Séminaire Lotharingien de Combinatoire **91B** (2024)
Article #97, 12 pp.

Proceedings of the 36th Conference on Formal Power Series and Algebraic Combinatorics (Bochum)

Jack Derangements

Nathan Lindzey*[1]

[1]*Henry & Marilyn Taub Faculty of Computer Science, Technion – Israel Institute of Technology*

Abstract.

For each integer partition $\lambda \vdash n$ we give a simple combinatorial expression for the sum of the Jack character θ_α^λ over the integer partitions of n with no singleton parts. For $\alpha = 1, 2$ this gives closed forms for the eigenvalues of the permutation and perfect matching derangement graphs, resolving an open question in algebraic graph theory. A byproduct of the latter is a simple combinatorial formula for the immanants of the matrix $J - I$ where J is the all-ones matrix, which might be of independent interest. Our proofs center around a Jack analogue of a hook product related to Cayley's Ω–process in classical invariant theory, which we call *the principal lower hook product*.

Keywords: Symmetric Functions, Jack Polynomials, Derangements, Algebraic Graph Theory, Young Tableaux, Umbral Calculus.

1 Introduction

Let $\lambda \vdash n$ be an integer partition and consider the power sum expansion of the *Jack polynomials*, i.e., $J_\lambda = \sum_{\mu \vdash n} \theta_\alpha^\lambda(\mu) p_\mu$ [32]. The θ_α^λ's are often called the *Jack characters* because they are a deformation of a normalization of the *irreducible characters* χ^λ of *the symmetric group* S_n. In particular, the Jack polynomials at $\alpha = 1, 2$ recover the integral forms of the *Schur* and *Zonal polynomials* respectively. These specializations have been widely studied in algebraic combinatorics due to their connections with S_n and the set $\mathcal{M}_{2n}$ of *perfect matchings of the complete graph* K_{2n}, but for arbitrary $\alpha \in \mathbb{R}$ many open questions remain [2, 32, 22]. This state of affairs has led to an investigation of the Jack characters since they provide *dual* information about Jack polynomials that may shed light on these open questions; however, the dual path towards understanding Jack polynomials is paved with its own conjectures [10, 16, 17]. We make some progress in this vein by taking *sums* of $\theta_\alpha^\lambda(\mu)$'s rather than single $\theta_\alpha^\lambda(\mu)$'s.

Let $\mathrm{fp}(\mu)$ be the number of singleton parts of μ. Define the λ*-Jack derangement sum*

$$\eta_\alpha^\lambda := \sum_{\substack{\mu \vdash n \\ \mathrm{fp}(\mu)=0}} \theta_\alpha^\lambda(\mu)$$

*lindzeyn@gmail.com. For the full version of this extended abstract, see `https://doi.org/10.48550/arXiv.2304.06629`. All proofs in this extended abstract have been deferred to the full version.

to be the sum of the Jack character θ^λ_α over the *derangements*, i.e., partitions $\mu \vdash n$ with no singleton parts. To motivate this definition, recall that if $\lambda \vdash n$ is the cycle type of a permutation $\pi \in S_n$, then π is a *derangement* if and only if $\mathrm{fp}(\lambda) = 0$. Let $D_n \subseteq S_n$ be the set of derangements of S_n. One can show that η^λ_1 is a scaled character sum over D_n, i.e.,

$$\eta^\lambda_1 = \sum_{\substack{\mu \vdash n \\ \mathrm{fp}(\mu)=0}} \theta^\lambda_1(\mu) = \sum_{\substack{\mu \vdash n \\ \mathrm{fp}(\mu)=0}} \frac{|C_\mu|}{\chi^\lambda(1)} \chi^\lambda(\mu) = \frac{1}{\chi^\lambda(1)} \sum_{\pi \in D_n} \chi^\lambda(\pi)$$

where $C_\mu \subseteq S_n$ is the conjugacy class corresponding to $\mu \vdash n$. For $\alpha = 2$, an analogous result holds for the so-called *perfect matching derangements* of $\mathcal{M}_{2n}$ (see [18], for example). We are unaware of combinatorial models for $\alpha \neq 1, 2$, but it is natural to view η^λ_α as the α-analogue of the character sum over derangements, which is our main focus.

While little is known about the Jack derangement sums for arbitrary $\alpha \in \mathbb{R}$, the $\alpha = 1, 2$ cases have received special attention in algebraic graph theory because they are in fact the eigenvalues of the so-called *derangement graphs*. The set $\{\eta^\lambda_1\}_{\lambda \vdash n}$ is the spectrum of the *permutation derangement graph*: $\Gamma_{n,1} := (S_n, E)$ where $\pi\sigma \in E \Leftrightarrow \sigma\pi^{-1} \in D_n$, i.e., the normal Cayley graph of S_n generated by D_n. See [7, Ch. 14] or [29] for more details on the permutation derangement graph. The set $\{\eta^\lambda_2\}_{\lambda \vdash n}$ is the spectrum of the *perfect matching derangement graph*: $\Gamma_{n,2} := (\mathcal{M}_{2n}, E)$ where $mm' \in E \Leftrightarrow m \cap m' = \emptyset$. For more details on the perfect matching derangement graph, see [7, Ch. 15] or [18].

These graphs made their debut in *Erdős–Ko–Rado combinatorics*, a branch of extremal combinatorics that studies how large families of combinatorial objects can be subject to the restriction that any two of its members intersect. By design, the *independent sets* (sets of vertices that are pairwise non-adjacent) of $\Gamma_{n,\alpha}$ are in one-to-one correspondence with the so-called *intersecting families* of permutations and perfect matchings for $\alpha = 1, 2$, and the spectra of these graphs have been used to give tight upper bounds and characterizations of the largest intersecting families of S_n and $\mathcal{M}_{2n}$. We refer the reader to [7] for a comprehensive account of algebraic techniques in Erdős–Ko–Rado combinatorics.

The derangement graphs are interesting in their own right since they are natural analogues of the celebrated *Kneser graph*, a cornerstone of algebraic graph theory [9]. Because the algebraic combinatorics of permutations and perfect matchings are more baroque than that of subsets, the eigenvalues of the derangement graphs have proven to be far more challenging to understand. We briefly overview the results in this area.

The first non-trivial recursion for the eigenvalues of the permutation derangement graph was derived by Renteln [29] using determinantal formulas for the *shifted Schur functions* [26], which he used to calculate the minimum eigenvalue of the permutation derangement graph. Using different techniques, Ellis [5] later computed the minimum eigenvalue of the permutation derangement graph. Deng and Zhang [4] determined the second largest eigenvalue. In [13], Ku and Wales investigated some interesting properties of the eigenvalues of the permutation derangement graph. In particular, they proved

The Alternating Sign Theorem, namely, that sgn $\eta_1^\lambda = (-1)^{|\lambda|-\lambda_1}$ for all λ, and they offered a conjecture on the magnitudes of the eigenvalues known as the *Ku–Wales Conjecture*. In [14], Ku and Wong proved this conjecture by deriving another recursive formula using shifted Schur functions that led to a simpler proof of the Alternating Sign Theorem.

It was soon noticed that the algebraic properties of the perfect matching derangement graph parallel those of the permutation derangement graph. The minimum eigenvalue of the perfect matching derangement graph was computed by Godsil and Meagher [8] and later by Lindzey [19, 20]. An analogue of the Alternating Sign Theorem was conjectured in [18, 7] which was recently proven by both Renteln [30] and Koh et al [12]. In an earlier effort to prove this conjecture, Ku and Wong [15] give recursive formulas for η_2^λ and a few closed forms for select shapes. In [31], Srinivasan gives more computationally efficient formulas for the eigenvalues of the perfect matching derangement graph. Godsil and Meagher ask whether an analogue of the Ku–Wales conjecture holds for the perfect matching derangement graph [7, pg. 316]. The latter has remained open since the eigenvalues of the perfect matching derangement graph have defied nice recursive expressions akin to permutation derangement graph. This is because the aforementioned determinantal formulas for shifted Schur functions do not exist for shifted Zonal polynomials or shifted Jack polynomials.

The main shortcoming of the known eigenvalue formulas for the derangement graphs is that they cannot be evaluated efficiently, i.e., they lack “good formulas". Indeed, finding closed forms was deemed a difficult open problem [7, pg. 316], perhaps due to the formal hardness of evaluating the irreducible characters of the symmetric group [28, 11, 27]. Our results show that good formulas do in fact exist.

To state our main results we need a few definitions. Let $h_*^\lambda(i,j) := \alpha a_\lambda(i,j) + l_\lambda(i,j) + 1$ be the *lower hook length* of the cell $(i,j) \in \lambda$ where $a_\lambda(i,j)$ and $l_\lambda(i,j)$ denote *arm length* and *leg length* respectively. We define $H_*^1(\lambda) := h_*^\lambda(1,1)h_*^\lambda(1,2)\cdots h_*^\lambda(1,\lambda_1)$ to be the *principal lower hook product* of the integer partition λ. For $\alpha = 1$, the lower hook length is just the usual notion of hook length, in which case we call $H_*^1(\lambda)$ the *principal hook product*. Note that the principal hook product for $\lambda = (n)$ is simply $n!$.

It turns out that the principal hook product for arbitrary λ arises naturally in classical invariant theory, namely, in the evaluation of a differential operator known as *Cayley's* Ω–*process* (see [3]). Independently, Filmus and Lindzey [6] observe a similar phenomenon in their study of harmonic polynomials on perfect matchings, wherein they show that the principal lower hook product appears in the evaluation of a family of differential operators acting polynomial spaces associated with perfect matchings. From the results of [6], we show in Section 3 that the principal hook product $H_*^1(\lambda)$ counts an interesting class of colored permutations $\mathcal{S}_\lambda$, defined as follows.

For each $i \in [n] := \{1, 2, \ldots, n\}$, we assign a list of colors $L(i) \subseteq [m]$ for some $m \in \mathbb{N}$. We define a *colored permutation* (c, σ) to be an assignment of colors $c = c_1, c_2, \ldots, c_n$ such that $c_i \in L(i)$ and a permutation $\sigma \in \mathrm{Sym}([n])$ such that $\sigma(i) = j \Rightarrow c_i = c_j$, i.e., each cycle of the permutation is monochromatic. Any partition λ defines a color list on each element i of the symbol set $[\lambda_1]$ by setting $L(i) := [\lambda'_i]$ where λ' denotes the *transpose* or *conjugate* partition of λ. We define $\mathcal{S}_\lambda$ to be the set of all such colored permutations, formally, $\mathcal{S}_\lambda := \{(c \in [\lambda'_1] \times \cdots \times [\lambda'_{\lambda_1}], \sigma \in S_{\lambda_1}) : \sigma(i) = j \Rightarrow c_i = c_j \text{ for all } i \in [\lambda_1]\}$. We say that a colored permutation $(c, \sigma) \in \mathcal{S}_\lambda$ is a *derangement* if $\sigma(i) = i \Rightarrow c_i \neq 1$ for all $1 \leq i \leq \lambda_1$. In other words, these are the colored permutations that have no colored cycles in common with $(1, \ldots, 1, ()) \in \mathcal{S}_\lambda$. Let $\mathcal{D}^\lambda$ be the set of derangements of $\mathcal{S}_\lambda$, and let $\mathcal{D}^\lambda_k$ be the set of derangements of $\mathcal{S}_\lambda$ with exactly k disjoint cycles. We define $D^\lambda := |\mathcal{D}^\lambda|$ and $d^\lambda_k := |\mathcal{D}^\lambda_k|$, so that $D^\lambda = d^\lambda_1 + d^\lambda_2 + \cdots + d^\lambda_{\lambda_1}$. For any $\alpha \in \mathbb{R}$, let $D^\lambda_\alpha := \sum_{k=1}^{\lambda_1} d^\lambda_k \alpha^{\lambda_1 - k}$ be the *λ-Jack derangement number*. Our first main result is Theorem 1, that the Jack derangement sums equal the Jack derangement numbers (up to sign).

Theorem 1. *For all $\alpha \in \mathbb{R}$, we have $\eta^\lambda_\alpha = (-1)^{|\lambda| - \lambda_1} D^\lambda_\alpha$*

Theorem 1 gives cleaner and more general proofs of all the previous results on the derangement graphs.

Corollary 1 (Alternating Sign Theorem)**.** *For all $\alpha \geq 0$, we have* $\operatorname{sgn} \eta^\lambda_\alpha = (-1)^{|\lambda| - \lambda_1}$.

Corollary 2 (Ku–Wales Theorem)**.** *For all $\mu, \lambda \vdash n$ such that $\mu_1 = \lambda_1$ and $\alpha \geq 0$, we have $\mu \trianglelefteq \lambda \Rightarrow |\eta^\mu_\alpha| \leq |\eta^\lambda_\alpha|$.*

Setting $\alpha = 2$ in Corollary 2 answers Godsil and Meagher's question on the Ku–Wales conjecture for the perfect matching derangement graph [7, pg. 316].

Corollary 3. *For all $\alpha \geq 1$ and $n \geq 6$, we have $(n) = \arg\max_{\lambda \vdash n} \eta^\lambda_\alpha$, $(n-1, 1) = \arg\min_{\lambda \vdash n} \eta^\lambda_\alpha$, and $(n-1, 1) = \arg\max_{\substack{\lambda \vdash n \\ \lambda \neq (n)}} |\eta^\lambda_\alpha|$.*

Our second main result is a closed-form expression for the eigenvalues of $\Gamma_{n,1}$ and $\Gamma_{n,2}$.

This work can be seen as a companion paper to [21], where less explicit but more general formulas for a variety of different "disjointness" and derangement graphs are given.

2 Shifted Jack Polynomials

We overview standard terminology associated with Jack polynomials. For any cell $(i, j) \in \lambda$, the *leg length* $l_\lambda(i, j)$ of (i, j) is the number of cells below (i, j) in the same column of λ, and the *arm length* $a_\lambda(i, j)$ of (i, j) is the number of cells to the right of (i, j) in the same row of λ, i.e., $a_\lambda(i, j) = |\{(i, k) \in \lambda : k > j\}|$ and $l_\lambda(i, j) = |\{(k, j) \in \lambda : k > i\}|$. Note that arm length and leg length remain well-defined even when λ is replaced by a set of cells that does not form an integer partition. Let $h^\lambda_*(i, j) := \alpha a_\lambda(i, j) + l_\lambda(i, j) + 1$ and $h^*_\lambda(i, j) := \alpha(a_\lambda(i, j) + 1) + l_\lambda(i, j)$ be the *lower hook length* and *upper hook length* of

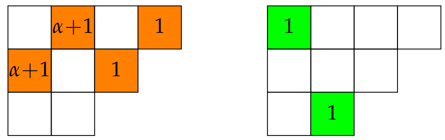


Figure 1: Let $\mu = (4,3,2) \vdash 9$. The colored cells $S = \{(2,1),(1,2),(2,3),(1,4)\}$ on the left is a 4-transversal of μ with α-weight $w_\alpha(S) = (\alpha+1)^2$. The colored cells $S' = \{(1,1),(3,2)\}$ on the right is a 2-transversal of μ with α-weight $w_\alpha(S') = 1$. Each colored cell is labeled with its lower hook length with respect to S and S'.

$(i,j) \in \lambda$, respectively. Let $H^\lambda_* = \prod_{(i,j)\in\lambda} h^\lambda_*(i,j)$ and $H^*_\lambda = \prod_{(i,j)\in\lambda} h^*_\lambda(i,j)$ be the *lower hook product* and *upper hook product* of λ, respectively. Note that the lower and upper hook product remain well-defined even when λ is replaced by a set of cells that does not form an integer partition.

Theorem 2 is a simple but opaque expression for η^λ_α in terms of the (integral form) shifted Jack polynomials $J^\star_\lambda(x;\alpha)$ (see [25], for example). These expressions are already known for η^λ_1 and η^λ_2 in terms of the determinantal formula for the shifted Schur polynomials [29] and more recently for the shifted Zonal polynomials [30]. Theorem 2 is simply the Jack analogue of these results.

Theorem 2. *For all λ and $\alpha \in \mathbb{R}$, we have $\eta^\lambda_\alpha = \sum_{k=0}^{|\lambda|}(-1)^{|\lambda|-k} J^\star_k(\lambda)/k!$.*

3 Tableau Transversals and Principal Hook Products

We now leverage some combinatorial results of [1, 6] to give a more tractable combinatorial formulation of Theorem 2, which we use to prove Theorem 1 for $\alpha = 1, 2$.

A *k-transversal* T of λ is a set of k cells of T which forms a partial transversal of the columns of λ, that is, no two cells of T lie in the same column of λ. Define the *α-weight* of a k-transversal T to be the lower hook product of T, i.e., $w_\alpha(T) = H^T_*$, with the convention that $w_\alpha(\varnothing) = 1$ (see Figure 1 for examples). Let $\mathcal{T}^k_\lambda$ be the collection of k-transversals of λ.

In [1, Theorem 5.12], Alexandersson and Féray show that $J^\star_k(\lambda)/k! = \sum_{T\in\mathcal{T}^k_\lambda} w_\alpha(T)$. Independently, Filmus and Lindzey [6] prove the following identity: $J^\star_{\lambda_1}(\lambda)/\lambda_1! = \sum_{T\in\mathcal{T}^{\lambda_1}_\lambda} w_\alpha(T) = H^1_*(\lambda)$. For $\alpha = 1$, we note that this identity can be deduced from Naruse's hook-length formula for standard skew-tableaux [23]. We write $\mu \preceq_k \lambda$ if μ is a subshape λ obtained by removing k columns of λ. There are $\binom{\lambda_1}{k}$ such subshapes, and we let the sigma notation $\sum_{\mu\preceq_k\lambda}$ denote the sum over all $\binom{\lambda_1}{k}$ subshapes μ of λ obtained by removing k columns.

Theorem 3. *For any shape λ and $\alpha \in \mathbb{R}$, we have $\eta_\alpha^\lambda = (-1)^{|\lambda|-\lambda_1} \sum_{k=0}^{\lambda_1} (-1)^k \sum_{\mu \preceq_k \lambda} H_*^1(\mu)$.*

Theorem 3 and Theorem 4 can now already be used to give an elementary combinatorial proof of Theorem 1 for $\alpha = 1, 2$ via the principle of inclusion-exclusion. This is because λ-colored permutations $\mathcal{S}_\lambda$ (see Section 1) and *λ-colored perfect matchings* $\mathcal{M}_\lambda$ (see full version) are *bona fide* combinatorial objects, and their sizes are counted by the principal hook product $H_*^1(\lambda)$.

Theorem 4. [6] *For any shape λ, we have $|\mathcal{S}_\lambda|, |\mathcal{M}_\lambda| = H_*^1(\lambda)$ for $\alpha = 1, 2$, respectively.*

In Section 5 we generalize this proof of Theorem 1 to all $\alpha \in \mathbb{R}$, but along the way we collect several results concerning principal lower hook products, perhaps of independent interest, that allow us to give more explicit expressions of Theorem 1. Specializing these expressions to $\alpha = 1, 2$ yields closed-form expressions for the eigenvalues of derangement graphs, our second main result.

4 Minors of the Principal Hook Product

In this section we prove a few technical lemmas concerning the principal hook product that are needed for closed-form expressions of Theorem 1. Let λ^{-i} be the shape obtained by removing the ith column of λ. Let $\lambda^{-i_1-i_2-\cdots-i_k}$ be the shape obtained by removing (distinct) columns $i_1, i_2, \ldots, i_k$ of λ. It is useful to think of the $H_*^1(\lambda^{-i})$'s as the *first minors* of λ, and the $H_*^1(\lambda^{-i_1-\cdots-i_k})$'s as *$k$-minors* of λ. The ordering of the i_j's is immaterial, i.e., $\lambda^{-i_1-i_2-\cdots-i_k} = \lambda^{-i_{\sigma(1)}-i_{\sigma(2)}-\cdots-i_{\sigma(k)}}$ for all $\sigma \in S_k$. Let $\lambda^{\underline{k}}$ be the shape obtained by removing the last k columns of λ. We adopt the shorthand $h_j := h_*^\lambda(1, j)$ henceforth. Lemma 1 gives a Laplace-like expansion that relates the principal lower hook product to its first minors.

Lemma 1 (Laplace Expansion)**.** *We have $\sum_{i=1}^{\lambda_1} H_*^1(\lambda^{-i}) = \frac{1}{\alpha}\left(H_*^1(\lambda) + (\alpha - h_{\lambda_1}) H_*^1(\lambda^{\underline{1}})\right)$, equivalently, $H_*^1(\lambda) = \sum_{i=1}^{\lambda_1 - 1} \alpha H_*^1(\lambda^{-i}) + h_{\lambda_1} H_*^1(\lambda^{-\lambda_1})$.*

For $\alpha \geq 1$, we are now in a position to give a short proof of both the Alternating Sign Theorem and a useful upper bound on the magnitudes of the Jack derangement sums.

Proposition 1. *For all $\alpha \geq 1$, we have* $\operatorname{sgn} \eta_\alpha^\lambda = (-1)^{|\lambda|-\lambda_1}$*. Moreover, $|\eta_\alpha^\lambda| \leq H_*^1(\lambda)$.*

For any λ and integer $0 \leq j \leq \lambda_1 - 1$, let $f_\lambda^*(j) := \prod_{i=0}^{j} ((j+1)\alpha - h_{\lambda_1 - i})$, and define $f_\lambda^*(j) := 1$ for all negative integers j. Lemma 2 is a generalization of Lemma 1 that we will be needed in order to give a more explicit version of [1, Theorem 5.12].

Lemma 2. *For all shapes λ and $0 \leq j \leq \lambda_1 - 1$, we have $\sum_{i=1}^{\lambda_1} f_{\lambda^{-i}}^*(j-1)\ H_*^1((\lambda^{-i})^{\underline{j}}) = \frac{1}{\alpha}\left(f_\lambda^*(j-1)\ H_*^1(\lambda^{\underline{j}}) + f_\lambda^*(j)\ H_*^1(\lambda^{\underline{j+1}})\right)$.*

Theorem 5 is a more explicit form for [1, Theorem 5.12], perhaps of independent interest.

Theorem 5. *For all $\alpha \in \mathbb{R}$, we have* $\frac{J^{\star}_{\lambda_1-k}(\lambda)}{(\lambda_1-k)!} = \sum_{\mu \preceq_k \lambda} H^1_*(\mu) = \frac{1}{\alpha^k}\sum_{j=0}^{k}(-1)^j \frac{\prod_{i=1}^{\lambda_1}(h_i - j\alpha)}{(k-j)!j!}$, *equivalently,* $\frac{H^*_k}{(\lambda_1-k)!}J^{\star}_{\lambda_1-k}(\lambda) = \sum_{j=0}^{k}(-1)^j\binom{k}{j}\prod_{i=1}^{\lambda_1}(h_i - j\alpha)$.

Those familiar with the umbral calculus or the calculus of finite differences may recognize the right-hand side of the second equation in Theorem 5 as essentially the kth-order forward difference Δ^k of the univariate degree-λ_1 polynomial $\mathbf{H}^1_*(\lambda, x) := \prod_{i=1}^{\lambda_1}(h_i - x\alpha)$ in x at the origin, i.e., $H^*_{(k)}J^{\star}_{\lambda_1-k}(\lambda)/(\lambda_1-k)! = (-1)^k\Delta^k[\mathbf{H}^1_*(\lambda,x)](0)$ where we define $\Delta^k[f](x) := \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}f(x+i)$ for any function $f(x)$. Forward differences of this kind are connected to polynomial interpolation in the falling factorial basis $x^{\underline{k}} := x(x-1)(x-2)\cdots(x-k+1)$, in particular, the *Newton (interpolation) polynomial* $N(x)$ of a set of points $S = \{(x_i, p(x_i))\}_{i=0}^{d}$:

$$N(x) := [p(x_0)]x^{\underline{0}} + [p(x_0), p(x_1)]x^{\underline{1}} + \cdots + [p(x_0), p(x_1), \ldots, p(x_d)]x^{\underline{d}}$$

where $[p(x_0), \ldots, p(x_j)]$ is the notation for the so-called *jth divided difference*. Note that if $p(x)$ is a degree-d polynomial and $|S| > d+1$, then $[p(x_0), \ldots, p(x_j)] = 0$ for all $j > d$.

Finally, we recall the fact that if $x_i = i$ for all $0 \leq i \leq d$, then $[p(x_0), p(x_1), \ldots, p(x_j)] = \Delta^j[p](0)/j!$, and the Newton interpolation polynomial is of the form

$$N(x) = \frac{p(0)}{0!}x^{\underline{0}} + \frac{\Delta^1[p](0)}{1!}x^{\underline{1}} + \cdots + \frac{\Delta^d[p](0)}{d!}x^{\underline{d}}. \tag{4.1}$$

See Stanley [33, Ch. 1.9] for a more in-depth discussion of the calculus of finite differences and its connections to combinatorics. In the next section, we show that each Jack derangement number is the sum of the coefficients of a Newton polynomial (Theorem 6).

5 Proof of Theorem 1

Building off the results of the previous sections, we sketch a proof of Theorem 1 in this section. For all $j > 0$, define $H^1_*(\lambda, j) := \prod_{i=1}^{\lambda_1}(h_i - j\alpha)$ to be the *j-shifted principal lower hook product*. It will be convenient to think of the shifted principal lower hook product as a univariate polynomial in x, i.e., $\mathbf{H}^1_*(\lambda, x) := \prod_{i=1}^{\lambda_1}(h_i - x\alpha)$. We let $d^{(\alpha)}_{n,k}$ denote the α-generalization of the *rencontres numbers*, that is, $d^{(\alpha)}_{n,k} := \frac{\alpha^n n!}{\alpha^k k!}\sum_{i=0}^{n-k}\frac{(-1)^i}{\alpha^i i!}$.

Theorem 6. *For all λ, $\alpha \in \mathbb{R}$, and $n \geq \lambda_1$, we have $\eta^{\lambda}_{\alpha} = (-1)^{|\lambda|-\lambda_1}\frac{1}{\alpha^n n!}\sum_{j=0}^{n} d^{(\alpha)}_{n,j}H^1_*(\lambda, j)$.*

Theorem 6 allows us to connect the Jack derangement sums to the Poisson distribution. For all $\alpha \in \mathbb{R}$, a simple induction shows that $\sum_{j=0}^{n} d^{(\alpha)}_{n,j}/\alpha^n n! = 1$, and moreover, that

$\lim_{n\to\infty} d_{n,k}^{(\alpha)}/\alpha^n n! = e^{-1/\alpha}/\alpha^k k!$. For $\alpha > 0$, the limiting distribution is the Poisson distribution with expected value $1/\alpha$. After taking limits, for all $\alpha \in \mathbb{R}$, we have

$$\eta_\alpha^\lambda = (-1)^{|\lambda|-\lambda_1} e^{-1/\alpha} \sum_{x=0}^{\infty} \frac{H_*^1(\lambda, x)}{\alpha^x x!}. \tag{5.1}$$

For $\alpha > 0$, we may interpret the Jack derangement sum as some type of "generalized factorial moment" of the Poisson distribution (up to sign), i.e., $\eta_\alpha^\lambda = (-1)^{|\lambda|-\lambda_1}\mathbb{E}[\mathbf{H}_*^1(\lambda, x)]$. A combinatorial interpretation of these moments will follow as a corollary of Theorem 1. Recall that the factorial moments of the Poisson distribution have a remarkably simple form, namely, for all $\alpha \in \mathbb{R}$, we have $\lim_{x\to\infty} x^{\underline{k}_\alpha}/\alpha^x x! = e^{1/\alpha}$ where $x^{\underline{k}_\alpha} := \alpha^k x^{\underline{k}}$. In light of Equation (5.1), the foregoing suggests that we should express the polynomial $\mathbf{H}_*^1(\lambda, x)$ in the *α-falling factorial basis* $\{x^{\underline{k}_\alpha}\}$, which we determine below for λ such that $\lambda_1 = 1, 2, 3$. Let λ' denote the transpose of λ. If $\lambda_1 = 1$, then we have $\mathbf{H}_*^1(\lambda, x) = -x^{\underline{1}_\alpha} + \lambda_1' x^{\underline{0}_\alpha}$. If $\lambda_1 = 2$, then we have $\mathbf{H}_*^1(\lambda, x) = x^{\underline{2}_\alpha} - (\lambda_2' + \lambda_1')x^{\underline{1}_\alpha} + \lambda_2'(\alpha + \lambda_1')x^{\underline{0}_\alpha}$. If $\lambda_1 = 3$, then we may write $\mathbf{H}_*^1(\lambda, x)$ as

$$-x^{\underline{3}_\alpha} + (\lambda_3' + \lambda_2' + \lambda_1')x^{\underline{2}_\alpha} - ((\alpha + \lambda_1')\lambda_3' + (\alpha + \lambda_1')\lambda_2' + (\alpha + \lambda_2')\lambda_3')x^{\underline{1}_\alpha} + \lambda_3'(\alpha + \lambda_2')(2\alpha + \lambda_1').$$

Indeed, the following proposition shows that each coefficient of $\mathbf{H}_*^1(\lambda, x)$ expressed in the α-falling factorial basis is a polynomial $c_k^\lambda(\alpha)$ that admits a combinatorial interpretation.

Proposition 2. *Let $\hat{\lambda}$ be the partition obtained by removing the first column of λ, and let $\#\mathrm{cyc}(\sigma)$ denote the number of cycles of a permutation σ. For all shapes λ and $\alpha \in \mathbb{R}$, we have $\mathbf{H}_*^1(\lambda, x) = \sum_{k=0}^{\lambda_1} c_k^\lambda(\alpha) x^{\underline{k}_\alpha}$ where $c_k^\lambda(\alpha) = (\alpha(\lambda_1 - 1 - k) + \lambda_1')c_k^{\hat{\lambda}}(\alpha) - c_{k-1}^{\hat{\lambda}}(\alpha)$, $c_k^\lambda(\alpha) := 0$ if $k > \lambda_1$, $c_{-1}^\lambda(\alpha) := 0$. Moreover, we have*

$$(-1)^k[\alpha^{\lambda_1-k-j}]c_k^\lambda(\alpha) = \sum_{\substack{I\subseteq[\lambda_1]\\|I|=k}} |\{(c,\sigma) \in \mathcal{S}_\lambda : \#\mathrm{cyc}(\sigma) = k+j \text{ and } c_i = 1, \sigma(i) = i\ \forall i \in I\}|.$$

Upon expressing Equation (5.1) in the α-falling factorial basis via the Proposition 2, the proof of Theorem 1 becomes straightforward (see the full version for more details).

6 Eigenvalues of the Permutation Derangement Graph

The known recursive expressions for the eigenvalues of the permutation derangement graph originate from [34, Ex. 7.63a], where Stanley considers the sum $d_\lambda := \sum_{\pi\in D_n} \chi^\lambda(\pi)$ and shows it can be written in terms of the complete homogeneous symmetric functions:

$$\sum_{\lambda\vdash n} d_\lambda s_\lambda = \sum_{k=0}^{n} (-1)^{n-k} n^{\underline{k}} h_1^{n-k} h_{n-k}.$$

For hook shapes, both Stanley [34, Ex. 7.63b] and Okazaki [24, Corollary 1.3] prove that

$$d_{(j,1^{n-j})} = (-1)^{n-j}\binom{n}{j}|D_j| + (-1)^{n-1}\binom{n-1}{j} = (-1)^{n-j}\binom{n-1}{j}((n-j)|D_{j-1}| + |D_j|).$$

Recalling that $\eta_1^\lambda = d_\lambda / f^\lambda$ where $f^\lambda := \chi^\lambda(1)$ is the number of standard Young tableaux of shape λ, the following generalizes Stanley and Okazaki's results to all partitions λ.

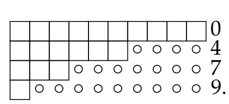
Corollary 4. $d_\lambda = (-1)^{|\lambda|-\lambda_1} f^\lambda D^\lambda$.

This suggests a natural combinatorial interpretation of $|d_\lambda|$ in terms of standard Young tableaux t of shape λ and colored derangements $(c,\sigma) \in \mathcal{D}^\lambda$. Indeed, the set $\mathcal{D}^\lambda$ is in bijection with permutations σ' defined on λ_1 cells of a fixed Young diagram t of shape λ that satisfy the following criteria: if $\sigma'(i) = j$, then the cells containing i and j belong to the same row of t; no two cells involved in the permutation σ' lie in the same column of t; and if $\sigma'(i) = i$, then the cell containing i does not belong to the first row of t. We obtain the desired count by letting t vary over all standard Young tableaux of shape λ. For $\lambda = (1^n)$ this gives a notably different proof of the well-known identity $d_{1^n} = \sum_{\pi \in D_n} \operatorname{sgn}(\pi) = \sum_{\pi \in D_n} (-1)^{\operatorname{inv}(\pi)} = (-1)^{n-1}(n-1)$, i.e., that the number of odd derangements versus even derangements differ by $\pm(n-1)$. More generally, for any integer partition $\lambda \vdash n$, we define the *immanant* of a $n \times n$ matrix A to be $\operatorname{Imm}_\lambda(A) := \sum_{\pi \in S_n} \chi^\lambda(\pi) A_{i,\pi(i)}$. If we consider the adjacency matrix of the complete graph $K_n = J_n - I_n$ where J_n is the $n \times n$ all-ones matrix, then we see that the immanants of the complete graph admit an elegant combinatorial interpretation:

$$\operatorname{Imm}_\lambda(K_n) = \sum_{\pi \in S_n} \chi^\lambda(\pi) \prod_{i=1}^{n} (K_n)_{i,\pi(i)} = \sum_{\pi \in D_n} \chi^\lambda(\pi) = d_\lambda.$$

Recall that Theorem 1 gives an expression for the Jack derangement numbers as a polynomial in α with non-negative coefficients $D_\alpha^\lambda = d_1^\lambda \alpha^{\lambda_1 - 1} + d_2^\lambda \alpha^{\lambda_1 - 2} + \cdots + d_{\lambda_1}^\lambda$ where d_k^λ is the number of colored permutations of $\mathcal{D}^\lambda$ that have precisely k disjoint cycles. One issue with this formula is that the d_k^λ's are hard to compute for general shapes λ, as they are at least as difficult as the associated Stirling numbers of the first kind. Theorem 6 offers a more concrete but less combinatorial form, which for arbitrary α seems to be as good as it gets; however, for $\alpha = 1, 2$, we show that Theorem 6 can be massaged into an explicit combinatorial closed form in terms of what we call *extended hook products*. Before we begin, we require a few more tableau-theoretic definitions.

Let $\lambda^c := (\lambda_1 - \lambda_1, \lambda_1 - \lambda_2, \cdots, \lambda_1 - \lambda_{\ell(\lambda)})$ be the *complement* of λ. In other words, the complement of λ is the subset of cells of the shape $(\lambda_1)^{\ell(\lambda)}$ that do not lie in λ. For $\lambda = (10, 6, 3, 1)$, the complement $\lambda^c = (0, 4, 7, 9)$ is the set of dots below:



Let $\mathrm{rev}(\lambda^c)$ be the partition obtained by reversing the order of the rows of λ^c. We also let $\mathrm{rev} : \lambda^c \to \mathrm{rev}(\lambda^c)$ denote the natural bijection defined on their cells, e.g.,

$$\mathrm{rev}\left(\begin{array}{cccccccccc} & & & & & & u & t & s & r \\ & & & q & p & o & n & m & l & k \\ j & i & h & g & f & e & d & c & b & a \end{array}\right) = \begin{array}{cccccccccc} a & b & c & d & e & f & g & h & i & j \\ k & l & m & n & o & p & q & & & \\ r & s & t & u & & & & & & \end{array}.$$

For any cell $\square \in \lambda^c$, we define its *upper hook length* to be $h^*_{\lambda^c}(\square) = h^*_{\mathrm{rev}(\lambda^c)}(\mathrm{rev}(\square))$, and similarly for lower hook lengths. For example, we have the following upper hook lengths for $\alpha = 1$ and $\mu = (10, 6, 3, 1)$:

13	11	10	8	7	6	4	3	2	1
8	6	5	3	2	1	1	2	3	4
4	2	1	1	2	3	5	6	7	8
1	1	2	4	5	6	8	9	10	11

.

Let $H^*_i(\lambda)$ be the *ith principal upper hook product*, i.e., the product of the upper hook lengths along the ith row of λ. We define the *extended ith principal upper hook product* to be $H^+_i(\lambda) := H^*_i(\lambda)H^*_i(\lambda^c)$. Continuing the example above, we see that $H^+_3(\mu) = 4 \cdot 2 \cdot 1 \cdot 8!/4 = 80640$. Note that $H^*_1(\lambda) = H^+_1(\lambda)$ for all λ since $(\lambda^c)_1 = 0$.

Let $d_{n,k}$ be the *kth rencontres number*, i.e., the number of permutations of S_n with precisely k fixed points. Let $p_{n,k} = d_{n,k}/n!$ be the probability of drawing a permutation (uniformly at random) from S_n with precisely k fixed points. The *Frobenius coordinates* of λ are given by $\lambda = (a_1, \ldots, a_d \mid b_1, \ldots, b_d)$ where $a_i := \lambda_i - i$ is the number of boxes to the right of the diagonal in row i, and $b_i := \lambda'_i - i$ is the number of boxes below the diagonal in column i. By default, we define $a_{d+1} := -1$. We are finally in a position to state our second main result, namely, good closed forms for the eigenvalues of $\Gamma_{n,1}$.

Theorem 7 (Eigenvalues of $\Gamma_{n,1}$)**.** *For all $\lambda = (\lambda_1, \ldots, \lambda_\ell) = (a_1, \ldots, a_d \mid b_1, \ldots, b_d) \vdash n$, we have $\eta^\lambda_1 = (-1)^n \sum_{i \leq \lambda_i + 1} (-1)^{\lambda_i} p_{\lambda_1, a_1 - a_i}\, H^+_i(\lambda)$.*

Explicit closed-form expressions for the eigenvalues of the perfect matching derangement graph $\Gamma_{n,2}$ can be derived in a similar manner, which we defer to the full version.

References

[1] P. Alexandersson and V. Féray. "Shifted symmetric functions and multirectangular coordinates of Young diagrams". *Journal of Algebra* **483** (2017), pp. 262–305. DOI.

[2] P. Alexandersson, J. Haglund, and G. Wang. "Some conjectures on the Schur expansion of Jack polynomials". *Journal of Combinatorics* **12**.2 (2021), pp. 215–233.

[3] E. Canfield and S. Williamson. "Hook length products and Cayley operators of classical invariant theory". *Linear and Multilinear Algebra* **9**.4 (1981), pp. 289–297. DOI.

[4] Y.-P. Deng and X.-D. Zhang. "A Note on Eigenvalues of the Derangement Graph". *Ars Combinatoria* **101** (July 2011).

[5] D. Ellis. "A proof of the Cameron-Ku conjecture". *Journal of the London Mathematical Society* **85**.1 (2012), pp. 165–190. DOI.

[6] Y. Filmus and N. Lindzey. "Harmonic Polynomials on Perfect Matchings (Extended Abstract)". *Proceedings of the 34th International Conference on"Formal Power Series and Algebraic Combinatorics"*. FPSAC '22. 2022.

[7] C. Godsil and K. Meagher. *Erdős–Ko–Rado Theorems: Algebraic Approaches*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2015.

[8] C. Godsil and K. Meagher. "An algebraic proof of the Erdös-Ko-Rado theorem for intersecting families of perfect matchings". *ARS MATHEMATICA CONTEMPORANEA* **12**.2 (2016), pp. 205–217.

[9] C. Godsil and G. Royle. *Algebraic Graph Theory*. Vol. 207. Graduate Texts in Mathematics. New York: Springer-Verlag, 2001, pp. xx+439.

[10] P. Hanlon. "Jack symmetric functions and some combinatorial properties of Young symmetrizers". *Journal of Combinatorial Theory, Series A* **47**.1 (1988), pp. 37–70. DOI.

[11] C. Ikenmeyer, I. Pak, and G. Panova. "Positivity of the symmetric group characters is as hard as the polynomial time hierarchy". 2022. arXiv:2207.05423.

[12] Z. K. S. Koh, C. Y. Ku, and K. B. Wong. "Alternating sign property of the perfect matching derangement graph". *Journal of Combinatorial Theory, Series A* **194** (2023), p. 105706. DOI.

[13] C. Y. Ku and D. B. Wales. "Eigenvalues of the derangement graph". *Journal of Combinatorial Theory, Series A* **117**.3 (2010), pp. 289 –312. DOI.

[14] C. Y. Ku and K. B. Wong. "Solving the Ku–Wales conjecture on the eigenvalues of the derangement graph". *European Journal of Combinatorics* **34**.6 (2013), pp. 941 –956. DOI.

[15] C. Y. Ku and K. B. Wong. "Eigenvalues of the Matching Derangement Graph". *Journal of Algebraic Combinatorics* **48**.4 (2018), 627–646. DOI.

[16] M. Lassalle. "Some combinatorial conjectures for Jack polynomials". *Annals of Combinatorics* **2** (1998), pp. 61–83.

[17] M. Lassalle. "Jack polynomials and free cumulants". *Advances in Mathematics* **222**.6 (2009), pp. 2227–2269. DOI.

[18] N. Lindzey. "Erdös–Ko–Rado for Perfect Matchings". *European Journal of Combinatorics* **65** (2017), pp. 130 –142. DOI.

[19] N. Lindzey. "Matchings and Representation Theory". PhD thesis. University of Waterloo, 2018.

[20] N. Lindzey. "Stability for 1-intersecting families of perfect matchings". *European Journal of Combinatorics* **86** (2020), p. 103091. DOI.

[21] N. Lindzey. "Disjointness Graphs (Extended Abstract)". *Proceedings of the 36th International Conference on"Formal Power Series and Algebraic Combinatorics"*. FPSAC '24. 2024.

[22] I. Macdonald. *Symmetric Functions and Hall Polynomials*. Oxford mathematical monographs. Clarendon Press, 1995.

[23] H. Naruse. "Schubert calculus and hook formula". *73rd Séminaire Lotharingien de Combinatoire* (2014).

[24] S. Okazaki. "Cycle Types of Permutations with Restricted Positions and a Characterization of a New Class of Interval Orders". PhD thesis. Massachusetts Institute of Technology, 1996.

[25] A. Okounkov and G. Olshanski. "Shifted Jack polynomials, binomial formula, and applications". *Mathematical Research Letters* **4**.1 (1997), pp. 69–78.

[26] A. Y. Okun'kov and G. Ol'shanskij. "Shifted Schur functions". *St. Petersbg. Math. J.* **9**.2 (1997), p. 1.

[27] I. Pak. "What is a combinatorial interpretation?" 2022. arXiv:2209.06142.

[28] I. Pak and G. Panova. "On the complexity of computing Kronecker coefficients". *computational complexity* **26** (2017), pp. 1–36.

[29] P. Renteln. "On the Spectrum of the Derangement Graph". *The Electronic Journal of Combinatorics* **14**.1 (2007), Research Paper 82, 17 pp. (electronic). Link.

[30] P. Renteln. "On the Spectrum of the Perfect Matching Derangement Graph". *Journal of Algebraic Combinatorics* **56**.1 (2022), 215–228. DOI.

[31] M. K. Srinivasan. "The perfect matching association scheme". *Algebraic Combinatorics* **3**.3 (2020), pp. 559–591. DOI.

[32] R. P. Stanley. "Some combinatorial properties of Jack symmetric functions". *Advances in Mathematics* **77**.1 (1989), pp. 76 –115. DOI.

[33] R. P. Stanley. *Enumerative Combinatorics: Volume 1*. 2nd. USA: Cambridge University Press, 2011.

[34] R. Stanley. *Enumerative Combinatorics: Volume 2*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2001.