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Universal Plücker coordinates for the Wronski map and positivity in real Schubert calculus

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Abstract. Given a d-dimensional vector space $V \subset \mathbb{C}[u]$, its Wronskian is the polynomial $(u+z_1)\cdots(u+z_n)$ whose zeros $-z_i$ are the points of $\mathbb{C}$ such that V contains a nonzero polynomial with a zero of order at least d at $-z_i$. Equivalently, V is a solution to the Schubert problem defined by osculating planes to the moment curve at $z_1, \dots, z_n$. The *inverse Wronski problem* involves finding all V with a given Wronskian. We solve this problem by providing explicit formulas for the Grassmann–Plücker coordinates of the general solution V, as commuting operators in the group algebra $\mathbb{C}[\mathfrak{S}_n]$ of the symmetric group. The Plücker coordinates of individual solutions over $\mathbb{C}$ are obtained by restricting to an eigenspace and replacing each operator by its eigenvalue. This generalizes work of Mukhin, Tarasov, and Varchenko (2013) and of Purbhoo (2022), which give formulas in $\mathbb{C}[\mathfrak{S}_n]$ for the differential equation satisfied by V. Moreover, if $z_1, \dots, z_n$ are real and nonnegative, then our operators are positive semidefinite, implying that the Plücker coordinates of V are all real and nonnegative. This verifies several outstanding conjectures in real Schubert calculus, including the positivity conjectures of Mukhin and Tarasov (2017) and of Karp (2021), the disconjugacy conjecture of Eremenko (2015), and the divisor form of the secant conjecture of Sottile (2003). The proofs involve the representation theory of $\mathfrak{S}_n$, symmetric functions, and τ-functions of the KP hierarchy.

Keywords: Wronskian, Schubert calculus, symmetric group, symmetric functions, KP hierarchy, total positivity

1 Introduction

For a system of real polynomial equations with finitely many solutions, we normally expect that some — but not all — of the solutions are real, while the remaining solutions come in complex-conjugate pairs. The precise number of real solutions usually depends in a complicated way on the coefficients of the equations. However, in some rare cases, it is possible to obtain a better understanding of the real solutions. A remarkable example occurs in the Schubert calculus of the Grassmannian $\mathrm{Gr}(d,m)$, for Schubert problems defined by flags osculating a rational normal curve. In 1993, Boris and Michael Shapiro

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conjectured that all such Schubert problems with real parameters have only real solutions. The corresponding systems of equations arise in various guises throughout mathematics, from algebraic curves [6, 20] to differential equations [30] to pole-placement problems [35, 7]. The conjecture was eventually proved by Mukhin, Tarasov, and Varchenko [31], using a reformulation in terms of Wronski maps, and machinery from quantum integrable systems and representation theory.

While the details of the Mukhin–Tarasov–Varchenko proof are rather intricate, the basic idea is relatively straightforward. They consider a family of commuting linear operators arising from the Gaudin model, and show that they satisfy algebraic equations defining a Schubert problem. Hence, by considering the spectra of these operators, they are able to infer some basic properties of the solutions to the Schubert problem. In this paper we extend these results, making the connection between the commuting operators and the corresponding solutions more explicit and concrete. Consequently, we obtain stronger results in real algebraic geometry, including several generalizations of the Shapiro–Shapiro conjecture. Namely, we resolve the divisor form of the secant conjecture of Sottile (2003), the disconjugacy conjecture of Eremenko [10], and the positivity conjectures of Mukhin–Tarasov (2017) and Karp [18]. Proofs and further details appear in the paper [19].

2 The Wronski map and the Bethe algebra

Let $\mathrm{Gr}(d,m)$ denote the Grassmannian of all d-dimensional linear subspaces of $\mathbb{C}^m$. It is often more convenient to work with the m-dimensional vector space $\mathbb{C}_{m-1}[u]$, of univariate polynomials of degree at most $m-1$, rather than $\mathbb{C}^m$. We explicitly identify $\mathbb{C}^m$ with $\mathbb{C}_{m-1}[u]$, via the isomorphism

$$(a_1,\ldots,a_m) \leftrightarrow \sum_{j=1}^{m} a_j \frac{u^{j-1}}{(j-1)!}\,. \tag{2.1}$$

In particular, we also view $\mathrm{Gr}(d,m)$ as the space of d-dimensional subspaces of $\mathbb{C}_{m-1}[u]$.

Now fix a nonnegative integer n, and let ν be a partition of n with at most d parts whose sizes are at most $m-d$. The ***Schubert cell*** $\mathcal{X}^\nu \subseteq \mathrm{Gr}(d,m)$ is the space of all d-dimensional linear subspaces of $\mathbb{C}[u]$ that have a basis $(f_1,\ldots,f_d)$, with $\deg(f_i) = \nu_i + d - i$. As a scheme, $\mathcal{X}^\nu$ is isomorphic to n-dimensional affine space.

Let $\mathcal{P}_n \subseteq \mathbb{C}[u]$ denote the n-dimensional affine space of monic polynomials of degree n. Given $V \in \mathcal{X}^\nu$, choose any basis $(f_1,\ldots,f_d)$ for V. We define $\mathrm{Wr}(V)$ to be the unique monic polynomial which is a scalar multiple of the Wronskian

$$\mathrm{Wr}(f_1, \dots, f_d) := \begin{vmatrix} f_1 & f_1' & f_1'' & \cdots & f_1^{(d-1)} \\ f_2 & f_2' & f_2'' & \cdots & f_2^{(d-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_d & f_d' & f_d'' & \cdots & f_d^{(d-1)} \end{vmatrix} .$$

It is not hard to see that $\mathrm{Wr}(V) \in \mathcal{P}_n$ is a polynomial of degree n, and is independent of the choice of basis. Thus we obtain a map $\mathrm{Wr} : \mathcal{X}^\nu \to \mathcal{P}_n$, called the ***Wronski map*** on $\mathcal{X}^\nu$. Abstractly, this is a finite morphism from n-dimensional affine space to itself.

Suppose $g(u) = (u + z_1) \cdots (u + z_n) \in \mathcal{P}_n$, where $z_1, \dots, z_n$ are complex numbers. The ***inverse Wronski problem*** is to compute the fibre $\mathrm{Wr}^{-1}(g) \subseteq \mathcal{X}^\nu$.

In their study of the Gaudin model for $\mathfrak{gl}_n$, Mukhin, Tarasov, and Varchenko [30, 32, 26, 31, 28] discovered a connection between the inverse Wronski problem, and the problem of diagonalizing the Gaudin Hamiltonians [16]. We will focus on the version of this story from [29], in which the Gaudin Hamiltonians generate the *Bethe algebra (of Gaudin type)* $\mathcal{B}_n(z_1, \dots, z_n) \subseteq \mathbb{C}[\mathfrak{S}_n]$, which is a commutative subalgebra of the group algebra of the symmetric group.

Let M^ν be the Specht module (i.e. irreducible $\mathfrak{S}_n$-representation) associated to the partition ν. Then $\mathcal{B}_n(z_1, \dots, z_n)$ acts on M^ν, and the image of this action defines a commutative subalgebra $\mathcal{B}_\nu(z_1, \dots, z_n) \subseteq \mathrm{End}(M^\nu)$. We have the following correspondence:

Theorem 2.1 (Mukhin, Tarasov, and Varchenko [29])**.** *The eigenspaces $E \subseteq M^\nu$ of the algebra $\mathcal{B}_\nu(z_1, \dots, z_n)$ are in one-to-one correspondence with the points $V_E \in \mathrm{Wr}^{-1}(g)$. The eigenvalues of the generators of $\mathcal{B}_\nu(z_1, \dots, z_n)$ are coordinates for V_E in some coordinate system.*

Unfortunately, Theorem 2.1 is poorly suited to studying certain properties of the Wronski map. This is because the generators of $\mathcal{B}_n(z_1, \dots, z_n)$ correspond to a somewhat unusual coordinate system for $\mathcal{X}^\nu$. Namely, given $V \in \mathcal{X}^\nu$, there is a unique ***fundamental differential operator*** $D_V = \partial_u^d + \psi_1(u)\partial_u^{d-1} + \cdots + \psi_d(u)$ with coefficients $\psi_j(u) \in \mathbb{C}(u)$, such that V is the space of solutions to the differential equation $D_V f(u) = 0$. The coefficients of D_V can be regarded as a coordinate system on $\mathcal{X}^\nu$. In the precise formulation of Theorem 2.1, the point $V_E \in \mathrm{Wr}^{-1}(g)$ is computed in these coordinates. In order to express V_E in standard coordinates, we need to solve a differential equation, resulting in highly non-linear formulas.

Our main result is Theorem 3.2 below, which is a new version of Theorem 2.1. Rather than using the fundamental differential operator coordinates, it computes $V_E \in \mathrm{Wr}^{-1}(g)$ in the ***Plücker coordinates***, which are the $d \times d$ minors of a $d \times m$ matrix whose rows form a basis for V_E. We introduce (by explicit formulas) a new set of generators β^λ for $\mathcal{B}_n(z_1, \dots, z_n)$, which are indexed by partitions λ. For any eigenspace $E \subseteq M^\nu$, the corresponding eigenvalues of the β^λ's are the Plücker coordinates of V_E.

There are three major advantages of this formulation. First, we obtain a more direct description of V_E which does not require solving a differential equation; the implicit part of our construction lies entirely in understanding the representation theory of $\mathfrak{S}_n$. Second, many natural objects of interest are given by *linear* functions of the Plücker coordinates. For example, we readily obtain explicit bases for V_E; the Wronskian and the fundamental differential operator coordinates are given as linear functions of the Plücker coordinates; and Schubert varieties and Schubert intersections are defined by linear equations in the Plücker coordinates. Third, basic properties of the operators β^λ imply positivity results about the Plücker coordinates of V_E. This enables us to resolve several conjectures in real algebraic geometry, as we explain in Section 4.

3 Universal Plücker coordinates

We now state our main theorem. For every partition λ, define

$$\beta^\lambda(t) := \sum_{\substack{X\subseteq[n],\\ |X|=|\lambda|}} \sum_{\sigma\in\mathfrak{S}_X} \chi^\lambda(\sigma)\sigma \prod_{i\in[n]\setminus X} (z_i + t)\,. \tag{3.1}$$

Here $[n] = \{1, \dots, n\}$, $\mathfrak{S}_X \subseteq \mathfrak{S}_n$ is the group of permutations of X, and $\chi^\lambda : \mathfrak{S}_X \to \mathbb{C}$ is the character of the Specht module M^λ. We note that χ^λ is integer-valued, so $\beta^\lambda(t)$ is in fact defined over $\mathbb{Z}$. Also, $\beta^\lambda(t)$ is nonzero if and only if $|\lambda| \le n$. Set $\beta^\lambda := \beta^\lambda(0)$.

Example 3.1. If $\lambda = (1, 1)$, then χ^λ is the sign character on $\mathfrak{S}_2$. When $n = 3$, we get

$$\beta^{11} = (\mathbf{1}_{\mathfrak{S}_3} - \sigma_{1,2})z_3 + (\mathbf{1}_{\mathfrak{S}_3} - \sigma_{1,3})z_2 + (\mathbf{1}_{\mathfrak{S}_3} - \sigma_{2,3})z_1\,,$$

where $\mathbf{1}_{\mathfrak{S}_3}$ denotes the identity element of $\mathfrak{S}_3$, and $\sigma_{i,j} := (i \quad j)$ is the transposition swapping i and j.

Theorem 3.2. *Let $z_1, \dots, z_n \in \mathbb{C}$, and set $g(u) := (u + z_1)\cdots(u + z_n) \in \mathbb{C}[u]$. The operators $\beta^\lambda(t) \in \mathbb{C}[\mathfrak{S}_n]$ satisfy the following algebraic identities:*

(i) Commutativity relations:

$$\beta^\lambda(s)\beta^\mu(t) = \beta^\mu(t)\beta^\lambda(s) \qquad \text{for all partitions } \lambda \text{ and } \mu\,. \tag{3.2}$$

(ii) Translation identity:

$$\beta^\mu(s + t) = \sum_{\lambda\supseteq\mu} \frac{\mathsf{f}^{\lambda/\mu}}{|\lambda/\mu|!} t^{|\lambda/\mu|} \beta^\lambda(s) \qquad \text{for all partitions } \mu\,, \tag{3.3}$$

where $\mathsf{f}^{\lambda/\mu}$ denotes the number of standard Young tableaux of shape λ/μ.

(iii) The quadratic Plücker relations.

Furthermore:

(iv) For every partition λ and $t \in \mathbb{C}$, we have $\beta^\lambda(t) \in \mathcal{B}_n(z_1, \ldots, z_n)$. The set $\{\beta^\lambda \mid |\lambda| \leq n\}$ generates $\mathcal{B}_n(z_1, \ldots, z_n)$ as an algebra.

(v) If $E \subseteq M^\nu$ is any eigenspace of $\mathcal{B}_\nu(z_1, \ldots, z_n)$, then the corresponding eigenvalues of the operators β^λ are the Plücker coordinates of a point $V_E \in \mathcal{X}^\nu \subseteq \mathrm{Gr}(d, m)$ such that $\mathrm{Wr}(V_E) = g$. Every point of $\mathrm{Wr}^{-1}(g)$ corresponds to some eigenspace $E \subseteq M^\nu$ of $\mathcal{B}_\nu(z_1, \ldots, z_n)$.

(vi) The multiplicity of V_E as a point of $\mathrm{Wr}^{-1}(g)$ is equal to $\dim \widehat{E}$, where $\widehat{E} \subseteq M^\nu$ is the generalized eigenspace of $\mathcal{B}_\nu(z_1, \ldots, z_n)$ containing E.

We note that while the translation identity in part (ii) is linear, parts (i) and (iii) both involve quadratic expressions in $\mathcal{B}_n(z_1, \ldots, z_n)$, making them intractable to prove directly. In both of these cases we proceed by reducing the problem to — and then proving — an easier identity, using a diverse set of algebraic tools. For part (i), we use properties of $\mathcal{B}_n(z_1, \ldots, z_n)$ and combinatorial ideas which appeared in [34]. For part (iii), we employ the translation identity, properties of the exterior algebra, new combinatorial identities of symmetric functions, and the theory of τ-functions of the KP hierarchy. Once identities (i)–(iii) are established, parts (iv)–(vi) are relatively straightforward consequences. See [19, Sections 3–4] for the details.

There is a precise scheme-theoretic formulation of Theorem 3.2(v); see [19, Section 5.1]. In [19, Section 5.2], we also use Theorem 3.2 to give two explicit bases for any element $V \in \mathrm{Wr}^{-1}(g)$, in terms of our operators $\beta^\lambda(t)$ acting on the associated eigenspace E.

Example 3.3. We illustrate Theorem 3.2 in the case $n = 2$, for the Grassmannian $\mathrm{Gr}(2, 4)$. Writing $\mathfrak{S}_2 = \{\mathbf{1}_{\mathfrak{S}_2}, \sigma_{1,2}\}$, we have

$$\beta^0 = \mathbf{1}_{\mathfrak{S}_2} z_1 z_2\,, \qquad \beta^1 = \mathbf{1}_{\mathfrak{S}_2}(z_1 + z_2)\,, \qquad \beta^2 = \mathbf{1}_{\mathfrak{S}_2} + \sigma_{1,2}\,, \qquad \beta^{11} = \mathbf{1}_{\mathfrak{S}_2} - \sigma_{1,2}\,,$$

and $\beta^\lambda = 0$ for all other partitions λ. Note that the β^λ's satisfy the equation

$$-\beta^0\beta^{22} + \beta^1\beta^{21} - \beta^{11}\beta^2 = 0\,,$$

which is the first non-trivial Plücker relation.

There are two Specht modules for $\mathfrak{S}_2$, namely M^2 and M^{11}, which are both 1-dimensional. In M^2, both $\mathbf{1}_{\mathfrak{S}_2}$ and $\sigma_{1,2}$ act with eigenvalue 1, and so

$$\beta^0 \rightsquigarrow z_1 z_2\,, \qquad \beta^1 \rightsquigarrow z_1 + z_2\,, \qquad \beta^2 \rightsquigarrow 2\,, \qquad \beta^{1,1} \rightsquigarrow 0\,. \tag{3.4}$$

These are the Plücker coordinates of $V = \left\langle 1,\, z_1 z_2 u + \frac{z_1+z_2}{2} u^2 + \frac{1}{3} u^3 \right\rangle \in \mathcal{X}^2 \subseteq \mathrm{Gr}(2, 4)$.

That is, when we represent V as the row span of the 2×4 matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z_1 z_2 & z_1 + z_2 & 2 \end{pmatrix}$$

(where vectors correspond to polynomials as in (2.1)), the maximal minors are precisely the β^λ's, where we read off the column set of a minor from λ as in Figure 1.

On the other hand, in M^{11}, the element $\mathbf{1}_{\mathfrak{S}_2}$ acts with eigenvalue 1 and $\sigma_{1,2}$ acts with eigenvalue -1, giving the solution $V = \left\langle \frac{z_1+z_2}{2} + u,\ -z_1z_2 + u^2 \right\rangle \in \mathcal{X}^{1,1} \subseteq \mathrm{Gr}(2,4)$. We can check that both elements of $\mathrm{Gr}(2,4)$ have Wronskian $g(u) = (u+z_1)(u+z_2)$.

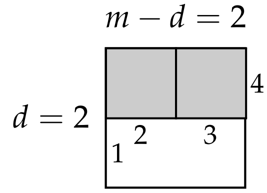

Figure 1: The partition $\lambda = (2)$ corresponds to the column set $\{1,4\}$, where $d = 2$ and $m = 4$. When we label the edges of the border of the diagram of λ by $1, \dots, m$ from southwest to northeast, the elements of I are the labels of the vertical edges.

Example 3.4. We illustrate parts (i) and (iii) of Theorem 3.2 in the case $n = 4$. Consider the 2-dimensional representation M^ν of $\mathfrak{S}_4$, $\nu = (2,2)$. The simple transpositions $\sigma_{1,2}$ and $\sigma_{3,4}$ both act as $\left(\begin{smallmatrix} 1 & 0 \\ 1 & -1 \end{smallmatrix}\right)$, and $\sigma_{2,3}$ acts as $\left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$. Let $\beta^\lambda_\nu \in \mathrm{End}(M^\nu)$ denote the operator β^λ acting on M^ν, which we regard as a 2×2 matrix. Then

$$\beta^0_\nu = z_1z_2z_3z_4 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \beta^1_\nu = (z_1z_2z_3 + z_1z_2z_4 + z_1z_3z_4 + z_2z_3z_4) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\beta^2_\nu = \begin{pmatrix} 2z_1z_2 + z_1z_4 + z_2z_3 + 2z_3z_4 & z_1z_3 - z_1z_4 - z_2z_3 + z_2z_4 \\ z_1z_2 - z_1z_4 - z_2z_3 + z_3z_4 & 2z_1z_3 + z_1z_4 + z_2z_3 + 2z_2z_4 \end{pmatrix},$$

$$\beta^{11}_\nu = \begin{pmatrix} 2z_1z_3 + z_1z_4 + z_2z_3 + 2z_2z_4 & -z_1z_3 + z_1z_4 + z_2z_3 - z_2z_4 \\ -z_1z_2 + z_1z_4 + z_2z_3 - z_3z_4 & 2z_1z_2 + z_1z_4 + z_2z_3 + 2z_3z_4 \end{pmatrix},$$

$$\beta^{21}_\nu = 3(z_1 + z_2 + z_3 + z_4) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \beta^{22}_\nu = 12 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and $\beta^\lambda_\nu = 0$ for all other partitions λ. We can see that the β^λ_ν's pairwise commute and satisfy the Plücker relation $-\beta^0_\nu\beta^{22}_\nu + \beta^1_\nu\beta^{21}_\nu - \beta^{11}_\nu\beta^2_\nu = 0$.

4 Conjectures in real algebraic geometry

We continue to work with the Schubert cell $\mathcal{X}^\nu \subseteq \mathrm{Gr}(d,m)$, where ν is a partition of n. The Schubert variety $\overline{\mathcal{X}^\nu} \subseteq \mathrm{Gr}(d,m)$ is the closure of $\mathcal{X}^\nu$. We write $\square$ for the rectangular partition $(m-d)^d = (m-d, \dots, m-d)$. In this case, $\overline{\mathcal{X}^\square} = \mathrm{Gr}(d,m)$.

We will be mainly concerned with the following Schubert problem. Given $W_1, \dots, W_n$ in $\mathrm{Gr}(m-d,m)$, determine all d-planes V such that

$$V \in \overline{\mathcal{X}^\nu} \qquad \text{and} \qquad V \cap W_i \neq \{0\} \ \text{ for all } i = 1, \dots, n\,. \tag{4.1}$$

When $W_1, \ldots, W_n$ are sufficiently general, the number of distinct solutions V to the Schubert problem (4.1) is exactly $\mathsf{f}^\nu = \dim M^\nu$.

We will be concerned with solving (4.1) over the real numbers when $W_1, \ldots, W_n$ are real, and especially with instances for which *all* the solutions are real. The interest in algebraic problems with only real solutions dates back at least to Fulton [13], who wrote, "The question of how many solutions of real equations can be real is still very much open, particularly for enumerative problems." Note that the property of having only real solutions is extremely rare; for example, for a 'random' Schubert problem on $\mathrm{Gr}(d,m)$ defined over $\mathbb{R}$, the number of real solutions is roughly the square root of the number of complex solutions [4]. We refer to [38] for a detailed survey of real enumerative geometry.

4.1 The Shapiro–Shapiro conjecture

The ***moment curve*** $\gamma : \mathbb{C} \to \mathbb{C}_{m-1}[u]$ is the parametric curve

$$\gamma(t) := \frac{(u+t)^{m-1}}{(m-1)!}\,. \tag{4.2}$$

The closure of the image of γ in $\mathbb{P}^{m-1}$ is a rational normal curve. A d-plane $V \in \mathrm{Gr}(d,m)$ ***osculates*** γ at $w \in \mathbb{C}$ if $(\gamma(w), \gamma'(w), \gamma''(w), \ldots, \gamma^{(d-1)}(w))$ is a basis for V. The ***Shapiro–Shapiro conjecture*** can be stated as follows:

Theorem 4.1 (Mukhin, Tarasov, and Varchenko [31])**.** *Let $z_1, \ldots, z_n$ be distinct real numbers. For $i = 1, \ldots, n$, let $W_i \in \mathrm{Gr}(m-d, m)$ be the osculating $(m-d)$-plane to γ at z_i. Then there are exactly f^ν distinct solutions to the Schubert problem (4.1), and all solutions are real.*

Theorem 4.1 was conjectured by Boris and Michael Shapiro in 1993, and extensively tested and popularized by Sottile [37]. It was proved in the cases $d \le 2$ and $m - d \le 2$ by Eremenko and Gabrielov [8], and in general by Mukhin, Tarasov, and Varchenko [31]. Their proof was later restructured and simplified in [34]. A very different proof, based on geometric and topological arguments, is given in [23].

Using Theorem 3.2, we obtain a number of generalizations of Theorem 4.1:

4.2 The divisor form of the secant conjecture

Let $I \subseteq \mathbb{R}$ be an interval. An $(m-d)$-plane $W \in \mathrm{Gr}(m-d,m)$ is a ***secant*** to γ along I if there exist distinct points $w_1, \ldots, w_{m-d} \in I$ such that $(\gamma(w_1), \ldots, \gamma(w_{m-d}))$ is a basis for W. More generally, W is a ***generalized secant*** to γ along I if there exist distinct points $w_1, \ldots, w_k \in I$ and positive integers $m_1, \ldots, m_k$, such that $m_1 + \cdots + m_k = m - d$ and $\big(\gamma(w_1), \gamma'(w_1), \ldots, \gamma^{(m_1-1)}(w_1),\ \ldots\ , \gamma(w_k), \gamma'(w_k), \ldots, \gamma^{(m_k-1)}(w_k)\big)$ is a basis for W.

Around 2003, Frank Sottile formulated the ***secant conjecture***, which asserts in particular that Theorem 4.1 remains true when $W_1, \ldots, W_n$ are generalized secants to γ along disjoint intervals of $\mathbb{R}$. This statement is what we call the ***divisor form*** of the secant conjecture, since it arises from intersecting Schubert varieties of codimension one; the general form of the secant conjecture involves intersecting Schubert varieties of arbitrary codimension. This case of the secant conjecture is a generalization of the Shapiro–Shapiro conjecture, since an osculating plane to γ is a special case of a generalized secant.

The secant conjecture appeared in [36] (cf. [38, Section 13.4]), and it was extensively tested experimentally in a project led by Sottile [15], as described in [17]. It has also been proved in special cases: Eremenko, Gabrielov, Shapiro, and Vainshtein [9] established the case $m - d \leq 2$; and Mukhin, Tarasov, and Varchenko [27] (cf. [15]) verified the case of the divisor form when there exists $r > 0$ such that every W_i is a (non-generalized) secant where $w_1, \ldots, w_{m-d} \in I_i$ are an arithmetic progression of step size r.

We show that the divisor form of the secant conjecture is true in general:

Theorem 4.2 (Secant conjecture, divisor form)**.** *Let $I_1, \ldots, I_n \subseteq \mathbb{R}$ be pairwise disjoint real intervals. For $i = 1, \ldots, n$, let $W_i \in \mathrm{Gr}(m-d, m)$ be a generalized secant to γ along I_i. Then there are exactly f^ν distinct solutions to the Schubert problem (4.1), and all solutions are real.*

This verifies the secant conjecture in the first non-trivial case of interest for a Schubert problem on an arbitrary Grassmannian. We do not yet know how to address the general form of the secant conjecture with our methods.

4.3 The disconjugacy conjecture

Suppose that V is a d-dimensional vector space of real analytic functions, defined on an interval $I \subseteq \mathbb{R}$. Disconjugacy is concerned with the question of how many zeros a function in V can have. By linear algebra, there always exists a nonzero function $f \in V$ such that f has at least $d - 1$ zeros on I. We say that V is ***disconjugate*** on I if every nonzero function in V has at most $d - 1$ zeros on I (counted with multiplicities). Disconjugacy has long been studied because it is related to explicit solutions for linear differential equations; see [5], as well as [18, Section 4.1] and the references therein.

It is not always straightforward to decide if V is disconjugate on I. However, a necessary condition is that $\mathrm{Wr}(V)$ has no zeros on I. This is because $\mathrm{Wr}(V)$ has a zero at w if and only if there exists a nonzero $f \in V$ such that f has a zero at w of multiplicity at least d. In general, the converse is false; for example, $V = \langle \cos u, \sin u \rangle$ is not disconjugate on $I = \mathbb{R}$, and $\mathrm{Wr}(V) = 1$. Eremenko [10, 11] conjectured that the converse statement is actually correct under very special circumstances. This is known as the ***disconjugacy conjecture***, which we state now as a theorem:

Theorem 4.3 (Disconjugacy conjecture)**.** *Let $V \subseteq \mathbb{R}[u]$ be a finite-dimensional vector space of polynomials such that $\mathrm{Wr}(V)$ has only real zeros. Then V is disconjugate on every interval which avoids the zeros of $\mathrm{Wr}(V)$.*

4.4 Positivity conjectures

A d-plane $V \in \mathrm{Gr}(d, m)$ is called ***totally nonnegative*** if all of its Plücker coordinates are real and nonnegative (up to rescaling). Similarly, V is called ***totally positive in $\mathcal{X}^\nu$*** if $V \in \mathcal{X}^\nu$ and all of its Plücker coordinates which are not trivially zero on $\mathcal{X}^\nu$ are positive:

$$\Delta^\lambda > 0 \;\text{ for all } \lambda \subseteq \nu \qquad \text{and} \qquad \Delta^\lambda = 0 \;\text{ for all } \lambda \not\subseteq \nu\,. \tag{4.3}$$

For example, each element $V \in \mathrm{Gr}(2, 4)$ from Example 3.3 is totally nonnegative if and only if $z_1, z_2 \geq 0$, and is totally positive in its Schubert cell if and only if $z_1, z_2 > 0$.

The totally nonnegative part of $\mathrm{Gr}(d, m)$ is a totally nonnegative partial flag variety in the sense of Lusztig [24, 25] (see [3, Section 1] for further discussion), and was studied combinatorially by Postnikov [33]. Total positivity in Schubert cells was considered by Berenstein and Zelevinsky [2]. These and similar totally positive spaces have been extensively studied in the past few decades, with connections to representation theory [24], combinatorics [33], cluster algebras [12], soliton solutions to the KP equation [22], scattering amplitudes [1], Schubert calculus [21], topology [14], and many other topics.

Mukhin–Tarasov and Karp conjectured that the reality statements from Sections 4.1 and 4.2 have totally positive analogues. We verify this in slightly greater generality:

Theorem 4.4 (Positive Shapiro–Shapiro conjecture)**.** *Let $z_1, \ldots, z_n$ and $W_1, \ldots, W_n$ be as in Theorem 4.1.*

(i) If $z_1, \ldots, z_n \in [0, \infty)$, then all solutions to the Schubert problem (4.1) are real and totally nonnegative.

(ii) If $z_1, \ldots, z_n \in (0, \infty)$, then all solutions to the Schubert problem (4.1) are real and totally positive in $\mathcal{X}^\nu$.

Theorem 4.5 (Positive secant conjecture, divisor form)**.** *Let $I_1, \ldots, I_n$ and $W_1, \ldots, W_n$ be as in Theorem 4.2.*

(i) If $I_1, \ldots, I_n \subseteq [0, \infty)$, then there are exactly f^ν distinct solutions to the Schubert problem (4.1), and all solutions are real and totally nonnegative.

(ii) If $I_1, \ldots, I_n \subseteq (0, \infty)$, then there are exactly f^ν distinct solutions to the Schubert problem (4.1), and all solutions are real and totally positive in $\mathcal{X}^\nu$.

In the special case $\nu = \square$, Theorem 4.4(i) was conjectured by Evgeny Mukhin and Vitaly Tarasov in 2017, and Theorems 4.4 and 4.5 were conjectured independently in [18].

4.5 Relationships between conjectures

We now explain how the conjectures stated in this section are related to each other, and why they follow from our main result Theorem 3.2. We have already noted that the divisor form of the secant conjecture (Theorem 4.2) implies the Shapiro–Shapiro conjecture

(Theorem 4.1). Eremenko showed that the disconjugacy conjecture (Theorem 4.3) implies the divisor form of the secant conjecture; in fact, his motivation was to generalize the argument used to prove the $m - d \leq 2$ case of the secant conjecture [9, Section 3]. Moreover, it was shown in [18] using topological arguments that the four statements in Theorems 4.4 and 4.5 in the case $\nu = \square$ are all pairwise equivalent, and that they are moreover equivalent to the disconjugacy conjecture. We can similarly show that Theorem 4.4 implies Theorem 4.5. Therefore to prove all of these statements, it suffices to establish Theorem 4.4. This is a direct consequence of Theorem 3.2; we briefly sketch the argument (see [19, Section 1] for the details).

If $W \in \mathrm{Gr}(m-d, m)$ osculates γ at $w \in \mathbb{C}$, then $V \cap W \neq \{0\}$ if and only if $-w$ is a zero of $\mathrm{Wr}(V)$. Hence in the setting of Theorem 4.4, the Schubert problem (4.1) is equivalent to $V \in \mathcal{X}^\nu$ and $\mathrm{Wr}(V) = g$, where $g(u) = (u + z_1) \cdots (u + z_n)$. By Theorem 3.2(v), we can write any such solution V as V_E for some eigenspace $E \subseteq M^\nu$ of $\mathcal{B}_\nu(z_1, \ldots, z_n)$. This means that the Plücker coordinates $[\Delta^\lambda : \lambda \subseteq \square]$ of V are the eigenvalues of the operators β^λ on E. If $z_1, \ldots, z_n \in [0, \infty)$, then one can show that each β^λ is positive semidefinite. Therefore the eigenvalues of β^λ are real and nonnegative, so V is totally nonnegative. This proves part (i) of Theorem 4.4. Similarly, if $z_1, \ldots, z_n \in (0, \infty)$, then each β^λ with $\lambda \subseteq \nu$ is positive definite, and hence has positive eigenvalues. This implies that V is totally positive in $\mathcal{X}^\nu$, proving part (ii).

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