu,k,l)$ (for sminv we simply order segments right to left).

Added in revision: this is done explicitly in the long version of this work.

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