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Two-Row Set-Valued Tableaux: Catalan^{+k} Combinatorics

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Abstract. Set-valued standard Young tableaux are a generalization of standard Young tableaux due to Buch (2002) with applications in algebraic geometry. The enumeration of set-valued SYT is significantly more complicated than in the ordinary case, although product formulas are known in certain special cases. In this work we study the case of two-rowed set-valued SYT with a fixed number of entries. These tableaux are a new combinatorial model for the Catalan, Narayana, and Kreweras numbers, and can be shown to be in correspondence with both 321-avoiding permutations and a certain class of bicolored Motzkin paths. We also introduce a generalization of the set-valued comajor index studied by Hopkins, Lazar, and Linusson (2023), and use this statistic to find seemingly new q-analogs of the Catalan and Narayana numbers.

Keywords: Catalan numbers, set-valued tableaux, pattern-avoidance

1 Introduction

1.1 Set-Valued Tableaux

Let $\lambda \vdash n$. A *set-valued Young tableau* of shape λ is a filling S of the cells of the Ferrers diagram of λ with nonempty sets of positive integers. They were introduced by Buch [2] to study the K-theory of the Grassmannian, and have since appeared in both algebro-geometric and combinatorial contexts (see, inter alia, [1, 4, 5, 6, 8, 11]).

A set-valued Young tableau is *standard* if:

1. The sets in the cells of λ form a set partition of $[n+k]$ for some $k \geq 0$, and
2. If u is (weakly) northwest of v in λ then $\max S(u) < \min S(v)$.

We write $\mathrm{SYT}^{+k}(\lambda)$ for the set of set-valued standard Young tableaux of λ with entries in $[n+k]$. Intuitively, a set-valued SYT S can be thought of an integer filling of λ (filling each cell u with $\min S(u)$) along with k *extra elements*. The combinatorics of these objects

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is much more intricate than in the ordinary case; there is no known analog of the hook-length formula for counting set-valued SYT in general, although Anderson, Chen, and Tarasca [1] proved a determinantal formula for counting them.

For the purposes of enumerating the elements of $\mathrm{SYT}^{+k}(\lambda)$, it is sometimes useful to view a set-valued tableaux S from a different perspective.

Proposition 1. *A standard set-valued Young tableau of shape λ is equivalent to the following data:*

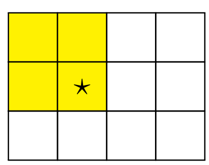
1. *A standard Young tableau S^* of shape λ,*
2. *A weak chain $\lambda^\bullet$ of subshapes $\emptyset = \lambda_0 \subsetneq \lambda_1 \subseteq \cdots \subseteq \lambda_k \subseteq \lambda_{k+1} = \lambda$,*
3. *A choice of a corner cell u_i of λ_i for each $1 \le i \le k$.*

In lieu of a proof, consider the following illustrative example.

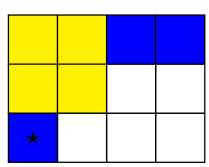
Example 2. *Consider the following set-valued SYT $T \in \mathrm{SYT}^{+4}(3 \times 4)$:*

1	2	7	8
3	4, 5	11	13
6, 9, 10	12	14, 15	16

There are cells with extra entries at matrix coordinates $(2,2)$, $(3,1)$, and $(3,3)$. Among these, the cell at $(2,2)$ has the smallest extra entry: 5. We define λ_1 to be the subshape of 3×4 for which the entries of T are between 1 and 5:

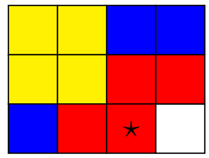


The starred cell at matrix position $(2,2)$ is u_1. The next extra entry is 9, at matrix position $(3,1)$. We define λ_2 to be the subshape for which the entries of T are between 1 and 9:



Then since 9 belongs to the starred cell $(3,1)$, we define that cell to be u_2. The next extra entry is 10, which is in the same cell as the extra entry 9. Then $\lambda_3 = \lambda_2$ and $u_3 = u_2$.

The last extra entry is 15, at matrix position $(3,3)$. We have that λ_4 is the subshape $(4,4,3)$ consisting of the cells of T whose entries are between 1 and 15.



Since 15 belongs to the cell at position $(3,3)$, we define u_4 to be that corner cell. Finally, λ_5 is the entire shape.

We obtain T^* from T by removing the 4 extra entries from T and decrementing the remaining entries of the cells in $\lambda_i \setminus \lambda_{i-1}$ by $i-1$ for each i:

1	2	6	7
3	4	8	10
5	9	11	12

The entries in the yellow cells are decremented by 0; those in the blue cells are decremented by 1; those in the red cells are decremented by 3 (notice there are no cells in $\lambda_3 \setminus \lambda_2$); and the entry of the bottom right square is decremented by 4.

This construction allows us to define a version of the *comajor index* for set-valued tableaux. Let $S \in \mathrm{SYT}^{+k}(\lambda)$, and decompose S into ℓ chunks $T_1, \ldots, T_\ell$ and k additional elements $x_1, \ldots, x_k$ as in Example 2. A *(natural) descent* of T_i is an entry j of T_i such that $j+1$ is also an entry of T_i and is in a higher row[1].

We write $\mathrm{D}(T_i)$ for the descent set of T_i, and we define the *set-valued descents* of S to be

$$\mathrm{D}^{+k}(S) := \bigsqcup D(T_i) \sqcup \{x_1, \ldots, x_k\}.$$

The *set-valued comajor index* of S is then defined as

$$\mathrm{comaj}^{+k}(S) := \sum_{x \in \mathrm{D}^{+k}(S)} (n+k-x).$$

Example 3. *Continuing from Example 2: $D^{+4}(S) = \{5, 6, 9, 10, 12, 15\}$, so $\mathrm{comaj}^{+4}(S) = 38$.*

The $k=1$ version set-valued comajor index was recently used by Hopkins, Lazar, and Linusson [7] to find a product formula for $\sum_{S \in \mathrm{SYT}^{+1}(a \times b)} q^{\mathrm{comaj}^{+1}(S)}$ analogous to Stanley's hook-content formula. Our generalized version is motivated by the probabilistic reasoning used in [7] — when one attempts to extend their arguments to general SYT^{+k}, the comaj^{+k} statistic emerges quite naturally and yields extensions of some of the results of [7] to the general case (to appear in forthcoming work).

[1]This is different from the usual definition of a descent in a Young tableau; our definition instead comes from the theory of P-partitions.

1.2 Main Results

The present work considers set-valued SYT from a different perspective. Rather than fixing the shape and the number of extra elements of a set-valued tableau S, we instead fix the number of rows and total number of elements.

This change of perspective has proven to be fruitful; if we restrict our attention to the case of two-row tableaux and fix the total number of elements while letting the number of columns vary, we obtain several new results:

- For fixed n and i, exact counts of $\bigsqcup_{2b-i+k=n} \mathrm{SYT}^{+k}(b, b-i)$ for all $0 \leq i \leq b$. For $i = 0$, that is, rows of equal length, it is the Catalan number (Equation (2.1)) and for general i it is a ballot number plus a binomial coefficient (Theorem 13).
- New models for the Catalan (Proposition 7), Narayana, and Kreweras (Proposition 8) numbers (proved via a bijection with 321-avoiding permutations).
- A new summation formula for the 321-avoiding permutations by the number of peaks (Corollary 10).
- Exact counts of several families of lattice paths arising from these tableaux (Proposition 12 and Theorem 13).
- Seemingly-new families of q-Catalan and q-Narayana numbers (Section 5.1).

2 Bijection to 321-avoiding Permutations

In this section we will use a bijection to 321-avoiding permutations of length $n-1$ prove that for any $n \geq 2$

$$\sum_{2b+k=n} \#\mathrm{SYT}^{+k}(2 \times b) = \mathrm{Cat}(n-1). \tag{2.1}$$

A permutation $\pi = \pi_1 \ldots \pi_n$ is called *321-avoiding* if it does not have three elements $\pi_i > \pi_j > \pi_k$ for $1 \leq i < j < k \leq n$. Another well-known way to describe 321 avoiding permutations is as follows. Recall that a *right-to-left minimum* in a permutation π is an element π_i such that $\pi_i < \pi_j$ for all $j > i$. The right-to-left minima of any permutation form an increasing sequence (when read from the left). The condition that a permutation is 321-avoiding is equivalent to asking that the elements that are not right-to-left minima also form an increasing sequence. This characterization dates back to the early 1900s; see [10, Vol. I, Section V, Chapter III]. [2] Visualising the permutation with a permutation matrix, the right-to-left minima will be on or below the main diagonal and the other elements above the diagonal. Forming a lattice path around the elements on or above

[2] The text considers 123-avoiding permutations, which are the reverses of 321-avoiding permutations.

$$\begin{array}{c}3\\5\\1\\2\\7\\8\\4\\10\\11\\6\\9\end{array}\begin{pmatrix}0&0&1&0&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0&0\\1&0&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&1&0&0&0\\0&0&0&1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0&1&0&0\end{pmatrix}$$

Figure 1: The permutation matrix of π from Example 4, along with its associated lattice path.

the diagonal gives a direct bijection to south-east lattice paths above the diagonal, which is one of the many standard representations of Catalan objects. Alternatively, one can draw a lattice path below the right-to left minima and then rotate the drawing by a half turn. We also need to define an *inner valley*[3] in a permutation $\pi \in S_n$ as an element $\pi_j, 1 < j < n$ such that $\pi_{j-1} > \pi_j < \pi_{j+1}$.

Example 4. *The permutation* $\pi = \ 3\ 5\ \overline{1}\ \overline{2}\ 7\ 8\ \overline{4}\ 10\ 11\ \overline{6}\ \overline{9}$ *is* 321*-avoiding. We overline the right-to-left-minima.*

For a fixed n we now define a map

$$\alpha : \bigsqcup_{2b+k=n} \mathrm{SYT}^{+k}(2 \times b) \mapsto \{321\text{-avoiding permutations of } [n-1]\}.$$

Let $T \in \mathrm{SYT}^{+k}(2 \times b)$, with $k + 2b = n$.

(1) Remove the largest element n from T, so it contains the numbers from 1 to $n - 1$.

(2) The permutation $\alpha(T)$ starts with all except the largest element in the top left box, followed by the entries of the box directly below it, and then the largest element of the top left box. The permutation continues with the elements in the second box in the top row except the largest, then all elements in the box below it, then the largest element in the second box in the top row. We continue in this way, placing the elements of the ith box from the left in the bottom row immediately before the largest element of the ith box in the top row.

[3] An inner valley differs from an ordinary valley in that neither the first nor the last position can be an inner valley.

Example 5.

1, 2	3, 4, 6	7	10
5, 8	9	11, 12	13, 14

$$\overset{\alpha}{\mapsto} \pi = \overline{1}\ 5\ 8\ \overline{2}\ \overline{3}\ \overline{4}\ 9\ \overline{6}\ 11\ 12\ \overline{7}\ 13\ \overline{10}$$

The resulting permutation $\alpha(T)$ will by construction have the numbers in the top row as its right-to-left minima. The elements in the bottom row (except n, which has been deleted) will form another increasing sequence. The permutation formed is thus 321-avoiding. Note that the largest elements in the top boxes in columns $1, \ldots, b-1$ will be inner valleys in the permutation and there are no other inner valleys.

The inverse of α is reasonably simple; however, the full description requires checking several cases so we omit most of the details here. Intuitively, given a 321-avoiding permutation π, the right-to-left minima are inserted into the top row (with the first run of right-to-left minima needing special handling), while the ith run of elements that are *not* right-to-left minima is inserted into the ith box of the bottom row.

Example 6.

$$\pi = \ 3\ 5\ \overline{1}\ \overline{2}\ 7\ 8\ \overline{4}\ 9\ 11\ \overline{6}\ \overline{10} \overset{\alpha^{-1}}{\mapsto}$$

1	2, 4	6	10
3, 5	7, 8	9, 11	12

We summarize some basic properties of α.

Proposition 7. *For all $n \geq 2$:*

- *The map α is a bijection from $\displaystyle\bigsqcup_{2b+k=n} \mathrm{SYT}^{+k}(2 \times b)$ to the set of 321-avoiding permutations of $[n-1]$.*
- *The elements in the top row of T form the sequence of right-to-left minima in $\alpha(T)$.*
- *If T has b columns, then $\alpha(T)$ will have $b-1$ inner valleys.*

An *inner peak* in a permutation $\pi \in S_n$ is an element $\pi_j, 1 < j < n$ such that $\pi_{j-1} < \pi_j > \pi_{j+1}$. For the set of 321-avoiding permutations the involution formed by rotating the permutation matrix a half turn shows that inner peaks and inner valleys are equidistributed for 321-avoiding permutations.[4]

Recall from the theory of Catalan numbers that the number of Dyck paths of length $2n$ is counted by the Catalan number $\mathrm{Cat}(n)$, also the number of such paths with m peaks

[4] We thank FindStat [12], which helped us find that the refinement into columns was equidistributed with number of inner peaks. This equidistribution was a key insight into finding the bijection α.

is enumerated by the Narayana number $N_{n,m} = \frac{1}{m}\binom{n}{m-1}\binom{n-1}{m-1}$. There is even one further refinement. Let c_i be the number of upsteps in the Dyck paths directly before peak number i in the path, which gives a partition $\mathbf{c} = (c_1, \ldots, c_m)$ of n, that is $\sum_i c_i = n$. Further let μ_j be the number of c_i that equals j. Thus μ (or sometimes written $[1^{\mu_1}2^{\mu_2}\ldots n^{\mu_n}]$) is the **type** of the composition $\mathbf{c}$ and of the Dyck path. The number of Dyck paths with m peaks and of type μ is known to be the Kreweras number $\mathrm{Krew}(n,m,\mu) = \frac{n(n-1)\ldots(n-m+1)}{\prod_j \mu_j!}$[9]. In the bijection α, a tableau with m elements in the top row will be mapped to a 321-avoiding permutation with m right-to-left minima. As discussed above, we can draw a lattice path under these in the permutation matrix and by rotating half a turn obtain a bijection to Dyck paths with m peaks. The distance between two consecutive elements in the top row is mapped to the number of upsteps c_i of the Dyck path. This proves the following proposition.

Proposition 8. *For any $b, k \geq 1$ we have the refinements:*

- *The number of tableaux in $\cup_{2b+k=n}\mathrm{SYT}^{+k}(2\times b)$ with m elements in the top row will be enumerated by the Narayana number $N_{n-1,m} = \frac{1}{m}\binom{n-1}{m-1}\binom{n-2}{m-1}$*
- *The number of tableaux in $\cup_{2b+k=n}\mathrm{SYT}^{+k}(2\times b)$ with m elements $a_1, \ldots, a_m$ in the top row will be enumerated by the Kreweras number $\mathrm{Krew}(n,m,\mu)$, where μ is the type of $(c_1, \ldots, c_m)$ with $c_i = a_i - a_{i-1}$, setting $a_0 = 0$.*

The bijection α also implies the following.

Proposition 9. *For $n \geq 3$*

1. $|\cup_{2b+k+1=n}\mathrm{SYT}^{+k}(b+1,b)| = \mathrm{Cat}(n) - \mathrm{Cat}(n-1) = \frac{3}{n+1}\binom{2n-2}{n}$.
2. $|\cup_{2b+k=n}\mathrm{SYT}^{+k}(b+1,b)/(1)| = \mathrm{Cat}(n) - 2\,\mathrm{Cat}(n-1) + \mathrm{Cat}(n-2)$.

3 Enumeration According to Peaks

In [1, Corollary 5.4], Anderson, Chen, and Tarasca give a formula for the Euler characteristic of a certain *Brill–Noether space*, which they had earlier shown to be equal to the (signed) count of a certain class of set-valued tableaux. Specializing to the two row case and translating into our notation, their formula becomes:

$$\#\{\mathrm{SYT}^{+k}(2\times b)\} = \frac{1}{k!}\sum_{c=0}^{\lfloor \frac{k}{2}\rfloor} f^{(k-c,c)} f^{(b+k-c,b+c)} (b+k-c-1)_{(k-c)} (b+c-2)_{(c)}, \tag{3.1}$$

where f^{λ} is the number of SYT of shape λ, and $(x)_a$ is the falling factorial $x(x-1)\cdots(x-a+1)$.

For our purposes, it is convenient to use the hook length formula to rewrite Equation 3.1 purely in terms of factorials and binomial coefficients:

$$\begin{aligned}
&\#\{\mathrm{SYT}^{+k}(2\times b)\} \qquad (3.2)\\
&= \frac{1}{k!}\sum_{c=0}^{\lfloor \frac{k}{2}\rfloor} \frac{k!(k-2c+1)}{c!(k-c+1)!}\,\frac{(2b+k)!(k-2c+1)}{(b+c)!(b+k-c+1)!}\,\frac{(b+k-c-1)!}{(b-1)!}\,\frac{(b+c-2)!}{(b-2)!}\\
&= \sum_{c=0}^{\lfloor \frac{k}{2}\rfloor} \frac{(k-2c+1)^2}{(k-c+1)(b+k-c+1)}\binom{b+c-2}{c}\binom{b+k-c-1}{b-1}\binom{2b+k}{b+c}. \qquad (3.3)
\end{aligned}$$

In light of our bijection between $\mathrm{SYT}^{+k}(2\times b)$ and the set of 321-avoiding permutations of $n = 2b+k$ with exactly k peaks, we immediately have the following result:

Corollary 10.

$$\#\{\mathfrak{S}^{321}_{2b+k}\} = \sum_{c=0}^{\lfloor \frac{k}{2}\rfloor} \frac{(k-2c+1)^2}{(k-c+1)(b+k-c+1)}\binom{b+c-2}{c}\binom{b+k-c-1}{b-1}\binom{2b+k}{b+c}.$$

This sort of closed form expression is a somewhat pleasant surprise; in [3, Theorem 3], the authors give the following generating function formula for the sequence $a^{\mathrm{pk}}_{n,k}(321)$ of 321-avoiding permutations of $[n]$ with k peaks:

$$\sum_{n\geq 0}\sum_{k\geq 0} a^{\mathrm{pk}}_{n,k}(321)q^k z^n = 1 + z\left(-\frac{-1+\sqrt{-4z^2q+4z^2-4z+1}}{2z(zq-z+1)}\right)^2, \qquad (3.4)$$

It is not at all obvious how one would obtain Equation (3.3) from Equation (3.4) (or vice-versa) using only elementary techniques.

4 Motzkinlike and Ballotlike Paths

In addition to 321-avoiding permutations, we can interpret the $\mathrm{SYT}^{+k}(2\times b)$ in terms of a certain class of bicolored Motzkin paths.

We recall that a *Motzkin path* of length n is a lattice path in $\mathbb{Z}^2$ from $(0,0)$ to $(n,0)$ consisting of *up steps* $U = (1,1)$, *down steps* $D = (1,-1)$, and *horizontal steps* $H = (1,0)$ in some order, with the property that the path never goes below the x-axis.

We will color the horizontal steps of the Motzkin paths with u (for upstairs or umber) and d (downstairs or denim). We will consider the following two restrictions on the coloring:

(1) umber horizontal steps do not occur at height zero;

(2) denim horizontal steps do not occur before the first down step.

We use $\mathrm{Motz}(n)$ to denote the set of bicolored Motzkin paths of length n, and $\mathrm{Motz}^1(n)$, $\mathrm{Motz}^2(n)$, and $\mathrm{Motz}^{1,2}(n)$ to denote the set of paths which satisfies the restrictions (1), (2), and both (1) and (2) respectively.

Extending the well-known bijection between two-rowed rectangular SYTs and Dyck paths, we have the following bijection.

Proposition 11. *There is a bijection β between $\mathrm{SYT}^k(2 \times b)$ and $\mathrm{Motz}^{1,2}(2b + k)$ with k horizontal steps. A tableau S maps to the path Γ for which:*

- *up steps of Γ occur at the minimal entries of boxes in the first row of S;*
- *down-steps of Γ occur at the minimal entries in the second row;*
- *we color a horizontal step of Γ umber if its index is a (non-minimal) entry of a box in the first row of S, and denim if its index is a (non-minimal) entry in the second row.*

The proof of Proposition 11 is reasonably straightforward and is omitted in this abstract.

The first two equalities in the following proposition are well-known but the other two seem to be new. We construct a bijection that, when concatenated with β in Proposition 11, gives us a second bijective proof of (2.1). The details are omitted in this extended abstract.

Proposition 12. *The Catalan numbers enumerate all four possible restriction on Motzkin paths:* $|\mathrm{Motz}(n)| = \mathrm{Cat}(n+1)$, $|\mathrm{Motz}^1(n)| = \mathrm{Cat}(n)$, $|\mathrm{Motz}^2(n)| = \mathrm{Cat}(n)$, *and* $|\mathrm{Motz}^{1,2}(n)| = \mathrm{Cat}(n-1)$, *for* $n \geq 2$.

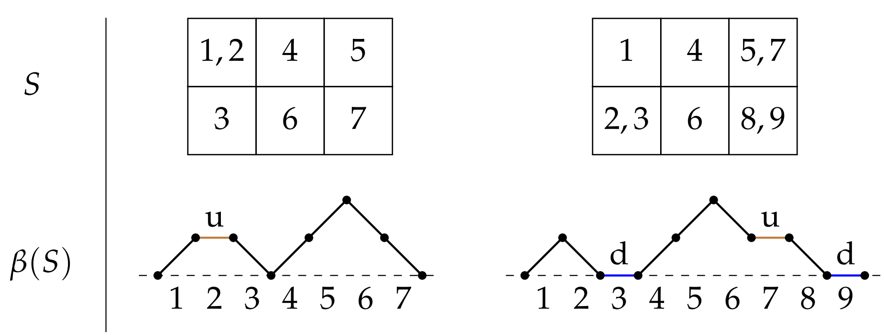


Figure 2: Examples of the bijection β between two-rowed rectangular set-valued SYTs and birestricted bicolored Motzkin paths.

4.1 Ballotlike paths

We can also consider a larger class of paths which we call *ballotlike*. A ballotlike path P is a lattice path in the 1st quadrant starting at $(0,0)$ and ending at (n,i) which uses the steps $U=(1,1)$, $D=(1,-1)$, $u=(1,0)^{\text{umber}}$ and $d=(1,0)^{\text{denim}}$, subject to the same conditions on u and d steps from the definition of $\mathrm{Motz}^{1,2}(n)$. We write $\mathrm{Bal}^*(n,i)$ for the set of ballotlike paths ending at (n,i). The enumeration turns out to be the sum of a classical ballot number and a binomial coefficient.

Theorem 13. *For any (n,i) with $0 \le i \le n$, we have*

$$\#\,\mathrm{Bal}^*(n,i) = \binom{2n-2}{n-i-1} - \binom{2n-2}{n-i-2} + \binom{n-2}{n-i}.$$

Moreover, if we take the obvious extension of the bijection between $\mathrm{Motz}^{1,2}(n)$ *and set-valued SYT of shape $2\times b$, we have for any (n,i) with $0 \le i \le n$,*

$$\#\bigsqcup_{2b+k-i=n} \mathrm{SYT}^{+k}(b,b-i) = \binom{2n-2}{n-i-1} - \binom{2n-2}{n-i-2} + \binom{n-2}{n-i}.$$

Example 14. *When $n=4$ and $i=2$ we have 6 set-valued SYT.*

$$\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1,2,3&4\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1,2&3,4\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&2,3,4\\\hline\end{array}$$

5 Future Work

5.1 q-Catalan and q-Narayana

Given the numerology for $\mathrm{SYT}^{+k}(2\times b)$, it is natural to consider the following q-analogs:

$$\widetilde{\mathrm{Cat}}_n(q) := \sum_{2b+k=n+1}\left(\sum_{S\in\mathrm{SYT}^{+k}(2\times b)} q^{\mathrm{comaj}^{+k}(S)}\right)$$

$$\widetilde{N}_{n,m}(q) := \sum_{2b+k=n+1}\left(\sum_{\substack{S\in\mathrm{SYT}^{+k}(2\times b)\\ m \text{ elts in top row}}} q^{\mathrm{comaj}^{+k}(S)}\right).$$

Using the bijection in Proposition 11 and a double recursion we can compute the polynomials $\widetilde{\mathrm{Cat}}_q$ and $\widetilde{N}_{n,m}(q)$ for small values of n,m (details omitted in this extended abstract). They do not seem to match any statistic we have found in the literature.

Question 15. *Are there better formulas for $\widetilde{\mathrm{Cat}}_n(q)$ and $\widetilde{N}_{n,m}(q)$?*

n	$\widetilde{\mathrm{Cat}}_n(q)$
1	1
2	$q+1$
3	q^3+2q^2+q+1
4	$q^6+2q^5+3q^4+3q^3+2q^2+2q+1$
5	$q^{10}+2q^9+3q^8+7q^7+6q^6+5q^5+6q^4+7q^3+3q^2+q+1$

Figure 3: The first six $\widetilde{\mathrm{Cat}}_n(q)$ polynomials.

n/m	**1**	**2**	**3**	**4**
1	1			
2	1	q		
3	q	$2q^2+1$	q^3	
4	q^3	$2q^4+q^3+q^2+q+1$	$2q^5+q^4+q^3+q^2+q$	q^6

Figure 4: $\widetilde{N}_{n,m}(q)$ for $2 \leq n \leq 5$.

5.2 Expected Number of Columns

The results of [7] draw heavily upon the language of probability theory. In particular, the authors consider several families of probability distributions on subshapes of the $a \times b$ rectangular partition, and compute the expected value of the number of corners of the subshapes with respect to these distributions. In this spirit, we consider the number of columns of a randomly-selected $S \in \bigsqcup_{2b+k=n} \mathrm{SYT}^{+k}(2 \times b)$ (equivalently, the number of inner peaks of a randomly-selected 321-avoiding permutation).

Conjecture 16. *If we sample such an S uniformly at random, we have for $n \geq 3$ that*

$$\mathbb{E}\left(\#\text{ of columns of } S\right) = \left(\binom{n}{2} + n - 3\right) \Big/ (2n-3).$$

Question 17. *Is there a nice formula for the q-version? Specifically, is there a better formula for*

$$\mathbb{E}_q\left(\#\text{ of columns of } S\right) = \sum_{2b+k=n+1} b \cdot \left(\sum_{S \in \mathrm{SYT}^{+k}(2\times b)} q^{\mathrm{comaj}^{+k}(S)}\right) \Big/ \widetilde{\mathrm{Cat}}_n(q)?$$

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