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Plane partitions and rowmotion on rectangular and trapezoidal posets

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Abstract. We define a birational map between labelings of a rectangular poset and its associated trapezoidal poset. This map tropicalizes to a bijection between the plane partitions of these posets of fixed height, giving a new bijective proof of a result by Proctor. We also show that this map is equivariant with respect to birational rowmotion, resolving a conjecture of Williams and implying that birational rowmotion on trapezoidal posets has finite order.

Keywords: birational rowmotion, plane partitions, trapezoidal posets

1 Introduction

For a finite poset P, a *plane partition* of P (also known as a *P-partition*) is an order-preserving labeling of P with nonnegative integers. When P is the *rectangular poset* $R_{r,s}$, the Cartesian product of two chains of r and s elements, an elegant product formula for the number of plane partitions of P with maximum label at most ℓ was given by MacMahon [16]. Surprisingly, Proctor [19] showed that there is another poset, namely the *trapezoidal poset* $T_{r,s}$, that has the same number of plane partitions with maximum label at most ℓ for all ℓ. (See Figure 1 for a depiction of $R_{4,3}$ and $T_{4,3}$.)

Proctor's proof relies on a branching rule for Lie algebra representations and is not bijective. Partial bijections were later constructed by Stembridge [22] and Reiner [20] for $\ell = 1$, and Elizalde [5] for $\ell = 2$, but a full bijection for all ℓ was not given until work of Hamaker, Patrias, Pechenik, and Williams [10] using K-theoretic jeu de taquin.

Although the bijection given in [10] has many nice properties, it also has some shortcomings. First, it cannot be extended in a natural way to a continuous piecewise-linear map on real-valued labelings of the rectangle and trapezoid. As a result, it cannot be written using expressions in the tropical semiring (that is, using the operations addition, subtraction, and maximum). Second, it does not appear to be generally well-behaved with respect to a certain map on posets called *rowmotion*.

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(Combinatorial) rowmotion is a term coined by Striker and Williams [23] to describe a map first studied by Brouwer and Schrijver [1] that permutes the set of order ideals of a poset, sending an order ideal $I \subseteq P$ to the order ideal generated by the minimal elements of $P \setminus I$. It was shown in [1] that the action of rowmotion on order ideals of the rectangle $R_{r,s}$ has order exactly $r+s$. Einstein and Propp [4] observed that one can generalize combinatorial rowmotion to a piecewise-linear map or birational map. Results about birational rowmotion then descend to results for piecewise-linear rowmotion (via tropicalization) and further to combinatorial rowmotion. Birational rowmotion on rectangular posets is closely related to the *birational Robinson–Schensted–Knuth (RSK) correspondence*, also known as *tropical* or *geometric RSK*—see [18] for some discussion.

For rectangular posets, birational rowmotion maintains many of the important dynamical properties of combinatorial rowmotion. For example, Grinberg and Roby [7] showed that birational rowmotion on $R_{r,s}$ still has finite order $r+s$, which was observed by Glick and Grinberg [9, 17] to be equivalent to a phenomenon from discrete dynamics known as *type AA Zamolodchikov periodicity* (see Volkov [24]). However, the class of posets for which birational rowmotion is known to have finite order is very small [7, 8]. Grinberg and Roby conjecture that birational rowmotion on $T_{r,s}$ likewise has finite order $r+s$. (See also [6] for more on conjectured good behavior of the related *R-systems*.)

Given the apparent close relationship between the rectangular and trapezoidal posets, Williams conjectures (as noted in [7], based on work in [25]) that there should exist a birational map between labelings of $R_{r,s}$ and $T_{r,s}$ that intertwines with the action of rowmotion (see also Hopkins [11] for further discussion). In particular, such a map would prove that birational rowmotion on $T_{r,s}$ has finite order $r+s$. In work of Dao, Wellman, Yost-Wolff, and Zhang [3], it was shown that the bijection given in [10] does intertwine with combinatorial rowmotion on plane partitions of height 1, thereby showing that combinatorial rowmotion on $T_{r,s}$ has the correct order. However, they also note that it does not respect piecewise-linear or birational rowmotion, so it cannot be used to prove periodicity on $T_{r,s}$ in these cases.

Our main result is to settle these questions. We construct a birational map between labelings of $R_{r,s}$ and $T_{r,s}$ using birational toggles [2, 4]. We then show that this map:

- tropicalizes to a continuous, piecewise-linear map that restricts to a bijection between plane partitions of $R_{r,s}$ and $T_{r,s}$ of height at most ℓ, and
- is equivariant with respect to rowmotion on $R_{r,s}$ and $T_{r,s}$, implying that birational (as well as piecewise-linear and combinatorial) rowmotion on $T_{r,s}$ has order $r+s$.

We also generalize the *chain shifting lemma* proved by the current authors in [12] (see also the noncommutative conversion lemma by Grinberg–Roby [9]), which is closely related to Schützenberger promotion on semistandard Young tableaux [13]. In particular, we derive a new, simple proof of this lemma based on the duality of plane trees.

The full version of this paper, which includes proofs, can be found at [14].

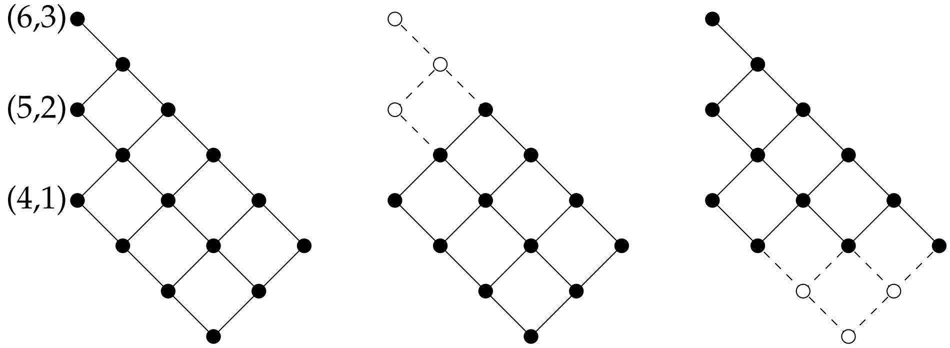

Figure 1: The right trapezoid $RT_{4,3}$, the rectangle $R_{4,3}$, and the trapezoid $T_{4,3}$.

2 Background

We begin with some background on posets and rowmotion. Fix positive integers $r \geq s$.

Definition 2.1. The *rectangle poset* $R_{r,s}$ is the Cartesian product of chains $[r] \times [s]$.

The *right trapezoid poset* $RT_{r,s}$ is the induced subposet $\{(i,j) \mid i-j<r\} \subseteq R_{r+s-1,s}$.

The *trapezoid poset* $T_{r,s}$ is the induced subposet $\{(i,j) \mid i+j>s\} \subseteq RT_{r,s}$.

See Figure 1. We draw our posets oriented in the plane so that the first coordinate increases to the northwest and the second coordinate increases to the northeast.

We can augment a poset P to a poset $\widehat{P}$ by adding minimum and maximum elements $\hat{0}$ and $\hat{1}$. Throughout, we will identify the labelings in $\mathbb{R}_+^P$ with the corresponding labelings in $\mathbb{R}_+^{\widehat{P}}$, where the labels at $\hat{0}$ and $\hat{1}$ are both 1.

Definition 2.2. For any $p \in P$, the *(birational) toggle* $t_p \colon \mathbb{R}_+^P \to \mathbb{R}_+^P$ is the map that acts on $y \in \mathbb{R}_+^P$ by fixing all coordinates except y_p and sending $y_p \mapsto \left(\sum_{q>p} \frac{1}{y_q}\right)^{-1}\left(\sum_{q<p} y_q\right)\frac{1}{y_p}$.

The *rowmotion map* $\rho \colon \mathbb{R}_+^P \to \mathbb{R}_+^P$ is the composition $\rho = t_{L^{-1}(1)} \circ t_{L^{-1}(2)} \circ \cdots \circ t_{L^{-1}(n)}$ for any linear extension $L \colon P \to [n]$.

Definition 2.3. The *transfer map* $\psi^{-1} \colon \mathbb{R}_+^P \to \mathbb{R}_+^P$ is defined coordinatewise by $\psi^{-1}(y)_p = \frac{y_p}{\sum_{q<p} y_q}$. Its inverse ψ acts by $\psi(x)_p = \sum_{\hat{0} \lessdot q_1 \lessdot \cdots \lessdot q_n = p} \prod_{i=1}^{n} x_{q_i}$.

It will be convenient for us to work with the conjugate of rowmotion under the transfer map, $\tilde{\rho} = \psi^{-1} \circ \rho \circ \psi$. (This map was called *birational antichain rowmotion* or *barmotion* by Joseph and Roby [15].) An important property of $\tilde{\rho}$ is the following identity.

Proposition 2.4. *Let $x \in \mathbb{R}_+^P$, $y = \psi(x)$, and $z = \tilde{\rho}^{-1}(x)$. Then for $p \in P$,*

$$x_p^{-1} = \sum_{q \lessdot p} \frac{y_q}{y_p} \qquad \textit{and} \qquad z_p^{-1} = \sum_{q \gtrdot p} \frac{y_p}{y_q}.$$

See Figure 2 for some example calculations involving these maps on $RT_{3,2}$.

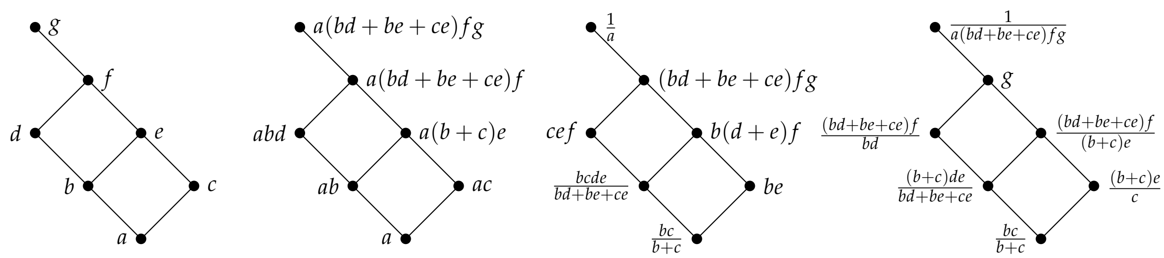

Figure 2: Labelings of $RT_{3,2}$: x, $y = \psi(x)$, $\rho^{-1}(y)$, and $z = \tilde{\rho}^{-1}(x) = \psi^{-1}(\rho^{-1}(y))$.

3 Chain Shifting in Skew Shapes

Let P be a poset and $x \in \mathbb{R}_+^P$. For a subset $S \subseteq P$, define the *weight* of S to be $w_S(x) = \prod_{p \in S} x_p$. We first show how to relate weights of certain subsets of P to weights of certain arborescences with respect to $y = \psi(x)$. (Recall that we set $y_{\hat{0}} = y_{\hat{1}} = 1$.)

Definition 3.1. An *upward arborescence of* P is a subgraph of $\widehat{P} \setminus \{\hat{1}\}$ such that every element of P has down degree 1. Similarly, a *downward arborescence of* P is a subgraph of $\widehat{P} \setminus \{\hat{0}\}$ such that every element of P has up degree 1. We denote the set of upward and downward arborescences of P by U_P and D_P, respectively.

Define the *weight* (with respect to y) of the edge e corresponding to the cover relation $p \lessdot q$ to be $\omega_e(y) = \frac{y_p}{y_q}$ and the *weight* of an arborescence T to be $\omega_T(y) = \prod_{e \in E(T)} \omega_e(y)$. (Note: The weight ω_T of an arborescence is different from the weight w_S of a subset.)

Example 3.2. The upward and downward arborescences of $RT_{3,2}$ are shown in Figure 3 with their weights. The weight of the first upward arborescence can be computed as

$$\frac{1}{y_{11}} \cdot \frac{y_{11}}{y_{21}} \cdot \frac{y_{11}}{y_{12}} \cdot \frac{y_{21}}{y_{31}} \cdot \frac{y_{21}}{y_{22}} \cdot \frac{y_{31}}{y_{32}} \cdot \frac{y_{32}}{y_{42}} = \frac{y_{11}y_{21}}{y_{12}y_{22}y_{42}}.$$

For any $y \in \mathbb{R}_+^P$, we can use $\omega_T(y)$ to define probability measures on U_P and D_P: for subsets $U \subseteq U_P$ and $D \subseteq D_P$, define

$$\mu_y(U) = \frac{\sum_{T \in U} \omega_T(y)}{\sum_{T \in U_P} \omega_T(y)}, \qquad \mu_y(D) = \frac{\sum_{T \in D} \omega_T(y)}{\sum_{T \in D_P} \omega_T(y)}.$$

Given a collection of saturated chains $\mathscr{C}$, let $U_P(\mathscr{C})$ and $D_P(\mathscr{C})$ denote the sets of all upward and downward arborescences that contain (the edges of) some chain in $\mathscr{C}$, and let $w_{\mathscr{C}}(x)$ denote the total weight of all chains in $\mathscr{C}$ (as subsets of P).

One can exploit the symmetry of Proposition 2.4 to compute the following result.

Corollary 3.3. *Let P be a poset, $x \in \mathbb{R}_+^P$, and $z = \tilde{\rho}^{-1}(x)$. Let $m, m', M, M' \in P$ such that m' is the unique element covered by m and M is the unique element covering M'. Let $\mathscr{C}$ be any collection of saturated chains from m to M, and similarly define $\mathscr{C}'$. Then* $\frac{\mu_y(U_P(\mathscr{C}))}{w_{\mathscr{C}}(x)} = \frac{\mu_y(D_P(\mathscr{C}'))}{w_{\mathscr{C}'}(z)}$.

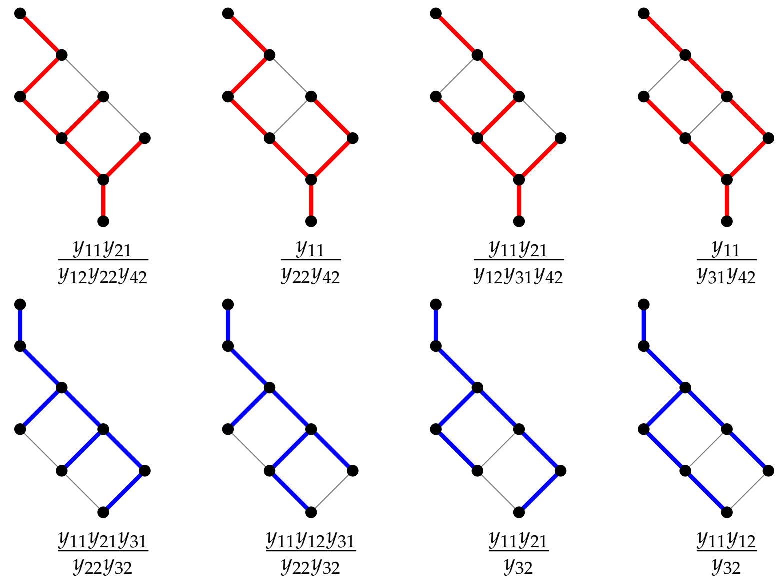

Figure 3: The four upward arborescences in $U_{RT_{3,2}}$ and the four downward arborescences in $D_{RT_{3,2}}$, together with their weights.

Let S be a *skew shape poset* as in Figure 4. Note that if $q \in S$ only covers a single element p, then any element of U_S must contain the edge $p \lessdot q$, so we call this edge *forced* for U_S. Likewise, the edge $p \lessdot q$ is *forced* for D_S if q is the only element covering p.

We define a bijection $\aleph \colon U_S \to D_S$ as follows. Translate $T \in U_S$ in the plane by the vector $(-\frac{1}{2}, -\frac{1}{2})$ (i.e., downward) to $\overline{T}$. Then form $\aleph(T)$ by taking all edges of S that do not intersect $\overline{T}$, together with all forced edges for D_S. See Figure 4 for an example.

We show that $\aleph$ affects the weight of each arborescence in a uniform way.

Lemma 3.4. *Let $T \in U_S$ and $y \in \mathbb{R}_+^S$. Then there exists a Laurent monomial $y^{\alpha(S)}$ depending only on S such that $\omega_{\aleph(T)}(y) = \omega_T(y) \cdot y^{\alpha(S)}$ for all $T \in U_S$.*

Corollary 3.5. *The bijection $\aleph \colon U_S \to D_S$ is measure-preserving: $\mu_y(U) = \mu_y(\aleph(U))$ for all $y \in \mathbb{R}_+^S$ and subsets $U \subseteq U_S$.*

Example 3.6. Consider again the arborescences for $RT_{3,2}$ in Figure 3. The bijection $\aleph$ sends each upward arborescence to the downward arborescence directly below it. In each case, $\aleph$ multiplies the weight by $y_{42} \cdot \frac{y_{12}y_{31}}{y_{32}}$, as predicted by Lemma 3.4.

We are now ready to prove a chain shifting lemma for skew shapes. Our first form is a generalization of the chain shifting lemma for rectangles proven by the current authors in [12] (and in the noncommutative setting by Grinberg and Roby [9]) to skew shapes S.

For $p \in S$, write $se(p) \neq \varnothing$ if p has a southeast neighbor and $se(p) = \varnothing$ otherwise. Given elements $p < q$ in S, let $\mathscr{C}_{p,q}$ denote the set of all saturated chains from p to q, and

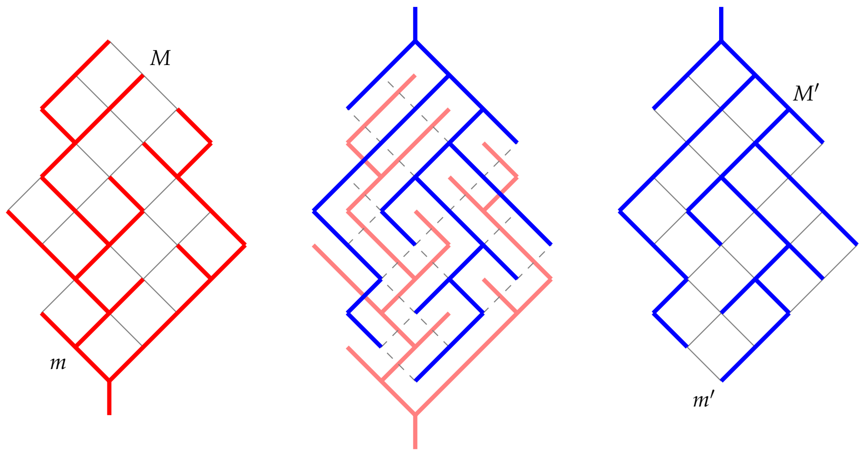

Figure 4: An upward arborescence T (in red) and its image $\aleph(T)$ (in blue).

let $\mathscr{C}^{se}_{p,q} \subseteq \mathscr{C}_{p,q}$ denote the subset consisting of those chains C for which $se(r) \neq \varnothing$ for all $r \in C$. (Also define the analogous notation for the directions sw, ne, and nw.)

Lemma 3.7. *Let $m' \lessdot m$ and $M' \lessdot M$ be elements of S such that $sw(m) = ne(M') = \varnothing$.*

(a) The bijection $\aleph$ restricts to a bijection from $U_S(\mathscr{C}^{se}_{m,M})$ to $D_S(\mathscr{C}^{nw}_{m',M'})$.

(b) Let $x \in \mathbb{R}^S_+$ and $z = \tilde{\rho}^{-1}(x)$. Then $w_{\mathscr{C}^{se}_{m,M}}(x) = w_{\mathscr{C}^{nw}_{m',M'}}(z)$.

Proof sketch. Let $T \in U_S(\mathscr{C}^{se}_{m,M})$, and let C be the chain in T from m to M. By the construction of $\aleph$, $\aleph(T) \in D_S$ contains a saturated chain upward from m' that does not cross C, so it must pass through M'. It follows that $\aleph(T) \in D_S(\mathscr{C}^{nw}_{m',M'})$. The reverse argument shows that $\aleph$ is a bijection, and (b) then follows from Corollaries 3.5 and 3.3. □

Example 3.8. Let $S = RT_{32}$ and take $m = (2,1)$, $M = (3,2)$, $m' = (1,1)$, and $M' = (2,2)$. We can verify that Lemma 3.7(b) holds in this case using the labels in Figure 2:

$$z_{11}z_{21}z_{22} + z_{11}z_{12}z_{22} = \frac{bcdf}{b+c} + \frac{b(bd+be+ce)f}{b+c} = bdf + bef = x_{21}x_{31}x_{32} + x_{21}x_{22}x_{32}.$$

We can also verify Lemma 3.7(a) using Figure 3 by noting that $U_S(\mathscr{C}^{se}_{m,M})$ and $D_S(\mathscr{C}^{nw}_{m',M'})$ are the arborescences in the leftmost three columns, which are in bijection via $\aleph$.

The bijection $\aleph$ is a powerful tool for relating weights of subsets of P with respect to x and $z = \tilde{\rho}(x)$. The general strategy is simple: relate the quantities of interest to the weights of certain subsets of U_P and D_P, then show that these subsets are in bijection via $\aleph$. In this way, one can easily prove many previously established results about rowmotion on rectangles as well as further generalizations.

For another example of this, define the *left border* of the right trapezoid $RT_{r,s}$ to be the set of elements of the form $L = \{(\ell + r - 1, \ell) \mid 1 \leq \ell \leq s\}$. For $p, q \in RT_{r,s}$, let $\mathscr{C}^{L}_{p,q}$ be the subset of $\mathscr{C}_{p,q}$ consisting of chains that intersect L.

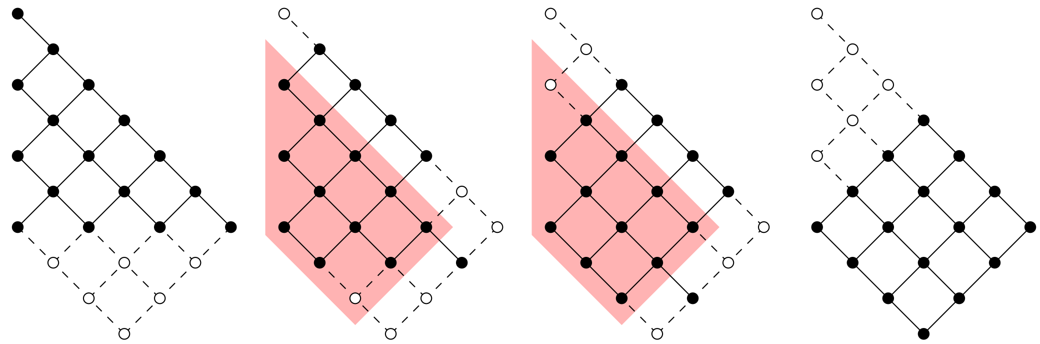

Figure 5: The four intermediate posets $I_4, \ldots, I_1$ lying inside of $RT_{4,4}$ with M_2 highlighted in red. The map ζ_2 acts on labelings of I_3 by applying $\tilde{\rho}_2^{-1}$ inside M_2 and shifting the other entries parallel to the sides.

Lemma 3.9. *Let $S = RT_{r,s}$, and let $m' \lessdot m$ and $M' \lessdot M$ such that $se(m) = ne(M') = \varnothing$.*

(a) The bijection $\aleph$ restricts to a bijection from $U_S(\mathscr{C}^L_{m,M})$ to $D_S(\mathscr{C}^L_{m',M'})$.

(b) Let $x \in \mathbb{R}^S_+$ and $z = \tilde{\rho}^{-1}(x)$. Then $w_{\mathscr{C}^L_{m,M}}(x) = w_{\mathscr{C}^L_{m',M'}}(z)$.

4 A map between the rectangle and trapezoid

In this section, we use the chain shifting lemmas (Lemmas 3.7 and 3.9) to define a birational map ζ between labelings of the rectangle $R_{r,s}$ and trapezoid $T_{r,s}$. In the tropical setting, this map will become a continuous, piecewise-linear, volume-preserving map between the chain polytopes of these two posets, which can be used to give a bijection between the plane partitions of $R_{r,s}$ and $T_{r,s}$ of height ℓ. To construct ζ, we need to construct certain intermediate posets as induced subposets of the right trapezoid $RT_{r,s}$.

Definition 4.1. Let $k \leq s \leq r$ be positive integers. The kth *intermediate poset* $I_k = I_{r,s,k}$ is the induced subposet of $RT_{r,s}$ on $T_{r,k} \cup [(k, k+1), (r+k-1, s)]$.

See Figure 5. Note that the leftmost minimal element of I_k is $(k, 1)$. One can easily verify that $I_1 = R_{r,s}$, $I_s = T_{r,s}$, and $|I_k| = rs$ for all k.

We now define maps $\zeta_k \colon \mathbb{R}^{I_{k+1}}_+ \to \mathbb{R}^{I_k}_+$ as follows. Consider the interval $M_k = [(k,1), (r+k, k+1)] \subseteq RT_{r,s}$. For any $x \in \mathbb{R}^{I_{k+1}}_+$, let $\bar{x} \in \mathbb{R}^{M_k}_+$ be obtained by restricting x to $M_k \setminus \{(k,1)\} \subseteq I_{k+1}$ and setting $\bar{x}_{k,1}$ to be an arbitrary $a \in \mathbb{R}_+$ (say, 1). Let $\tilde{\rho}_k \colon \mathbb{R}^{M_k}_+ \to \mathbb{R}^{M_k}_+$ be the antichain rowmotion map on M_k. Then we define $\zeta_k(x) \in \mathbb{R}^{I_k}_+$ by

$$\zeta_k(x)_{ij} = \begin{cases} \tilde{\rho}_k^{-1}(\bar{x})_{ij} & \text{if } (i,j) \in I_k \cap M_k = M_k \setminus \{(r+k, k+1)\}, \\ x_{i+1,j} & \text{if } (i,j) \in I_k \setminus M_k \text{ and } j > k+1, \\ x_{i,j+1} & \text{if } (i,j) \in I_k \setminus M_k \text{ and } i < k. \end{cases}$$

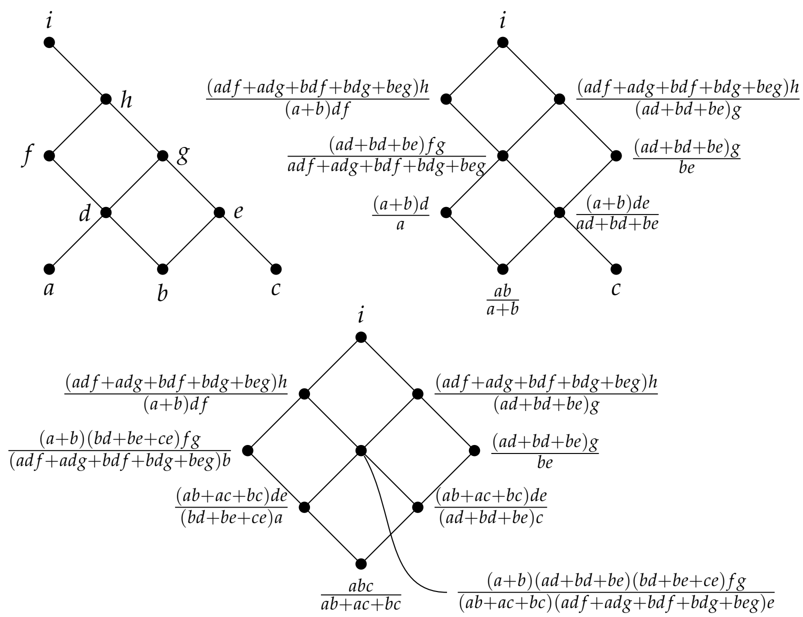

Figure 6: Applying ζ_2 and then ζ_1 to a labeling x of $T_{3,3}$, resulting in $\zeta(x)$.

See Figure 5. One can show that ζ_k does not depend on the choice of a.

We then define the birational map $\zeta = \zeta_1 \circ \zeta_2 \circ \cdots \circ \zeta_{s-1}$ from $\mathbb{R}_+^{T_{r,s}}$ to $\mathbb{R}_+^{R_{r,s}}$.

Example 4.2. Figure 6 shows the result of applying $\zeta = \zeta_1 \circ \zeta_2$ to a labeling $x \in \mathbb{R}_+^T$ when $T = T_{3,3}$. Note that $x_{13} = c = \zeta_2(x)_{12}$ as this label lies below M_2. Similarly $\zeta_2(x)_{j+1,3} = \zeta(x)_{j3}$ for $j = 1, 2, 3$ as these labels lie above M_1.

The key property of ζ that we will need to prove is that ζ preserves the total weight of all maximal chains. However, this does not hold for the maps ζ_k unless we restrict to a certain special class of polygonal chains.

Definition 4.3. A maximal chain $C \subseteq I_k \subseteq RT_{r,s}$ is *polygonal* if C intersects L (the left border of $RT_{r,s}$) or if $(k, 1) \in C$.

Note that all chains in the trapezoid and rectangle are polygonal (when $k = s$ or 1).

The following result relates the weights of polygonal chains under ζ_k. Since the only complicated part of ζ_k occurs inside M_k, it follows readily from Lemmas 3.7 and 3.9.

Proposition 4.4. *Let $\mathscr{P}_k$ be the collection of polygonal chains in I_k. For $x \in \mathbb{R}_+^{I_{k+1}}$, let $z = \zeta_k(x)$. Then $w_{\mathscr{P}_{k+1}}(x) = w_{\mathscr{P}_k}(z)$.*

It is now simple to deduce the following theorem.

Theorem 4.5. *Let $\mathscr{C}$ and $\mathscr{C}'$ be the sets of all maximal chains in $T_{r,s}$ and $R_{r,s}$, respectively. Then for all $x \in \mathbb{R}_+^{T_{r,s}}$, $w_{\mathscr{C}}(x) = w_{\mathscr{C}'}(\zeta(x))$.*

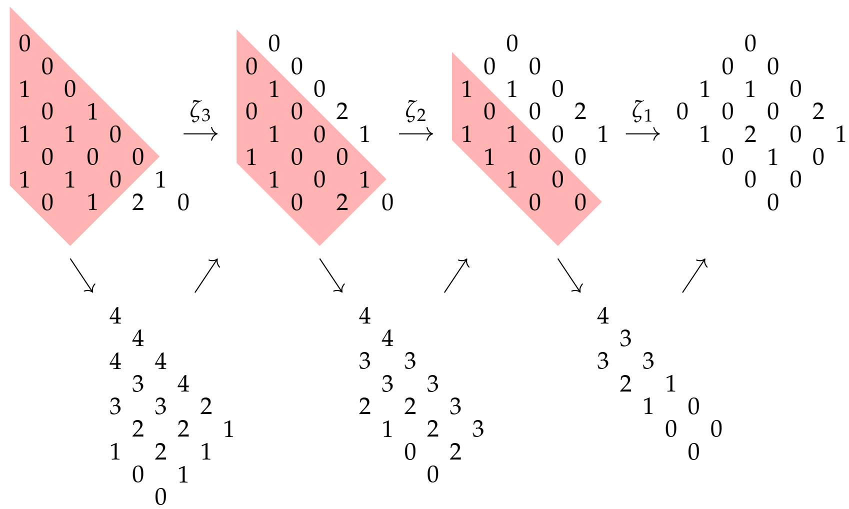

Figure 7: Example calculation of ζ for $r = 5$ and $s = 4$. The top row shows the labelings of the intermediate posets obtained from x when applying ζ_3, ζ_2, and ζ_1 with entries in M_k highlighted. The vertical maps show applications of (‡) and (†) on M_k.

4.1 Polytopes and plane partitions

We now examine the consequences of Proposition 4.4 and Theorem 4.5 in the piecewise-linear case. In this section, we will take all maps to be their piecewise-linear counterparts. In particular, the definition of ζ_k inside M_k utilizes the map $\tilde{\rho}^{-1}$ on M_k. By tropicalizing Proposition 2.4, we can compute $z = \tilde{\rho}^{-1}(x)$ via

$$z_p = -\max_{q \gtrdot p}\{y_p - y_q\} = \min_{q \gtrdot p}\{y_q\} - y_p, \quad \text{where} \tag{†}$$

$$y_p = \psi(x)_p = \max_{\hat{0} \lessdot q_1 \lessdot \cdots \lessdot q_n = p} \sum_i x_{q_i}. \tag{‡}$$

Example 4.6. An example calculation of ζ when $r = 5$ and $s = 4$ is given in Figure 7. Each ζ_k can be computed by applying (‡) and then (†) on M_k. (The minimum element of M_k is arbitrarily given the label 0, and the label of the maximum element of M_k is discarded after ζ_k is applied.) The entries outside of M_k are shifted downward appropriately.

To each intermediate poset $I_k \subseteq RT_{r,s}$, we can associate a *polygonal chain polytope*. This coincides with the chain polytope (as defined by Stanley [21]) when $k = 1$ or $k = s$.

Definition 4.7. The *polygonal chain polytope* $\widetilde{\mathcal{C}}(I_k) \subseteq \mathbb{R}^{I_k}$ is the set of all $\mathbb{R}$-labelings $x = (x_p)_{p \in I_k}$ such that $x_p \geq 0$ for all $p \in I_k$, and $\sum_{p \in C} x_p \leq 1$ for all polygonal chains $C \subseteq I_k$.

Although $\widetilde{\mathcal{C}}(I_k)$ is a lattice polytope when $k = 1$ or $k = s$ (when it is an ordinary chain polytope), this is not true in general. Nevertheless, for fixed r and s, these polytopes all

$$
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4 & & 4 & & & & & \\
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 & 2 & & 2 & & 3 & & \\
1 & & 2 & & 2 & & 3 & \\
 & 0 & & 1 & & 2 & & 0
\end{array}
\quad \xrightarrow{\psi \circ \zeta \circ \psi^{-1}} \quad
\begin{array}{cccccccc}
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 & & 4 & & 4 & & & \\
 & 4 & & 4 & & 3 & & \\
1 & & 3 & & 3 & & 3 & \\
 & 1 & & 3 & & 1 & & 1 \\
 & & 0 & & 1 & & 0 & \\
 & & & 0 & & 0 & & \\
 & & & & 0 & & &
\end{array}
$$

Figure 8: Example of the bijection $\psi \circ \zeta \circ \psi^{-1}$ obtained by applying ψ to the labelings in Figure 7. Note that both plane partitions have the same height.

have the same volume and Ehrhart polynomial. In particular, if $\widetilde{\mathcal{C}}(I_k)$ is not a lattice polytope, then it exhibits *period collapse* of its Ehrhart quasi-polynomial.

Theorem 4.8. *The map $\zeta_k \colon \mathbb{R}^{I_{k+1}} \to \mathbb{R}^{I_k}$ defines a continuous, piecewise-linear, and lattice-preserving bijection from $\ell \cdot \widetilde{\mathcal{C}}(I_{k+1})$ to $\ell \cdot \widetilde{\mathcal{C}}(I_k)$ for all $\ell \in \mathbb{Z}_{\geq 0}$. Hence, for fixed r and s, the rational polytopes $\widetilde{\mathcal{C}}(I_k)$ share the same Ehrhart polynomial for all k.*

The following corollary then gives a bijective proof of the result of Proctor [19].

Corollary 4.9. *The continuous, piecewise-linear map $\psi \circ \zeta \circ \psi^{-1} \colon \mathbb{R}^{T_{r,s}} \to \mathbb{R}^{R_{r,s}}$ defines a bijection between plane partitions of $T_{r,s}$ and $R_{r,s}$ of height ℓ for all $\ell \in \mathbb{Z}_{\geq 0}$.*

Example 4.10. An example application of $\psi \circ \zeta \circ \psi^{-1}$ is given in Figure 8 (obtained by applying ψ to Figure 7). As required, both plane partitions have the same height.

5 Rowmotion equivariance

To prove that the map ζ defined in the previous section is equivariant with respect to the action of rowmotion $\tilde{\rho}$ (or, equivalently, that $\psi \circ \zeta \circ \psi^{-1}$ is equivariant with respect to ρ), we define a modified version of rowmotion on the intermediate posets I_k that is respected by the maps ζ_k.

As above, let $k \leq s \leq r$ and consider $I_k \subseteq RT_{r,s}$. Let $\mathscr{P}_k$ denote the set of polygonal chains in I_k, and let $\mathscr{P}_k(p)$ denote the subset of those chains that contain p.

Definition 5.1. For any $p \in I_k$, the *(birational) polygonal toggle* $\tau'_p \colon \mathbb{R}_+^{I_k} \to \mathbb{R}_+^{I_k}$ is the map that changes the p-coordinate of $x \in \mathbb{R}_+^{I_k}$ by $x_p \mapsto w_{\mathscr{P}_k(p)}(x)^{-1}$ while keeping all other coordinates fixed. The *(birational) polygonal rowmotion* map $\tilde{\varrho}_k : \mathbb{R}_+^{I_k} \to \mathbb{R}_+^{I_k}$ is the composition $\tilde{\varrho}_k = \tau'_{L^{-1}(|I_k|)} \circ \cdots \circ \tau'_{L^{-1}(1)}$ for any linear extension L of I_k.

When $k = 1$ or s (that is, on the rectangle or trapezoid), the set of polygonal chains is just the set of all maximal chains, and so $\tilde{\varrho}_k = \tilde{\rho}_k$ as shown by Joseph and Roby [15].

We then prove the following theorem.

Theorem 5.2. *The maps $\zeta_k\colon \mathbb{R}_+^{I_{k+1}} \to \mathbb{R}_+^{I_k}$ are equivariant with respect to polygonal rowmotion:*

$$\zeta_k \circ \tilde{\varrho}_{k+1} = \tilde{\varrho}_k \circ \zeta_k.$$

The proof involves categorizing the possible bottom and top parts of polygonal chains and expressing their weights using "partial transfer maps". We then use the bijection $\aleph$ to formulate and apply chain shifting results for these maps. (This proof requires the use of subtraction.) The following corollary follows immediately.

Corollary 5.3. *Let $T = T_{r,s}$ and $R = R_{r,s}$ be the rectangle and trapezoid poset. Then the map $\zeta\colon \mathbb{R}_+^T \to \mathbb{R}_+^R$ is equivariant with respect to birational (antichain) rowmotion:*

$$\zeta \circ \tilde{\rho}_T = \tilde{\rho}_R \circ \zeta.$$

In particular, birational rowmotion on the trapezoid ($\tilde{\rho}_T$ or ρ_T) has order $r+s$.

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