Séminaire Lotharingien de Combinatoire **91B** (2024)
Article #59, 12 pp.

Proceedings of the 36th Conference on Formal Power Series and Algebraic Combinatorics (Bochum)

Projective dimension of weakly chordal graphic arrangements

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Abstract. A graphic arrangement is a subarrangement of the braid arrangement whose set of hyperplanes is determined by an undirected graph. A classical result due to Stanley, Edelman and Reiner states that a graphic arrangement is free if and only if the corresponding graph is chordal, i.e., the graph has no chordless cycle with four or more vertices. In this article we extend this result by proving that the module of logarithmic derivations of a graphic arrangement has projective dimension at most one if and only if the corresponding graph is weakly chordal, i.e., the graph and its complement have no chordless cycle with five or more vertices.

Keywords: Hyperplane arrangements, graph theory, projective resolutions

1 Introduction

The principal algebraic invariant associated to a hyperplane arrangement $\mathcal{A}$ is its module of logarithmic vectors fields or derivation module $D(\mathcal{A})$. Such modules provide an interesting class of finitely generated graded modules over the coordinate ring of the ambient space of the arrangement. The chief problem is to relate the algebraic structure of $D(\mathcal{A})$ to the combinatorial structure of $\mathcal{A}$, i.e., whether it is free or more generally to determine its projective dimension or even graded Betti numbers. In general, this is notoriously difficult and still wide open, at its center is Terao's famous conjecture which states that over a fixed field of definition, the freeness of $D(\mathcal{A})$ is completely determined

*abetaku@rikkyo.ac.jp. Takuro Abe was partially supported by JSPS KAKENHI Grant Number JP21H00975.

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by combinatorial data. Conversely, one might ask which combinatorial properties of $\mathcal{A}$ are determined by the algebraic structure of $D(\mathcal{A})$.

It is natural to approach these very intricate questions by restricting attention to certain distinguished classes of arrangements.

A prominent and much studied class are the *graphic arrangements*, around which our present article revolves. They are defined as follows.

Definition 1.1. Let $V \cong \mathbb{Q}^\ell$ be an ℓ-dimensional $\mathbb{Q}$-vector space. Let $x_1, .., x_\ell$ be a basis for the dual space V^*. Given an undirected graph $G = (\mathcal{V}, E)$ with $\mathcal{V} = \{1, \dots, \ell\}$, define an arrangement $\mathcal{A}(G)$ by

$$\mathcal{A}(G) := \{\ker(x_i - x_j) | \{i, j\} \in E\}.$$

Regarding the freeness of $D(\mathcal{A}(G))$, a nice complete answer is given by the following theorem, due to work by Stanley [15], and Edelman and Reiner [6].

Theorem 1.2 ([6, Thm. 3.3])**.** *The module $D(\mathcal{A}(G))$ is free if and only if the graph G is chordal, i.e., G does not contain a chordless cycle with four or more vertices.*

A recent refined result was established in [16] by Tran and Tsujie, who showed that the subclass of so-called strongly chordal graphs in the class of chordal graphs corresponds to the subclass of MAT-free arrangements, cf. [2], [4].

In this note, we will investigate the natural question raised by Kung and Schenck in [11] of whether it is possible to give a characterization of graphs G, similar to Theorem 1.2, for which the projective dimension of $D(\mathcal{A}(G))$ is bounded by a certain positive value. To this end, we consider the more general notion of *weakly chordal* graphs introduced by Hayward [9]:

Definition 1.3. A graph G is weakly chordal if G and its complement graph G^C do not contain a chordless cycle with five or more vertices.

It was subsequently discovered that many algorithmic questions that are intractable for arbitrary graphs become efficiently solvable within the class of weakly chordal graphs [10].

The main result of this paper is the following:

Theorem 1.4. *The projective dimension of $D(\mathcal{A}(G))$ is at most 1 if and only if the graph G is weakly chordal. Moreover, the projective dimension is exactly 1 if G is weakly chordal but not chordal.*

Along the way towards the preceding theorem, we will prove the following key result, yielding the more difficult implication of Theorem 1.4.

Theorem 1.5. *For $\ell \geq 6$, the projective dimension of $D(\mathcal{A}(C_\ell^C))$ is equal to 2, where C_ℓ^C is the complement of the cycle-graph with ℓ vertices, also called the (ℓ-)antihole.*

Moreover, we prove a refined result. Namely, in Theorem 5.10 we provide an explicit minimal free resolution of $D(\mathcal{A}(C_\ell^C))$.

Remark 1.6. This extended abstract corresponds to an article that is published as preprint on the arXiv ([3]).

2 Preliminaries – Graph Theory

In this section, we define objects of interest to us while studying graphic arrangements, notably specific graph classes and their attributes. The exposition is mostly based on [5]. We only consider simple, undirected graphs:

Definition 2.1. (i) A simple graph G on a set $\mathcal{V}$ is a tuple $(\mathcal{V}, E)$ with $E \subseteq \binom{\mathcal{V}}{2}$ the set of (undirected) edges connecting the vertices in $\mathcal{V}$.

(ii) The graph $G^C = \left(\mathcal{V}, \binom{\mathcal{V}}{2} \setminus E\right)$ is called the *complement graph* of G.

(iii) A graph $G' = (\mathcal{V}', E')$ with $\mathcal{V}' \subseteq \mathcal{V}, E' \subseteq E$ is called a *subgraph* of G. If E' is the set of all edges of E between vertices in $\mathcal{V}'$, i.e. $E' = \binom{\mathcal{V}'}{2} \cap E$, the graph G' is an *induced subgraph* of G.

Besides restricting the graph to a set of vertices, there are two basic operations we can perform on graphs, as described in [12]:

Definition 2.2. Let $G = (\mathcal{V}, E)$ be a graph and $e = \{i, j\} \in E$.
The graph $G' = (\mathcal{V}, E \setminus \{e\})$ is obtained from G through deletion of e and the graph $G'' = (\mathcal{V}'', E'')$ with V'' the vertex set obtained by identifying i and j and $E'' = \{\{\bar{p}, \bar{q}\} \mid \{p, q\} \in E'\}$ is obtained by contraction of G with respect to e.

We will define graph classes based on certain path or cycle properties:

Definition 2.3. 1. For $k \geq 2$, a path of length k is the graph $P_k = (\mathcal{V}, E)$ of the form

$$\mathcal{V} = \{v_0, \ldots, v_k\}\ ,\ E = \{\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{k-1}, v_k\}\}$$

where all v_i are distinct.

2. If $P_k = (\mathcal{V}, E)$ is a path, and $k \geq 3$, then the graph $C_k = (\mathcal{V}, E \cup \{v_{k-1}, v_0\})$ is called a (k-)cycle.

For $k \geq 6$, we call C_k^C the *k-antihole*.
The main objects of interest in this article are graphs that satisfy a weaker condition than chordality and were introduced by Hayward in [9]:

Definition 2.4. A graph is called *weakly chordal* (or *weakly triangulated*) if it contains no induced k-cycle with $k \geq 5$ and no complement of such a cycle as an induced subgraph.

We prove the following:

Lemma 2.5. *For a weakly chordal graph $G = (\mathcal{V}, E)$, there exists a sequence of edges $e_1, .., e_k \notin E$, such that*

1. *$G_i = (\mathcal{V}, E \cup \{e_1, \dots, e_i\})$ is weakly chordal for $i = 1, \dots, k-1$,*
2. *the edge e_i is not part of an induced cycle C_4 in G_i for $i = 1, \dots, k$ and*
3. *G_k is chordal.*

3 Preliminaries – Hyperplane Arrangements

In this section, we recall some fundamental notions form the theory of hyperplane arrangements. The standard reference is Orlik and Terao's book [12].

Definition 3.1. Let $\mathbb{K}$ be a field and let $V \cong \mathbb{K}^\ell$ be a $\mathbb{K}$-vector space of dimension ℓ. A hyperplane H in V is a linear subspace of dimension $\ell - 1$. A hyperplane arrangement $\mathcal{A} = (\mathcal{A}, V)$ is a finite set of hyperplanes in V.

Let V^* be the dual space of V and $S = S(V^*)$ be the symmetric algebra of V^*. Identify S with the polynomial algebra $S = \mathbb{K}[x_1, \dots, x_\ell]$.

Definition 3.2. Let $\mathcal{A}$ be a hyperplane arrangement. Each hyperplane $H \in \mathcal{A}$ is the kernel of a polynomial α_H of degree 1 defined up to a constant. The product

$$Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} \alpha_H$$

is called a *defining polynomial* of $\mathcal{A}$.

Define the *rank* of $\mathcal{A}$ as $\mathrm{rk}(\mathcal{A}) := \mathrm{codim}_V(\cap_{H \in \mathcal{A}} H)$. If $\mathcal{B} \subseteq \mathcal{A}$ is a subset, then $(\mathcal{B}, V)$ is called a subarrangement. The *intersection lattice* $L(\mathcal{A})$ of the arrangement is the set of all non-empty intersections of elements of $\mathcal{A}$ (including V as the intersection over the empty set), with partial order by reverse inclusion. For $X \in L(\mathcal{A})$ define the *localization* at X as the subarrangement $\mathcal{A}_X$ of $\mathcal{A}$ by

$$\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subseteq H\}$$

as well as the *restriction* $(\mathcal{A}^X, X)$ as an arrangement in X by

$$\mathcal{A}^X := \{X \cap H \mid H \in \mathcal{A} \backslash \mathcal{A}_X \text{ and } X \cap H \neq \emptyset\}.$$

Define

$$L_k(\mathcal{A}) := \{X \in L(\mathcal{A}) \mid \operatorname{codim}_V(X) = k\}$$

and $L_{\geq k}(\mathcal{A}), L_{\leq k}(\mathcal{A})$ analogously.

Definition 3.3. Let $\mathcal{A}$ be a non-empty arrangement and let $H_0 \in \mathcal{A}$. Let $\mathcal{A}' = \mathcal{A} \backslash \{H_0\}$ and let $\mathcal{A}'' = \mathcal{A}^{H_0}$. We call $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ a triple of arrangements with distinguished hyperplane H_0.

We can associate a special module to the hyperplane arrangement $\mathcal{A}$:

Definition 3.4. A $\mathbb{K}$-linear map $\theta : S \to S$ is a derivation if for $f, g \in S$:

$$\theta(f \cdot g) = f \cdot \theta(g) + g \cdot \theta(f).$$

Let $\operatorname{Der}_{\mathbb{K}}(S)$ be the S-module of derivations of S. This is a free S-module with basis the usual partial derivatives $\partial_{x_1}, \ldots, \partial_{x_\ell}$.

Define an S-submodule of $\operatorname{Der}_{\mathbb{K}}(S)$, called the module of $\mathcal{A}$-derivations, by

$$D(\mathcal{A}) := \{\theta \in \operatorname{Der}_{\mathbb{K}}(S) | \theta(Q(\mathcal{A})) \in Q(\mathcal{A})S\}.$$

The arrangement $\mathcal{A}$ is called free if $D(\mathcal{A})$ is a free S-module.

The class of arrangements we are interested in are graphic arrangements:

Definition 3.5. Given a graph $G = (\mathcal{V}, E)$ with $\mathcal{V} = \{1, \ldots, \ell\}$, define an arrangement $\mathcal{A}(G)$ by

$$\mathcal{A}(G) := \{\ker(x_i - x_j) | \{i, j\} \in E\}.$$

Remark 3.6. Note that for a graphic arrangement $\mathcal{A}(G)$, localizations exactly correspond to disconnected unions of induced subgraphs of G.

For given derivations $\theta_1, \ldots, \theta_\ell \in \operatorname{Der}(S)$ we define the the *coefficient matrix*

$$M(\theta_1, \ldots, \theta_\ell) := \left(\theta_j(x_i)\right)_{1 \leq i,j \leq \ell},$$

i.e., the matrix of coefficients with respect to the standard basis $\partial_{x_1}, \ldots, \partial_{x_\ell}$ of $\operatorname{Der}(S)$. We recall Saito's useful criterion for the freeness of $D(\mathcal{A})$, cf. [12, Thm. 4.19].

Theorem 3.7. *For $\theta_1, \ldots, \theta_\ell \in D(\mathcal{A})$, the following are equivalent:*

1. $\det(M(\theta_1, \ldots, \theta_\ell)) \in \mathbb{K}^\times Q(\mathcal{A})$,
2. $\theta_1, \ldots, \theta_\ell$ *is a basis of* $D(\mathcal{A})$.

3.1 Projective dimension

In this manuscript, we want to take a look at the non-free case of graphic arrangements and find a characterization for their different projective dimensions. For a comprehensive account of all the required homological and commutative algebra notions we refer to [17] respectively [7].

Definition 3.8. A *projective resolution* of a module M is a complex $P_\bullet$ with a map $\epsilon : P_0 \to M$, such that the augmented complex

$$\cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\epsilon} M \to 0$$

is exact and P_i is projective for all $i \in \mathbb{N}$.

We can define the notion of projective dimension:

Definition 3.9. Let M be an S-module. Its *projective dimension* $\operatorname{pd}(M)$ is the minimum integer n (if it exists), such that there is a resolution of M by projective S-modules

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

The projective dimension of an arrangement is the projective dimension of its derivation module and we simply write $\operatorname{pd}(\mathcal{A}) := \operatorname{pd}(D(\mathcal{A}))$. Note that since S is a polynomial ring, it follows from the Quillen-Suslin Theorem that in this case projective and free resolutions coincide. The following result is due to Terao, cf. [18, Lem. 2.1].

Proposition 3.10. *Let $X \in L(\mathcal{A})$. Then $\operatorname{pd}(\mathcal{A}_X) \leq \operatorname{pd}(\mathcal{A})$.*

An arrangement $\mathcal{A}$ is *generic*, if $|\mathcal{A}| > \operatorname{rk}(\mathcal{A})$ and for all $X \in L(\mathcal{A}) \setminus \{\cap_{H \in \mathcal{A}} H\}$ we have $|\mathcal{A}_X| = \operatorname{codim}_V(X)$. The next result, due to Rose and Terao [13], identifies generic arrangements as those with maximal projective dimension.

Theorem 3.11. *Let $\mathcal{A}$ be a generic arrangement. Then $\operatorname{pd}(\mathcal{A}) = \operatorname{rk}(\mathcal{A}) - 2$.*

Important for our present investigations are the following examples of generic arrangements.

Example 3.12. Let C_ℓ be the cycle graph with ℓ vertices. Then, for $\ell \geq 3$, the graphic arrangement $\mathcal{A}(C_\ell)$ is generic. In particular, we have $\operatorname{pd}(\mathcal{A}(C_\ell)) = \operatorname{rk}(\mathcal{A}(C_\ell)) - 2 = \ell - 3$.

Since arrangements of induced subgraphs correspond to localizations, from Example 3.12 and Proposition 3.10 we obtain the following, first observed by Kung and Schenck [11, Cor. 2.4].

Corollary 3.13. *If G contains an induced cycle of length m, then $\operatorname{pd}(\mathcal{A}(G)) \geq m - 3$.*

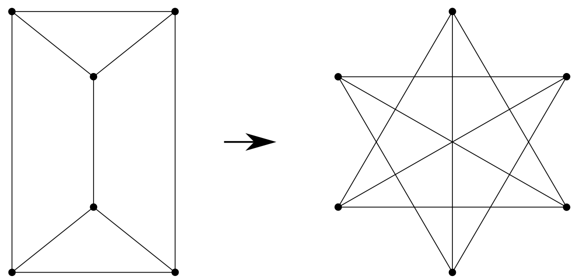

Figure 1: The triangular prism of [11] on the left is the same as ${C_6}^C$ on the right.

In [11], Kung and Schenck introduced a graph they called the "triangular prism" to serve as an example for a graphic arrangement $\mathcal{A}(G)$ whose projective dimension is strictly greater than $k-3$, k the length of the longest chordless cycle in G. Note that the graph they describe is the 6-antihole, see Figure 1. It does not have any cycle of length 5 or more, yet $\mathrm{pd}(\mathcal{A}(G)) = 2$ and it is not weakly chordal.

3.2 Terao's polynomial B

Let $\mathcal{A}$ be an arbitrary arrangement and H_0 a distinguished hyperplane. Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be the corresponding triple. Choose a map $\nu : \mathcal{A}'' \to \mathcal{A}'$ such that $\nu(X) \cap H_0 = X$ for all $X \in \mathcal{A}''$.

Terao defined the following polynomial

$$B(\mathcal{A}', H_0) = \frac{Q(\mathcal{A})}{\alpha_{H_0} \prod_{X \in \mathcal{A}''} \alpha_{\nu(X)}}.$$

The main properties of this polynomial can be summarized as follows:

Proposition 3.14. *[12, Lem. 4.39 and Prop. 4.41]*

1. $\deg B(\mathcal{A}', H_0) = |\mathcal{A}'| - |\mathcal{A}''|$.
2. *The ideal* $(\alpha_{H_0}, B(\mathcal{A}', H_0))$ *is independent of the choice of* ν.
3. *The polynomial* $\theta(\alpha_{H_0})$ *is contained in the ideal* $(\alpha_{H_0}, B(\mathcal{A}', H_0))$ *for all* $\theta \in D(\mathcal{A}')$.

In the following, we fix a hyperplane H_0 and simply write $B = B(\mathcal{A}', H_0)$ for Terao's polynomial.

By Proposition 3.14, we have an exact sequence:

$$0 \to D(\mathcal{A}) \hookrightarrow D(\mathcal{A}') \xrightarrow{\bar{\partial}'} \bar{S} \cdot \bar{B}, \tag{3.15}$$

where $\bar{S} = S/\alpha_{H_0}$, $\bar{B}$ is Terao's polynomial in $\bar{S}$ and $\overline{\partial'}(\theta) = \overline{\theta(\alpha_{H_0})}$.

The following new result regarding this sequence will be important in our subsequent proofs. It is a special case of "surjectivity theorems" for sequences of local functors recently obtained by the first author in [1].

Theorem 3.16. *Assume that* $\mathrm{pd}(\mathcal{A}_X) < \mathrm{codim}_V(X) - 2$ *for all* $X \in L_{\geq 2}(\mathcal{A}^{H_0})$. *Then the map* $\overline{\partial'}$ *in the sequence* (3.15) *is surjective. Hence, in this case, the sequence* (3.15) *is also right exact.*

Proof. This immediately follows from [1, Thm. 3.2, Thm. 3.3]. □

We record the following consequences of the preceding theorem.

Corollary 3.17. *Assume that* $\mathcal{A}_X$ *is free for all* $X \in L_2(\mathcal{A}^{H_0})$ *and* $\mathrm{pd}(\mathcal{A}) \leq 1$. *Then the sequence* (3.15) *is also right exact.*

Proof. This follows immediately from Theorem 3.16 and Proposition 3.10. □

Lemma 3.18. *Assume that* $\mathcal{A}_X$ *is free for all* $X \in L_2(\mathcal{A}^{H_0})$ *and* $\mathrm{pd}(\mathcal{A}) \leq 1$. *Then we also have* $\mathrm{pd}(\mathcal{A}') \leq 1$.

4 Weakly chordal graphic arrangements

The goal of this section is to show that a graphic arrangement of a weakly chordal graph has projective dimension at most 1, which gives one direction of our main Theorem 1.4.

Theorem 4.1. *Let* $G = (\mathcal{V}, E)$ *be a weakly chordal graph. Then* $\mathrm{pd}(\mathcal{A}(G)) \leq 1$.

Proof. Firstly, Lemma 2.5 implies that there exists a sequence of edges $e_1, \ldots, e_k$ such that $G_i = (\mathcal{V}, E \cup \{e_1, .., e_i\})$ is weakly chordal, the edge e_i is not the middle edge of any induced P_4 in G_i for $i = 1, .., k$, and G_k is chordal.

We prove that $\mathrm{pd}(\mathcal{A}(G_i)) \leq 1$ for all $i = 1, .., k$ by a descending induction. As G_k is chordal, the arrangement $\mathcal{A}(G_k)$ is free and hence $\mathrm{pd}(\mathcal{A}(G_k)) = 0$ by Theorem 1.2. So assume that $\mathrm{pd}(\mathcal{A}(G_j)) \leq 1$ for some $1 < j \leq k$. We will now argue that this implies $\mathrm{pd}(\mathcal{A}(G_{j-1})) \leq 1$ which finishes the proof.

Let H_0 be the hyperplane corresponding to the edge e_j in the arrangement $\mathcal{A}(G_j)$. We aim to apply Lemma 3.18 to $\mathcal{A}(G_j)$ and $\mathcal{A}(G_{j-1})$. To check the assumption of this result, we consider $X \in L_2(\mathcal{A}(G_j)^{H_0})$ and need to show that the arrangement $\mathcal{A}(G_j)_X$ is free.

Assume the contrary, i.e., that $\mathcal{A}(G_j)_X$ is not free. By definition of X, the arrangement $\mathcal{A}(G_j)_X$ is a graphic arrangement on an induced subgraph of G_j on four vertices containing the edge e_j. The assumption that this arrangement is not free implies that this induced subgraph is not chordal. As this subgraph only contains four vertices it must be the cycle C_4. This however contradicts condition (2) in Lemma 2.5 which states that the edge e_j cannot be an edge of an induced cycle C_4 in the graph G_j. Therefore, the arrangement $\mathcal{A}(G_j)_X$ is free for all $X \in L_2(\mathcal{A}(G_j)^{H_0})$.

Moreover, by the induction hypothesis, we have $\mathrm{pd}(\mathcal{A}(G_j)) \leq 1$. Thus, by Lemma 3.18, we also have $\mathrm{pd}(\mathcal{A}(G_{j-1})) \leq 1$ as desired. □

Let us record the following result which immediately follows from the previous theorem and Theorem 1.2.

Corollary 4.2. *Let G be a weakly chordal but not chordal graph. Then* $\mathrm{pd}(\mathcal{A}(G)) = 1$.

5 Graphic arrangements of antiholes

The main result of this section yields the other direction of implications in Theorem 1.4. Recall that the graph C_ℓ^C is the complement graph of a cycle with ℓ vertices which is called the *ℓ-antihole*.

Theorem 5.1. *For all $\ell \geq 6$ it holds that*

$$\mathrm{pd}(\mathcal{A}(C_\ell^C)) = 2.$$

Let us first explain how this concludes the proof of Theorem 1.4.

Proof of Theorem 1.4, using Theorem 5.1. By Theorem 4.1, we have $\mathrm{pd}(\mathcal{A}(G)) \leq 1$ for a weakly chordal graph G and $\mathrm{pd}(\mathcal{A}(G)) = 1$ if G is not chordal by Corollary 4.2.

Conversely, assume that G is a graph such that $\mathrm{pd}(\mathcal{A}(G)) = 1$. In particular, by Theorem 1.2, the graph G is not chordal. Suppose G is also not weakly chordal. Then, by definition, there is either an $m \geq 5$ such that C_m is an induced subgraph or there is an $\ell \geq 6$ such that C_ℓ^C is an induced subgraph of G. In the first case, by Corollary 3.13, we have $\mathrm{pd}(\mathcal{A}(G)) \geq \ell - 3 \geq 2$; in the second case, by Proposition 3.10 and Theorem 5.1, we also have $\mathrm{pd}(\mathcal{A}(G)) \geq 2$. Both cases contradict our assumption. Hence, G is weakly chordal. □

To prove Theorem 5.1, let us first introduce some notation for special derivations we will consider in this section. Let G be a graph with vertex set $\mathcal{V} = [\ell] := \{1, 2, \ldots, \ell\}$. Write $H_{ij} := \ker(x_i - x_j)$ for the hyperplane corresponding to the edge $\{i, j\}$ and let

$$\mathcal{A}_{\ell-1} := \{H_{ij} \mid 1 \leq i < j \leq \ell\}$$

be the graphic arrangement of the complete graph in $\mathbb{Q}^\ell$. We set

$$\theta_i := \sum_{j=1}^{\ell} x_j^i \partial_{x_j} \ (i \geq 0) \ \text{ and define } \ \varphi_i := \prod_{j \in [\ell] \setminus \{i-1, i, i+1\}} (x_i - x_j) \partial_{x_i}$$

for $i \neq 1, \ell$. Also define

$$\varphi_1 := \prod_{i=3}^{\ell-1} (x_1 - x_i) \partial_{x_1} \ \text{ and } \ \varphi_\ell := \prod_{i=2}^{\ell-2} (x_\ell - x_i) \partial_{x_\ell}.$$

In this section we always consider indices and vertices in $[\ell]$ in a cyclic way, i.e., we identify $i+\ell$ with i etc.

There is the following fundamental result due to K. Saito.

Theorem 5.2 ([14]). *$\mathcal{A}_{\ell-1}$ is free with basis $\theta_0, \dots, \theta_{\ell-1}$.*

With this, we can show the following.

Lemma 5.3. *Let*

$$\mathcal{B}_{i,j} := \mathcal{A}_{\ell-1} \setminus \{H_{s,s+1} \mid i \leq s \leq j\}.$$

Then $\mathcal{B}_{i,i+2}$ is free with basis

$$\theta_0, \dots, \theta_{\ell-3}, \varphi_{i+1}, \varphi_{i+2}.$$

With the same notation, we have:

Proposition 5.4. *If $i+2 \leq j$ and $(i,j) \neq (1,l)$, then $D(\mathcal{B}_{i,j})$ is generated by*

$$\theta_0, \dots, \theta_{\ell-3}, \varphi_{i+1}, \varphi_{i+2}, \dots, \varphi_j.$$

To obtain generators for $\mathcal{B}_{1,\ell}$ we need to modify the argument utilizing the polynomial B. For that purpose, we introduce the following new refined version of Proposition 3.14.

Theorem 5.5. *Let $\mathcal{A}$ be an arrangement, $H_1, H_2 \notin \mathcal{A}$ be distinct hyperplanes and let $\mathcal{A}_i := \mathcal{A} \cup \{H_i\}$. Assume that $H_1 = \ker(\alpha)$, $H_2 = \ker(\beta)$ and let B_i be the polynomial B with respect to $(\mathcal{A}, H_i)$. Assume that $\ker(\alpha+\beta) \in \mathcal{A}$, let b be the greatest common divisor of the reduction of B_1 and B_2 modulo (α, β) and let $b_2 b \equiv B_2$ modulo (α, β). Then for $\theta \in D(\mathcal{A})$ we have:*

$$\theta(\alpha) \in (\alpha, \beta B_1, b_2 B_1).$$

We can apply Theorem 5.5 to $\mathcal{B}_{1,\ell-1}$ and $\mathcal{B} := \mathcal{B}_{1,\ell} = \mathcal{A}_{\ell-1} \setminus \{H_{1,2}, \dots, H_{\ell-1,\ell}, H_{\ell,1}\}$. Namely, we can show the following:

Theorem 5.6. $D(\mathcal{B}) = \langle \theta_0, \dots, \theta_{\ell-3}, \varphi_1, \dots, \varphi_\ell \rangle_S$.

Note that $\psi_i := (x_{i-1} - x_i)\varphi_i - (x_{i+1} - x_{i+2})\varphi_{i+1} \in D(\mathcal{A}_{\ell-1}) = \langle \theta_0, \dots, \theta_{\ell-1} \rangle_S$ for $i = 1, \dots, \ell$, since

$$\psi_i(x_i - x_{i+1}) = -\prod_{j \in [\ell] \setminus \{i,i+1\}} (x_i - x_j) + \prod_{j \in [\ell] \setminus \{i,i+1\}} (x_{i+1} - x_j) \equiv 0 \mod (x_i - x_{i+1}).$$

Thus, there are f_{ij} such that

$$\psi_i - \sum_{j=0}^{\ell-3} f_{ij}\theta_j = -\theta_{\ell-2} \; (i = 1, 2, \dots, \ell). \tag{5.7}$$

So we have relations

$$\psi_i - \sum_{j=0}^{\ell-3} f_{ij}\theta_j = \psi_s - \sum_{j=0}^{\ell-3} f_{sj}\theta_s,$$

and they are generated by

$$\psi_1 - \sum_{j=0}^{\ell-3} f_{1j}\theta_j = \psi_i - \sum_{j=0}^{\ell-3} f_{ij}\theta_j \tag{5.8}$$

for $i = 2, \dots, \ell$. We now prove that they indeed generate all the relations among the generators of $D(\mathcal{B})$.

Theorem 5.9. *All relations among the set of generators $\theta_0, \dots, \theta_{\ell-3}, \varphi_1, \dots, \varphi_\ell$ are generated by the ones given in Equations (5.8).*

Now we are ready to prove the following, which immediately implies Theorem 1.5.

Theorem 5.10. *The module $D(\mathcal{B})$ has the following minimal free resolution:*

$$0 \to S[-\ell+1] \to S[-\ell+2]^{\ell-1} \to \oplus_{i=0}^{\ell-4} S[-i] \oplus S[-\ell+3]^{\ell+1} \to D(\mathcal{B}) \to 0. \tag{5.11}$$

In particular, $\mathrm{pd}(\mathcal{B}) = 2$.

6 Remark on generalization of the result

A natural question arising from our Theorem 1.4 would be if this generalizes to the remaining projective dimensions, i.e. if $\mathcal{A}(G)$ has projective dimension $\leq k$ if and only if G and its complement graph do not contain a chordless cycle with $k+4$ or more vertices. This is however not the case, first note that in the case of projective dimension 0, it suffices for the graph itself to have no chordless cycle of length 4 or more and chordality is not closed under taking the complement (The complement of the 4-cycle for instance, is chordal, whereas the 4-cycle itself is not). Moreover, since the arrangement of the k-cycle is generic of rank $k-1$, it has maximal projective dimension $k-3$ (see Example 3.12) and by Theorem 1.5 its complement has projective dimension 2. Moreover, we found two counterexamples to the other direction of this conjecture in dimension 7; both graphs and their complements have no induced cycle of length more than 5, yet have projective dimension 3, which was also found by Hashimoto in [8].

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