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Regular Schur labeled skew shape posets and their 0-Hecke modules

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Abstract. Assuming Stanley's P-partitions conjecture holds, the regular Schur labeled skew shape posets are precisely the finite posets P with underlying set $\{1, 2, \ldots, |P|\}$ such that the P-partition generating function is symmetric and the set of linear extensions of P, denoted $\Sigma_L(P)$, is a left weak Bruhat interval in the symmetric group $\mathfrak{S}_{|P|}$. We describe the permutations in $\Sigma_L(P)$ in terms of reading words of standard Young tableaux when P is a regular Schur labeled skew shape poset, and classify $\Sigma_L(P)$'s up to descent-preserving isomorphism as P ranges over regular Schur labeled skew shape posets. The results obtained are then applied to classify the 0-Hecke modules M_P associated with regular Schur labeled skew shape posets P up to isomorphism. Then we characterize regular Schur labeled skew shape posets as the finite posets P whose linear extensions form a dual plactic-closed subset of $\mathfrak{S}_{|P|}$. Using this characterization, we construct distinguished filtrations of M_P with respect to the Schur basis when P is a regular Schur labeled skew shape poset.

Keywords: labeled poset, P-partition, weak Bruhat order, 0-Hecke algebra, representation, skew Schur function

1 Introduction

Schur labeled skew shape posets naturally appear in the context of the celebrated Stanley's P-partition conjecture. Let P_n be the set of posets on $[n] := \{1, 2, \ldots, n\}$. To each poset $P \in \mathsf{P}_n$, one can associate a quasisymmetric function K_P, called the *P-partition generating function*. In 1972, Stanley [15, p. 81] proposed a conjecture stating that for $P \in \mathsf{P}_n$, K_P is

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a symmetric function if and only if P is a Schur labeled skew shape poset. While this conjecture has been verified to be true for all posets P with $|P| \leq 8$, it remains open in the general case (see [12]). For the definition of Schur labeled skew shape posets, see Subsection 2.3. We denote by SP_n the set of all Schur labeled skew shape posets in P_n.

On the other hand, *regular posets* were introduced by Björner–Wachs [4] during their investigation of the convex subsets of the symmetric group $\mathfrak{S}_n$ on $[n]$ under the right weak Bruhat order. For $P \in \mathsf{P}_n$ with the partial order $\preceq$, let $\Sigma_R(P)$ be the set of permutations $\pi \in \mathfrak{S}_n$ satisfying that if $x \preceq y$, then $\pi^{-1}(x) \leq \pi^{-1}(y)$. They observed that every convex subset of $\mathfrak{S}_n$ under the right weak Bruhat order appears as $\Sigma_R(P)$ for some $P \in \mathsf{P}_n$, and every right weak Bruhat interval in $\mathfrak{S}_n$ is convex. This observation led them to characterize the posets $P \in \mathsf{P}_n$ satisfying that $\Sigma_R(P)$ is a right weak Bruhat interval. They introduced the notion of regular posets, and proved that $P \in \mathsf{P}_n$ is a regular poset if and only if $\Sigma_R(P)$ is a right weak Bruhat interval in $\mathfrak{S}_n$. For the definition of regular posets, refer to Definition 2.1. We denote by RP_n the set of all regular posets in P_n. Let $\Sigma_L(P) := \{\gamma^{-1} \mid \gamma \in \Sigma_R(P)\}$. By considering the left Bruhat order and $\Sigma_L(P)$ instead of the right Bruhat order and $\Sigma_R(P)$, we can establish a similar characterization. However, we prefer the former over the latter as it is better suited for handling left $H_n(0)$-modules.

Let $\mathsf{RSP}_n := \mathsf{RP}_n \cap \mathsf{SP}_n$. In the following, we explain the reason why we study regular Schur labeled skew shape posets from the perspective of the representation theory of the 0-Hecke algebra.

In 1996, Duchamp–Krob–Leclerc–Thibon [7] introduced a ring isomorphism, called the *quasisymmetric characteristic*, from the Grothendieck ring $\mathcal{G}_0(H_\bullet(0))$ of the tower of 0-Hecke algebras to the ring QSym of quasisymmetric functions. For the definition of the quasisymmetric characteristic, see Subsection 2.4. In 2002, Duchamp–Hivert–Thibon [6] associated a right $H_n(0)$-module M_P with each poset $P \in \mathsf{P}_n$, such that the image of M_P under the quasisymmetric characteristic is K_P. This was achieved by defining a suitable right $H_n(0)$-action on $\Sigma_R(P)$. For technical reasons, we use a slightly different 0-Hecke module, denoted as M_P, instead of Duchamp–Hivert–Thibon's module M_P. Our M_P is a left $H_n(0)$-module with the basis $\Sigma_L(P)$. For the precise definition of M_P, refer to Definition 2.4.

Since the middle of 2010, various left 0-Hecke modules, each equipped with a tableau basis and yielding an important quasisymmetric characteristic image, have been constructed ([1, 3, 14, 16, 17]). In order to handle these modules in a uniform manner, Jung–Kim–Lee–Oh [9] introduced a left $H_n(0)$-module $\mathsf{B}(I)$, referred to as *the weak Bruhat interval module associated with I*, for each left weak Bruhat interval I in $\mathfrak{S}_n$. Furthermore, they showed that the Grothendieck ring $\bigoplus_{n\geq 0} \mathcal{G}_0(\mathscr{B}_n)$ is isomorphic to QSym as Hopf algebras, where $\mathscr{B}_n$ is the category direct sums of finitely many isomorphic copies of weak Bruhat interval modules of $H_n(0)$. Recently, Choi–Kim–Oh [5] clarified the exact relationship between the weak Bruhat interval modules and the 0-Hecke modules M_P, using Björner–Wachs' characterization.

The aim of this paper is to give a comprehensive investigation of regular Schur labeled skew shape posets and their associated 0-Hecke modules. Firstly, we provide an explicit description of $\Sigma_L(P)$ for $P \in \mathsf{RSP}_n$. Next, we study the classification of left weak Bruhat intervals in $\mathfrak{S}_n$ up to descent-preserving poset isomorphism. Using the classification, we classify the $H_n(0)$-modules M_P up to isomorphism as P ranges over RSP_n. Then, we characterize regular Schur labeled skew shape posets as the posets such that $\Sigma_L(P)$ is a dual plactic-closed subset of $\mathfrak{S}_n$. This characterization is applied to show that for $P \in \mathsf{RSP}_n$, M_P admits a distinguished filtration with respect to the Schur basis. A tableau description of M_P for $P \in \mathsf{RSP}_n$ is also provided. Lastly, we discuss further issues concerned with the classification of the $H_n(0)$-modules M_P.

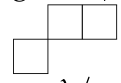
For details and more results, we refer the reader to [10].

2 Preliminaries

Throughout this paper, n will denote a nonnegative integer unless otherwise stated.

2.1 Compositions, Young diagrams, and bijective tableaux

A *composition* α of n, denoted by $\alpha \models n$, is a finite ordered list of positive integers $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ satisfying $\sum_{i=1}^{k} \alpha_i = n$. We call $k =: \ell(\alpha)$ the *length* of α and $n =: |\alpha|$ the *size* of α. Given $\alpha = (\alpha_1, \ldots, \alpha_{\ell(\alpha)}) \models n$, we define $\mathrm{set}(\alpha) := \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \sum_{i=1}^{\ell(\alpha)-1} \alpha_i\}$. The *reverse composition* α^{r} *of* α is the composition $(\alpha_k, \alpha_{k-1}, \ldots, \alpha_1)$ and the *complement* α^{c} *of* α is the unique composition satisfying $\mathrm{set}(\alpha^c) = [n-1] \setminus \mathrm{set}(\alpha)$. If a composition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \models n$ satisfies $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$, then it is called a *partition* of n and denoted as $\lambda \vdash n$. Given two partitions λ and μ with $\ell(\lambda) \geq \ell(\mu)$, we write $\lambda \supseteq \mu$ if $\lambda_i \geq \mu_i$ for all $1 \leq i \leq \ell(\mu)$. A *skew partition* λ/μ is a pair (λ, μ) of partitions with $\lambda \supseteq \mu$. We call $|\lambda/\mu| := |\lambda| - |\mu|$ the *size* of λ/μ.

Given a partition λ, we define the *Young diagram* $\mathtt{yd}(\lambda)$ *of* λ to be the left-justified array of n boxes, where the ith row from the top has λ_i boxes for $1 \leq i \leq k$. Similarly, given a skew partition λ/μ, we define the *Young diagram* $\mathtt{yd}(\lambda/\mu)$ *of* λ/μ to be the Young diagram $\mathtt{yd}(\lambda)$ with all boxes belonging to $\mathtt{yd}(\mu)$ removed. A skew partition is called *basic* if the corresponding Young diagram contains neither empty rows nor empty columns. In this paper, every skew partition is assumed to be basic unless otherwise stated. For skew partitions λ/μ and ν/κ, $\lambda/\mu \oplus \nu/\kappa$ is the skew partition whose Young diagram is obtained by taking a rectangle of empty squares with the same number of rows as $\mathtt{yd}(\lambda/\mu)$ and the same number of columns as $\mathtt{yd}(\nu/\kappa)$, and putting $\mathtt{yd}(\nu/\kappa)$ below and $\mathtt{yd}(\lambda/\mu)$ to the right of this rectangle. For instance, $\mathtt{yd}((2) \oplus (1)) =$  .

Given a skew partition λ/μ of size n, a *bijective tableau* of shape λ/μ is a filling of

$\mathtt{yd}(\lambda/\mu)$ with distinct entries in $[n]$. For later use, we denote by $\tau_0^{\lambda/\mu}$ the bijective tableau of shape λ/μ obtained by filling $1, 2, \ldots, n$ from right to left starting with the top row. A bijective tableau is referred to as a *standard Young tableau* if the elements in each row are arranged in increasing order from left to right, and the elements in each column are arranged in increasing order from top to bottom. We denote by $\mathrm{SYT}(\lambda/\mu)$ the set of all standard Young tableaux of shape λ/μ. And, we let $\mathrm{SYT}_n := \bigcup_{\lambda \vdash n} \mathrm{SYT}(\lambda)$.

2.2 Weak Bruhat orders on the symmetric group

Let $\mathfrak{S}_n$ denote the symmetric group on $[n]$. For $1 \leq i \leq n-1$, let s_i be the simple transposition $(i, i+1)$. For $\sigma \in \mathfrak{S}_n$, let

$$\mathrm{Des}_L(\sigma) := \{i \in [n-1] \mid \ell(s_i\sigma) < \ell(\sigma)\} \text{ and } \mathrm{Des}_R(\sigma) := \{i \in [n-1] \mid \ell(\sigma s_i) < \ell(\sigma)\},$$

where $\ell(\sigma)$ is the length of σ. The *left weak Bruhat order* $\preceq_L$ (resp., *right weak Bruhat order* $\preceq_R$) on $\mathfrak{S}_n$ is the partial order on $\mathfrak{S}_n$ whose covering relation $\preceq_L^c$(resp., $\preceq_R^c$) is given as follows: $\sigma \preceq_L^c s_i\sigma$ if and only if $i \notin \mathrm{Des}_L(\sigma)$ (resp, $\sigma \preceq_R^c \sigma s_i$ if and only if $i \notin \mathrm{Des}_R(\sigma)$). Given $\sigma, \rho \in \mathfrak{S}_n$, the *left weak Bruhat interval* $[\sigma, \rho]_L$ (resp., *right weak Bruhat interval* $[\sigma, \rho]_R$) denotes the closed interval $\{\gamma \in \mathfrak{S}_n \mid \sigma \preceq_L \gamma \preceq_L \rho\}$ (resp., $\{\gamma \in \mathfrak{S}_n \mid \sigma \preceq_R \gamma \preceq_R \rho\}$). Let $\mathrm{Int}(n)$ be the set of nonempty left weak Bruhat intervals in $\mathfrak{S}_n$.

2.3 Regular posets and Schur labeled skew shape posets

Let P_n be the set of posets on $[n]$. Given $P \in \mathsf{P}_n$, we write the partial order of P as $\preceq_P$.

Definition 2.1. A poset $P \in \mathsf{P}_n$ is said to be *regular* if the following holds: for all $x, y, z \in [n]$ with $x \preceq_P z$, if $x < y < z$ or $z < y < x$, then $x \preceq_P y$ or $y \preceq_P z$.

We denote by RP_n the set of all regular posets in P_n. From the result of Björner–Wachs [4, Theorem 6.8], it follows that

(i) for $P \in \mathsf{P}_n$, P is regular if and only if $\Sigma_L(P)$ is a left weak Bruhat interval, and

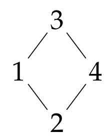
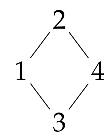
(ii) the map $\mathsf{RP}_n \to \mathrm{Int}(n)$ sending P to $\Sigma_L(P)$ is a one-to-one correspondence.

Here, $\Sigma_L(P) := \{\sigma \in \mathfrak{S}_n \mid \sigma(i) \leq \sigma(j) \text{ for all } i, j \in [n] \text{ with } i \preceq_P j\}$.

Next, let us introduce Schur labeled skew shape posets. Let λ/μ be a skew partition of size n. Given a bijective tableau τ of shape λ/μ, we define $\mathsf{poset}(\tau)$ to be the poset $([n], \preceq_\tau)$, where $i \preceq_\tau j$ if and only if i lies weakly northeast of j in τ. A *Schur labeling of shape* λ/μ is a bijective tableau of shape λ/μ such that the entries in each row decrease from left to right and the entries in each column increase from top to bottom. Let $\mathsf{S}(\lambda/\mu)$ be the set of all Schur labelings of shape λ/μ. Since $\tau_0^{\lambda/\mu}$ is a Schur labeling, $\mathsf{S}(\lambda/\mu)$ is nonempty. Set $\mathsf{SP}(\lambda/\mu) := \{\mathsf{poset}(\tau) \mid \tau \in \mathsf{S}(\lambda/\mu)\}$ and $\mathsf{SP}_n := \bigcup_{|\lambda/\mu|=n} \mathsf{SP}(\lambda/\mu)$.

Definition 2.2. A poset in P_n is said to be a *Schur labeled skew shape poset* if it is contained in SP_n.

Example 2.3. When $\lambda/\mu = (2,2)$, we have $\mathsf{S}(\lambda/\mu) = \left\{ \tau_1 := \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 4 & 3 \\ \hline \end{array} \,,\, \tau_2 := \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 4 & 2 \\ \hline \end{array} \right\}$. Therefore, $\mathsf{SP}(\lambda/\mu)$ consists of the following posets:

$\mathsf{poset}(\tau_1) =$  and $\mathsf{poset}(\tau_2) =$  .

For simplicity, we set $\mathsf{RSP}_n := \mathsf{RP}_n \cap \mathsf{SP}_n$.

2.4 The 0-Hecke algebra and the quasisymmetric characteristic

The 0-Hecke algebra $H_n(0)$ is the associative $\mathbb{C}$-algebra with 1 generated by $\pi_1, \pi_2, \ldots, \pi_{n-1}$ subject to the following relations: (1) $\pi_i^2 = \pi_i$ for $1 \le i \le n-1$, (2) $\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}$ for $1 \le i \le n-2$, and (3) $\pi_i \pi_j = \pi_j \pi_i$ if $|i-j| \ge 2$. According to [13], there are 2^{n-1} pairwise nonisomorphic irreducible $H_n(0)$-modules which are naturally parametrized by compositions of n. For each $\alpha \models n$, the irreducible module $\mathbf{F}_\alpha$ corresponding to α is the 1-dimensional $H_n(0)$-module spanned by a vector v_α, which is annihilated by π_i if $i \in \mathrm{set}(\alpha)$ or fixed by π_i otherwise for all $1 \le i \le n-1$.

Let $\mathcal{G}_0(H_n(0))$ be the *Grothendieck group* of the category of finitely generated left $H_n(0)$-modules and $\mathcal{G}_0(H_\bullet(0)) := \bigoplus_{n \ge 0} \mathcal{G}_0(H_n(0))$ the ring equipped with the induction product. In [7], Duchamp–Krob–Leclerc–Thibon showed that the linear map

$$\mathrm{ch} : \mathcal{G}_0(H_\bullet(0)) \to \mathrm{QSym}, \quad [\mathbf{F}_\alpha] \mapsto F_\alpha,$$

called *quasisymmetric characteristic*, is a ring isomorphism. Here, QSym is the ring of quasisymmetric functions and F_α is the *fundamental quasisymmetric function*.

2.5 Modules arising from posets and weak Bruhat interval modules

Definition 2.4. (cf. [6, Definition 3.18]) Let $P \in \mathsf{P}_n$. Define M_P to be the left $H_n(0)$-module with $\mathbb{C}\Sigma_L(P)$ as the underlying space and with the $H_n(0)$-action defined by

$$\pi_i \cdot \gamma := \begin{cases} \gamma & \text{if } i \in \mathrm{Des}_L(\gamma), \\ 0 & \text{if } i \notin \mathrm{Des}_L(\gamma) \text{ and } s_i\gamma \notin \Sigma_L(P), \\ s_i\gamma & \text{if } i \notin \mathrm{Des}_L(\gamma) \text{ and } s_i\gamma \in \Sigma_L(P). \end{cases}$$

For $P \in \mathsf{P}_n$, a map $f : [n] \to \mathbb{Z}_{\ge 0}$ is called a *P-partition* if (i) $f(i) \le f(j)$ for all $i \preceq_P j$ and (ii) $f(i) < f(j)$ for all $i \preceq_P j$ with $i > j$. The *P-partition generating function* is defined by $K_P := \sum_{f:P\text{-partition}} x_1^{|f^{-1}(1)|} x_2^{|f^{-1}(2)|} \cdots$.

Theorem 2.5. ([6, Theorem 3.21(i)]) *For $P \in \mathsf{P}_n$, we have* $\mathrm{ch}([\mathsf{M}_P]) = \psi(K_P)$*, where ψ is the involution of* QSym *defined by* $\psi(F_\alpha) = F_{\alpha^c}$.

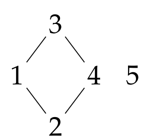
In order to provide a unified method for dealing with $H_n(0)$-modules constructed using tableaux in [1, 3, 14, 16, 17], Jung–Kim–Lee–Oh [9] introduced the *weak Bruhat interval module* $\mathsf{B}(I)$ associated with a left weak Bruhat interval I in $\mathfrak{S}_n$. For $I \in \mathrm{Int}(n)$, $\mathsf{B}(I)$ can be defined as M_P, where P is the unique poset in RP_n such that $\Sigma_L(P) = I$.

3 The weak Bruhat interval structure of $\Sigma_L(P)$ for $P \in \mathsf{RSP}_n$

First, we introduce a specific Schur labeling associated with a Schur labeled skew shape poset. For $P \in \mathsf{SP}_n$, we define τ_P to be the unique Schur labeling such that

$$\mathrm{sh}(\tau_P) \text{ is basic}, \quad \mathsf{poset}(\tau_P) = P, \quad \text{and} \quad \min_i(\tau_P) < \min_j(\tau_P) \ \text{ for } 1 \le i < j \le k. \tag{3.1}$$

Here, $\min_i(\tau_P)$ is the smallest entry in the ith connected component of τ_P from the top for all $1 \le i \le k$ and k is the number of connected components of P.

Example 3.1. Let $P =$  . There are two Schur labelings τ such that $\mathsf{poset}(\tau) = P$, more precisely,

$$\tau_1 := \begin{array}{ccc} & \boxed{2} & \boxed{1} \\ & \boxed{4} & \boxed{3} \\ \boxed{5} & & \end{array} \quad \text{and} \quad \tau_2 := \begin{array}{ccc} & & \boxed{5} \\ \boxed{2} & \boxed{1} & \\ \boxed{4} & \boxed{3} & \end{array} .$$

Since τ_1 is a Schur labeling and satisfies (3.1), $\tau_P = \tau_1$.

Definition 3.2. Let $P \in \mathsf{SP}_n$ and $\lambda/\mu = \mathrm{sh}(\tau_P)$. The τ_P*-reading* is the map

$$\mathsf{read}_{\tau_P} : \mathrm{SYT}(\lambda/\mu) \to \mathfrak{S}_n, \quad T \mapsto \mathsf{read}_{\tau_P}(T),$$

where $\mathsf{read}_{\tau_P}(T)$ is the permutation in $\mathfrak{S}_n$ given by $\mathsf{read}_{\tau_P}(T)(k) = T_{\tau_P{}^{-1}(k)}$, the entry of T in the box $\tau^{-1}(k)$, for $1 \le k \le n$.

With these notions, we state the following theorem.

Theorem 3.3. *Let $P \in \mathsf{SP}_n$ and $\lambda/\mu = \mathrm{sh}(\tau_P)$. Then, $\Sigma_L(P) = \mathsf{read}_{\tau_P}(\mathrm{SYT}(\lambda/\mu))$. In particular, if $P \in \mathsf{RSP}_n$, then*

$$\Sigma_L(P) = [\mathsf{read}_{\tau_P}(T_{\lambda/\mu}), \mathsf{read}_{\tau_P}(T'_{\lambda/\mu})]_L.$$

Here, $T_{\lambda/\mu}$(resp. $T'_{\lambda/\mu}$) is the standard Young tableau obtained by filling $\mathtt{yd}(\lambda/\mu)$ *by $1, 2, \ldots, n$ from left to right starting with the top row (resp. from top to bottom starting with leftmost column).*

Example 3.4. Let P be the poset given in Example 3.1 and $\lambda/\mu = (3,3,1)/(1,1)$. Note that

$$T_{\lambda/\mu} = \begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 1 & 2 \\ \cline{2-3} \multicolumn{1}{c|}{} & 3 & 4 \\ \hline 5 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}, \quad T'_{\lambda/\mu} = \begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 2 & 4 \\ \cline{2-3} \multicolumn{1}{c|}{} & 3 & 5 \\ \hline 1 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}, \quad \text{and} \quad \tau_P = \begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 2 & 1 \\ \cline{2-3} \multicolumn{1}{c|}{} & 4 & 3 \\ \hline 5 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}.$$

Since $\mathsf{read}_{\tau_P}(T_{\lambda/\mu}) = 21435$ and $\mathsf{read}_{\tau_P}(T'_{\lambda/\mu}) = 42531$, we have $\Sigma_L(P) = [21435, 42531]_L$.

4 An equivalence relation on $\mathrm{Int}(n)$

For $I_1, I_2 \in \mathrm{Int}(n)$, a poset isomorphism $f : (I_1, \preceq_L) \to (I_2, \preceq_L)$ is called *descent-preserving* if $\mathrm{Des}_L(\gamma) = \mathrm{Des}_L(f(\gamma))$ for all $\gamma \in I_1$. We define an equivalence relation $\overset{D}{\simeq}$ on $\mathrm{Int}(n)$ by $I_1 \overset{D}{\simeq} I_2$ if there is a descent-preserving poset isomorphism between I_1 and I_2. In [10, Section 4], we show that $\mathsf{B}(I_1) \cong \mathsf{B}(I_2)$ for all $I_1, I_2 \in \mathrm{Int}(n)$ with $I_1 \overset{D}{\simeq} I_2$. This leads us to study the equivalence classes under $\overset{D}{\simeq}$. The following theorem provides significant information regarding equivalence classes under $\overset{D}{\simeq}$.

Theorem 4.1. *Let C be an equivalence class under $\overset{D}{\simeq}$. Then, $\{\sigma \mid [\sigma, \rho]_L \in C\}$ is a right weak Bruhat interval in $\mathfrak{S}_n$.*

According to Theorem 4.1, every equivalence class C can be expressed as follows:

$$C = \{[\gamma, \xi_C \gamma]_L \mid \gamma \in [\sigma_0, \sigma_1]_R\},$$

where $\xi_C := \rho\sigma^{-1}$ for any $[\sigma, \rho]_L \in C$ and $\sigma_0, \sigma_1 \in \mathfrak{S}_n$ with $[\sigma_0, \sigma_1]_R = \{\sigma \mid [\sigma, \rho]_L \in C\}$. When $P \in \mathsf{RSP}_n$, we explicitly describe the equivalence class of $\Sigma_L(P)$ in the following theorem.

Theorem 4.2. *Let $P \in \mathsf{RSP}_n$ and C the equivalence class of $\Sigma_L(P)$ under $\overset{D}{\simeq}$. Then,*

$$C = \{\Sigma_L(Q) \mid Q \in \mathsf{RSP}_n \textit{ with } \mathrm{sh}(\tau_Q) = \mathrm{sh}(\tau_P)\}.$$

Theorem 4.2 tells us that $\{\Sigma_L(P) \mid P \in \mathsf{RSP}_n\}$ is closed under $\overset{D}{\simeq}$ and the equivalence classes in this set are parametrized by skew partitions of size n. To be precise, for any skew partition λ/μ of size n, let

$$C_{\lambda/\mu} = \{\Sigma_L(P) \mid P \in \mathsf{RSP}_n \text{ with } \mathrm{sh}(\tau_P) = \lambda/\mu\}.$$

This set is nonempty since $\mathsf{poset}(\tau_0^{\lambda/\mu}) \in C_{\lambda/\mu}$, and therefore it is an equivalence class by Theorem 4.1. To summarize, $\{\Sigma_L(P) \mid P \in \mathsf{RSP}_n\} = \bigsqcup_{|\lambda/\mu|=n} C_{\lambda/\mu}$ (disjoint union).

5 The classification of M_P's for $P \in \mathsf{RSP}_n$

In [15], Stanley proposed the following conjecture, called *Stanley's P-partitions conjecture*.

Conjecture 5.1. ([15, p. 81]) *For $P \in \mathsf{P}_n$, if K_P is symmetric, then $P \in \mathsf{SP}_n$.*

Assuming Stanley's P-partitions conjecture holds, Theorem 2.5 implies that for any $P \in \mathsf{RSP}_n$ and $Q \in \mathsf{RP}_n \setminus \mathsf{SP}_n$, $\mathrm{ch}([\mathsf{M}_P])$ is symmetric but $\mathrm{ch}([\mathsf{M}_Q])$ is not symmetric, thus $\mathsf{M}_P \not\cong \mathsf{M}_Q$. In addition, by the correspondence between RP_n and $\mathrm{Int}(n)$ in Subsection 2.3,

$$\{\mathsf{M}_P \mid P \in \mathsf{RSP}_n\} = \{\mathsf{B}(I) \mid I \in \mathrm{Int}(n) \text{ and } \mathrm{ch}([\mathsf{B}(I)]) \in \mathrm{Sym}\}.$$

This leads us to consider the classification problem for $\{\mathsf{M}_P \mid P \in \mathsf{RSP}_n\}$. We solve this problem by determining the projective cover and injective hull of M_P ($P \in \mathsf{RSP}_n$) up to isomorphism.

It is well known that there is a one-to-one correspondence between the set of irreducible $H_n(0)$-modules and that of projective indecomposable $H_n(0)$-modules. For $\alpha \models n$, let $\mathbf{P}_\alpha$ be the projective indecomposable module corresponding to $\mathbf{F}_\alpha$, that is, $\mathbf{P}_\alpha/\mathrm{rad}(\mathbf{P}_\alpha) \cong \mathbf{F}_\alpha$. In [6, Propsition 4.1], it was shown that $H_n(0)$ is a Frobenius algebra. Thus, an $H_n(0)$-module M is projective if and only if it is injective (see [2, Proposition 1.6.2]).

A *generalized composition* $\boldsymbol{\alpha}$ *of* n is a formal sum $\alpha^{(1)} \oplus \alpha^{(2)} \oplus \cdots \oplus \alpha^{(k)}$, where $\alpha^{(i)} \models n_i$ for positive integers n_i's with $n_1 + n_2 + \cdots + n_k = n$. For a generalized composition $\boldsymbol{\alpha} = \alpha^{(1)} \oplus \alpha^{(2)} \oplus \cdots \oplus \alpha^{(k)}$ of n, set $\boldsymbol{\alpha}^{\mathrm{c}} := (\alpha^{(1)})^{\mathrm{c}} \oplus (\alpha^{(2)})^{\mathrm{c}} \oplus \cdots \oplus (\alpha^{(k)})^{\mathrm{c}}$ and $\boldsymbol{\alpha}^{\mathrm{r}} := (\alpha^{(k)})^{\mathrm{r}} \oplus (\alpha^{(k-1)})^{\mathrm{r}} \oplus \cdots \oplus (\alpha^{(1)})^{\mathrm{r}}$. And, define $\mathbf{P}_{\boldsymbol{\alpha}} := \mathbf{P}_{\alpha^{(1)}} \otimes \mathbf{P}_{\alpha^{(2)}} \otimes \cdots \otimes \mathbf{P}_{\alpha^{(k)}} \uparrow_{H_{n_1}(0) \otimes H_{n_2}(0) \otimes \cdots \otimes H_{n_k}(0)}^{H_n(0)}$, where $n_i := |\alpha_i|$ for $1 \le i \le k$. This module is projective and its decomposition into projective indecomposable modules was provided in [8, Theorem 3.3].

For a connected skew partition λ/μ of size n, define

$$\boldsymbol{\alpha}_{\mathrm{proj}}(\lambda/\mu) := (\lambda_1 - \mu_1, \lambda_2 - \mu_2, \ldots, \lambda_{\ell(\lambda)} - \mu_{\ell(\lambda)}).$$

And, for a disconnected skew partition λ/μ of size n, define

$$\boldsymbol{\alpha}_{\mathrm{proj}}(\lambda/\mu) := \boldsymbol{\alpha}_{\mathrm{proj}}(\lambda^{(1)}/\mu^{(1)}) \oplus \boldsymbol{\alpha}_{\mathrm{proj}}(\lambda^{(2)}/\mu^{(2)}) \oplus \cdots \oplus \boldsymbol{\alpha}_{\mathrm{proj}}(\lambda^{(k)}/\mu^{(k)}),$$

where $\lambda/\mu = \lambda^{(1)}/\mu^{(1)} \oplus \lambda^{(2)}/\mu^{(2)} \oplus \cdots \oplus \lambda^{(k)}/\mu^{(k)}$ with connected $\lambda^{(i)}/\mu^{(i)}$'s ($1 \le i \le k$). Set

$$\boldsymbol{\alpha}_{\mathrm{inj}}(\lambda/\mu) := (\boldsymbol{\alpha}_{\mathrm{proj}}(\lambda^{\mathrm{t}}/\mu^{\mathrm{t}})^{\mathrm{c}})^{\mathrm{r}},$$

where λ^{t} and μ^{t} denote the transpose of λ and μ, respectively.

Lemma 5.2. *For $P \in \mathsf{RSP}_n$ and $\lambda/\mu = \mathrm{sh}(\tau_P)$, $\mathbf{P}_{\boldsymbol{\alpha}_{\mathrm{proj}}(\lambda/\mu)}$ (resp. $\mathbf{P}_{\boldsymbol{\alpha}_{\mathrm{inj}}(\lambda/\mu)}$) is the projective cover (resp. the injective hull) of M_P.*

Using this lemma, we establish the following classification theorem of M_P's for $P \in \mathsf{RSP}_n$ up to $H_n(0)$-module isomorphism.

Theorem 5.3. *Let $P, Q \in \mathsf{RSP}_n$. Then $\mathsf{M}_P \cong \mathsf{M}_Q$ if and only if $\mathrm{sh}(\tau_P) = \mathrm{sh}(\tau_Q)$.*

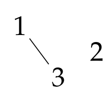
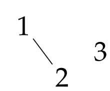
The "if" part can be derived from Theorem 4.2. Let us briefly explain how we prove the "only if" part. Considering Lemma 5.2 together with Huang's decomposition of $\mathbf{P}_\alpha$ in [8, Theorem 3.3], we prove that for $P, Q \in \mathsf{RSP}_n$, M_P and M_Q have either nonisomorphic projective covers or nonisomorphic injective hulls if τ_P and τ_Q have different shapes.

6 A characterization of regular Schur labeled skew shape posets P and distinguished filtrations of M_P

We first characterize regular Schur labeled skew shape posets from the viewpoint of dual plactic congruence. The *Robinson–Schensted correspondence* is a one-to-one correspondence between $\mathfrak{S}_n$ and $\bigcup_{\lambda \vdash n} \mathrm{SYT}(\lambda) \times \mathrm{SYT}(\lambda)$. For $\sigma \in \mathfrak{S}_n$, we use the notation $(\mathtt{ins}(\sigma), \mathtt{rec}(\sigma))$ to represent the image of σ under this bijection. The *dual plactic congruence* is an equivalence relation $\overset{K^*}{\cong}$ on $\mathfrak{S}_n$ defined by $\sigma \overset{K^*}{\cong} \rho$ if $\mathtt{rec}(\sigma) = \mathtt{rec}(\rho)$. A subset S of $\mathfrak{S}_n$ is called *dual plactic-closed* if S is a union of equivalence classes under the dual plactic congruence.

In [11, Theorem 1], Malvenuto proved that if $\Sigma_L(P)$ is dual plactic-closed, then $P \in \mathsf{SP}_n$. We improve Malvenuto's result by providing the following characterization of regular Schur labeled skew shape posets.

Theorem 6.1. *For $P \in \mathsf{P}_n$, P is a regular Schur labeled skew shape poset if and only if $\Sigma_L(P)$ is dual plactic-closed.*

Example 6.2. Consider the posets $P =$  and $Q =$  in SP_3. One sees that (i) P is non-regular and Q is regular and that (ii) $\Sigma_L(P) = \{231, 312, 321\}$ is not dual plactic-closed and $\Sigma_L(Q) = \{213, 312, 321\}$ is dual plactic-closed.

Using the characterization given in Theorem 6.1, we construct a filtration of M_P ($P \in \mathsf{RSP}_n$) which provides a representation theoretic interpretation of $s_{\lambda/\mu} = \sum_{\nu \vdash n} c^{\lambda}_{\mu,\nu} s_\nu$, the expansion of the skew Schur function $s_{\lambda/\mu}$ in the Schur basis $\{s_\nu \mid \nu \vdash n\}$. Here, $\lambda/\mu = \mathrm{sh}(\tau_P)$ and $c^{\lambda}_{\mu,\nu}$ is the Littlewood–Richardson coefficient. To handle such filtrations in a uniform manner, we introduce the notion of distinguished filtrations.

Definition 6.3. Let $\mathcal{B} = \{\mathcal{B}_\alpha \mid \alpha \in I\}$ be a linearly independent subset of QSym_n with the property that $\mathcal{B}_\alpha$ is F-positive for all $\alpha \in I$, where I is an index set. Given a finite dimensional $H_n(0)$-module M, a *distinguished filtration of M with respect to $\mathcal{B}$* is an $H_n(0)$-submodule series of M

$$0 =: M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_l := M$$

such that for all $1 \le k \le l$, $\mathrm{ch}([M_k/M_{k-1}]) = \mathcal{B}_\alpha$ for some $\alpha \in I$.

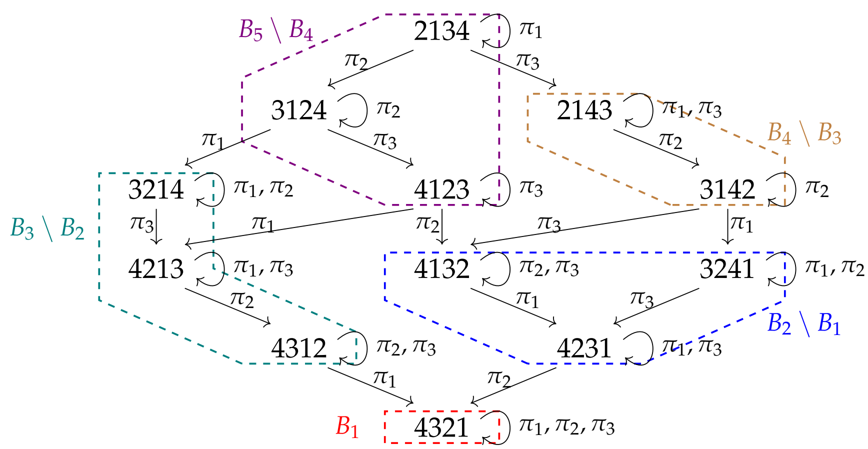
It should be remarked that a distinguished filtration of M with respect to $\mathcal{B}$ may not exist even if $\mathrm{ch}([M])$ expands positively in $\mathcal{B}$. For instance, see [10, Example 6.6]. If such a filtration exists, we have a representation theoretic interpretation of the expansion of $\mathrm{ch}([M])$ in $\mathcal{B}$.

Theorem 6.4. *For every $P \in \mathsf{RSP}_n$, M_P admits a distinguished filtration with respect to the Schur basis $\{s_\lambda \mid \lambda \vdash n\}$.*

Example 6.5. Let $P = \mathsf{poset}(\tau_0^{(4,2,1)/(2,1)})$. The set $\{\mathtt{rec}(\gamma) \mid \gamma \in \Sigma_L(P)\}$ is equal to

$$\left\{ Q_1 := \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array},\ Q_2 := \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array},\ Q_3 := \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array},\ Q_4 := \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array},\ Q_5 := \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array} \right\}.$$

For $0 \le k \le 5$, let $B_k := \{\gamma \in \mathfrak{S}_4 \mid \mathtt{rec}(\gamma) = Q_l \text{ for some } 1 \le l \le k\}$. Then, $0 = \mathbb{C}B_0 \subset \mathbb{C}B_1 \subset \mathbb{C}B_2 \subset \mathbb{C}B_3 \subset \mathbb{C}B_4 \subset \mathbb{C}B_5 = \mathsf{M}_P$ is a filtration of M_P, as seen in the figure:



Moreover, since $\mathrm{ch}([\mathbb{C}B_k/\mathbb{C}B_{k-1}]) = s_{\mathrm{sh}(Q_k)^{\mathrm{t}}}$ for all $1 \le k \le 5$, it is a distinguished filtration with respect to $\{s_\lambda \mid \lambda \vdash 4\}$.

7 A tableau description of M_P for $P \in \mathsf{RSP}_n$

Let λ/μ be a skew partition of size n. Define $X_{\lambda/\mu}$ to be the $H_n(0)$-module with the underlying space $\mathbb{C}\mathrm{SYT}(\lambda/\mu)$ and with the $H_n(0)$-action given by

$$\pi_i \cdot T = \begin{cases} T & \text{if } i \text{ is strictly left of } i+1 \text{ in } T, \\ 0 & \text{if } i \text{ and } i+1 \text{ are in the same column of } T, \\ s_i \cdot T & \text{if } i \text{ is strictly right of } i+1 \text{ in } T \end{cases}$$

for $1 \leq i \leq n-1$ and $T \in \mathrm{SYT}(\lambda/\mu)$. Here, $s_i \cdot T$ is the tableau obtained from T by swapping i and $i+1$. One can see that this $H_n(0)$-action is well defined and $X_{\lambda/\mu} \cong \mathsf{M}_{\mathsf{poset}(\tau_0^{\lambda/\mu})}$. Theorem 5.3 says that $\mathsf{M}_P \cong X_{\mathrm{sh}(\tau_P)}$ for $P \in \mathsf{RSP}_n$, and $X_{\lambda/\mu} \ncong X_{\nu/\kappa}$ for distinct skew partitions λ/μ, ν/κ of size n.

Proposition 7.1. *We have the following isomorphisms.*

(1) *For skew partitions λ/μ of size n and ν/κ of size m,*

$$X_{\lambda/\mu} \otimes X_{\nu/\kappa} \uparrow_{H_n(0)\otimes H_m(0)}^{H_{n+m}(0)} \cong X_{\lambda/\mu \oplus \nu/\kappa} \quad \text{as } H_{n+m}(0)\text{-modules.}$$

(2) *For a skew partition λ/μ of size n and $1 \leq k \leq n-1$,*

$$X_{\lambda/\mu} \downarrow_{H_k(0)\otimes H_{n-k}(0)} \cong \bigoplus_{\substack{|\nu/\mu|=k \\ \mu\subset\nu\subset\lambda}} X_{\overline{\nu/\mu}} \otimes X_{\overline{\lambda/\nu}} \quad \text{as } H_k(0)\otimes H_{n-k}(0)\text{-modules.}$$

Here, $\overline{\nu/\mu}$ and $\overline{\lambda/\nu}$ denote the basic skew partitions whose Young diagrams are obtained from $\mathtt{yd}(\nu/\mu)$ and $\mathtt{yd}(\lambda/\nu)$, respectively, by removing empty rows and empty columns.

8 Final remarks

In Theorem 5.3, we show that for $P, Q \in \mathsf{RSP}_n$,

$$\mathsf{M}_P \cong \mathsf{M}_Q \quad \text{if and only if} \quad \mathrm{sh}(\tau_P) = \mathrm{sh}(\tau_Q). \tag{8.1}$$

Since $\mathsf{RSP}_n = \mathsf{RP}_n \cap \mathsf{SP}_n$, it would be natural to consider the classification problem for $\{\mathsf{M}_P \mid P \in \mathsf{SP}_n\}$ and $\{\mathsf{M}_P \mid P \in \mathsf{RP}_n\}$.

(1) Although the notion 'the shape of τ_P' is available for $P \in \mathsf{SP}_n$, (8.1) does not hold for $P, Q \in \mathsf{SP}_n$ in general (see [10, Section 7.1.1]).

(2) Unlike (i), the notion 'the shape of τ_P' is not available for $P \in \mathsf{RP}_n$ in general. For this reason, we modify (8.1) in the following form: for $P, Q \in \mathsf{RSP}_n$,

$$\mathsf{M}_P \cong \mathsf{M}_Q \quad \text{if and only if} \quad \Sigma_L(P) \overset{D}{\simeq} \Sigma_L(Q), \tag{8.2}$$

which can be obtained by combining Theorem 4.2 and Theorem 5.3. Since the equivalence relation $\overset{D}{\simeq}$ is defined on $\mathrm{Int}(n) = \{\Sigma_L(P) \mid P \in \mathsf{RP}_n\}$, we expect that this classification can be extended to RP_n in its current form. The validity of this expectation has been checked for values of n up to 6 with the aid of the computer program SageMath. Also, we show that (8.2) holds when $P \in \mathsf{RSP}_n$, $Q \in \mathsf{RP}_n$, and $\mathrm{ch}([\mathsf{M}_P])$ is a Schur function. For more detail, see [10, Section 7.1.2].

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