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Real matroid Schubert varieties, zonotopes, and virtual Weyl groups

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Abstract. We show that the matroid Schubert variety of a real hyperplane arrangement is homeomorphic to the zonotope of the arrangement with parallel faces identified. Using this explicit model, we compute the homology and fundamental group of the matroid Schubert variety in terms of combinatorial data of the underlying oriented matroid. When the hyperplane arrangement is a Coxeter arrangement, we show that the equivariant fundamental group is a virtual analogue of the associated Weyl group.

Keywords: matroid Schubert varieties, zonotopes, virtual Weyl groups

1 Introduction

Several recent breakthroughs in matroid theory have come from understanding the topology of certain algebraic varieties associated to hyperplane arrangements. One such variety is the *matroid Schubert variety*, which compactifies the ambient vector space of a central essential hyperplane arrangement. Originally studied by Ardila–Boocher [1] and Li [8], it has most notably found applications to the Dowling–Wilson top-heavy conjecture for representable matroids [5].

We study the topology of matroid Schubert varieties $Y_{\mathcal{A}}$ associated with real hyperplane arrangements $\mathcal{A}$. The main result is an explicit homeomorphism from $Y_{\mathcal{A}}$ to a natural quotient of the zonotope associated with $\mathcal{A}$. As a consequence, we obtain presentations for the homology and fundamental group of $Y_{\mathcal{A}}$ that depend only on the oriented matroid data of $\mathcal{A}$. When $\mathcal{A}$ is a Coxeter arrangement, we also show that the equivariant fundamental groups are of independent interest. We call them virtual Weyl groups, since they are quotients of virtual Artin groups [3] (which themselves generalise the virtual braid group in type A).

Full details of the results in this extended abstract will be presented as part of the forthcoming paper [7].

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2 Setup

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let $\mathcal{A} = (\alpha_e)_{e\in E} \in (V^*)^E$ be a representation of an $\mathbb{F}$-linear matroid M with (finite) ground set E. The *rank* of $S \subseteq E$ is $\operatorname{rk} S = \dim \operatorname{span}\{\alpha_e \colon e \in S\}$, and a *flat* of M is a subset $F \subseteq E$ that is not strictly contained in another subset of the same rank. Flats of M are partially ordered by inclusion, and this poset is in fact a geometric lattice $\mathcal{L}(M)$ called the *lattice of flats* of M.

Remark 1. The α_e should be thought of as defining hyperplanes $H_e = \ker \alpha_e \subseteq V$. When M is simple, the α_e are all nonzero and no two are parallel. In this case, M is the matroid associated to the central hyperplane arrangement determined by the H_e.

For every $F \in \mathcal{L}(M)$, define the subspace $V_F = \bigcap_{e\in F} \ker \alpha_e \subseteq V$. The *localisation* M^F is the matroid on ground set F with flats $\{G \subseteq F \colon G \in \mathcal{L}(M)\}$. It has a representation $\mathcal{A}^F = (\alpha_e)_{e\in F} \in ((V/V_F)^*)^F$. The *contraction* M_F is the matroid on ground set $E \setminus F$ with flats $\{G \setminus F \colon F \subseteq G \in \mathcal{L}(M)\}$. It has a representation $\mathcal{A}_F = (\alpha_e|_{V_F})_{e\in E\setminus F} \in (V_F^*)^{E\setminus F}$.

Without loss of generality, assume that the α_e span V^*. (If M is simple, this would correspond to the hyperplane arrangement defined in Remark 1 being essential.) In this case, the choice of $\mathcal{A}$ defines an embedding $V \to \mathbb{F}^E$ by $v \mapsto (\alpha_e(v))_{e\in E}$. Considering $\mathbb{F}^E$ as the subset of $(\mathbb{P}^1)^E = (\mathbb{F} \cup \{\infty\})^E$ with all coordinates finite, the closure of V in $(\mathbb{P}^1)^E$ (in the Zariski topology) is the *matroid Schubert variety* $Y_{\mathcal{A}}$ of $\mathcal{A}$. A key property of matroid Schubert varieties is the existence of an affine paving.

Proposition 2 ([9, Lemmas 7.5 and 7.6])**.** *The matroid Schubert variety $Y_{\mathcal{A}}$ has a stratification $Y_{\mathcal{A}} = \bigsqcup_{F\in\mathcal{L}(M)} Y_{\mathcal{A}}^F$, where*

$$Y_{\mathcal{A}}^F = \{(y_e)_{e\in E} \in Y_{\mathcal{A}} \colon y_e = \infty \text{ if and only if } e \notin F\} \cong V/V_F \cong \mathbb{F}^{\operatorname{rk} F}.$$

Further, $\overline{Y_{\mathcal{A}}^G} = \bigsqcup_{F\subseteq G} Y_{\mathcal{A}}^F \cong Y_{\mathcal{A}^G}$ for every $G \in \mathcal{L}(M)$.

Henceforth we fix $\mathbb{F} = \mathbb{R}$. In this case much of the combinatorics of $\mathcal{A}$ (and hence M) can be encoded in the geometry of a convex polytope.

Definition 3. The *zonotope* associated to $\mathcal{A}$ is the Minkowski sum

$$\mathcal{Z}_{\mathcal{A}} = \sum_{e\in E} [-1, 1]\alpha_e = \left\{ \sum_{e\in E} c_e\alpha_e \colon -1 \leqslant c_e \leqslant 1 \text{ for all } e \in E \right\} \subset V^*.$$

Equivalently, it is the image of the cube $[-1, 1]^E$ under the projection $(c_e)_{e\in E} \mapsto \sum_{e\in E} c_e\alpha_e$.

The face structure of $\mathcal{Z}_{\mathcal{A}}$ can be understood explicitly using the oriented matroid structure of $\mathcal{A}$ in terms of *covectors*. Define a map $V \to \{+, -, 0\}^E$ by sending $v \in V$ to $C = (C_e)_{e\in E}$, where $C_e = +$ if $\alpha_e(v) > 0$, $-$ if $\alpha_e(v) < 0$, and 0 if $\alpha_e(v) = 0$. The image of this map is the set of covectors of $\mathcal{A}$. Each covector C gives a decomposition

$E = C_+ \sqcup C_- \sqcup C_0$, where $C_+ = \{e \in E\colon C_e = +\}$, $C_- = \{e \in E\colon C_e = -\}$, and $C_0 = \{e \in E\colon C_e = 0\}$. Observe that the flats of $\mathcal{A}$ are exactly the zero sets C_0 of covectors of $\mathcal{A}$. Further, the set of covectors forms a lattice (the *face lattice*) with the product order induced from the partial order $0 < +, -$ on each coordinate.

We can associate a face of $\mathcal{Z}_{\mathcal{A}}$ to each covector C as follows:

$$C \mapsto \sum_{e\in C_+} \alpha_e - \sum_{e\in C_-} \alpha_e + \sum_{e\in C_0} [-1,1]\alpha_e. \tag{2.1}$$

It follows immediately that the face associated to a covector C is (isometric to) $\mathcal{Z}_{\mathcal{A}^F}$, where F is the zero set of C.

Proposition 4 ([4, Proposition 2.2.2])**.** *The map* (2.1) *is an order-reversing bijection between the face lattice and faces of* $\mathcal{Z}_{\mathcal{A}}$ *(under inclusion).*

Example 5. In Figure 1 we visualise the rank 2 braid arrangement. Concretely, in the above notation we have $V^* = \mathbb{R}^3/\mathbb{R}(1,1,1)$, $E = \{1,2,12\}$, and $\alpha_1 = (1,-1,0)$, $\alpha_2 = (0,-1,1)$, and $\alpha_{12} = (1,0,-1) = \alpha_1 + \alpha_2$. By fixing an isomorphism $V \cong V^*$ using the dot product, we draw both the hyperplanes in V (in black) and the zonotope in V^* (in blue) in the same plane. Finally, we label the regions of V by their covectors.

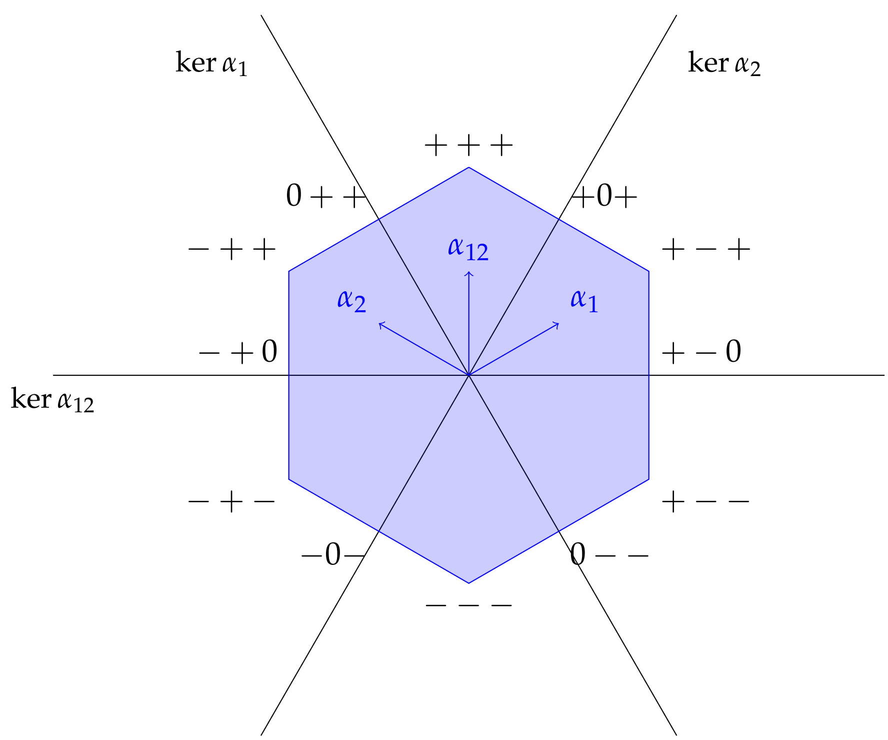

Figure 1: The rank 2 braid arrangement

3 The combinatorial model

Observe from the explicit equations (2.1) that faces of $\mathcal{Z}_{\mathcal{A}}$ whose covectors correspond to the same flat are translates of each other. Let $\widetilde{\mathcal{Z}}_{\mathcal{A}}$ be the quotient of $\mathcal{Z}_{\mathcal{A}}$ obtained by identifying two points if one is moved to the other by such a translation. Intuitively, $\widetilde{\mathcal{Z}}_{\mathcal{A}}$ is the result of identifying parallel faces in $\mathcal{Z}_{\mathcal{A}}$.

The stratification of $\mathcal{Z}_{\mathcal{A}}$ by (relatively open) faces descends to a stratification of $\widetilde{\mathcal{Z}}_{\mathcal{A}}$ indexed by $\mathcal{L}(M)$. Denote by $\mathcal{Z}^F_{\mathcal{A}}$ the stratum corresponding to $F \in \mathcal{L}(M)$.

Theorem 6. *The real matroid Schubert variety $Y_{\mathcal{A}}$ (with the analytic topology) is homeomorphic to $\widetilde{\mathcal{Z}}_{\mathcal{A}}$.*

Proof sketch. We claim that every homeomorphism $f\colon \mathbb{R} \to (-1,1)$ determines a homeomorphism $\phi\colon Y_{\mathcal{A}} \to \widetilde{\mathcal{Z}}_{\mathcal{A}}$. First observe that there is a map $V \to \mathbb{R}^E \to (-1,1)^E \to \mathcal{Z}^E_{\mathcal{A}}$ defined by $v \mapsto (\alpha_e(v)) \mapsto (f(\alpha_e(v))) \mapsto \sum_{e\in E} f(\alpha_e(v))\alpha_e$. Unfortunately this composition does not obviously extend to the desired map, as the projection $[-1,1]^E \to \mathcal{Z}_{\mathcal{A}}$ does not descend to a well-defined map after identifying parallel faces. Nevertheless, we claim that a suitable interpretation of the formula $(y_e)_{e\in E} \mapsto \sum_{e\in E} f(y_e)\alpha_e$ extends the map $V \to \mathcal{Z}^E_{\mathcal{A}}$ to the required continuous ϕ. In fact, if $y = (y_e)_{e\in E} \in Y^F_{\mathcal{A}}$, there are several possible values for $(f(y_e))_{e\in E} \in [-1,1]^E$ allowed by continuity, but they correspond to different covectors with the same zero set F and hence $\sum_{e\in E} f(y_e)\alpha_e$ is well-defined in the quotient $\widetilde{\mathcal{Z}}_{\mathcal{A}}$.

Since $Y_{\mathcal{A}}$ is compact and $\widetilde{\mathcal{Z}}_{\mathcal{A}}$ is Hausdorff, the continuous map ϕ is a homeomorphism if it is a bijection. By construction it maps $Y^F_{\mathcal{A}}$ to $\mathcal{Z}^F_{\mathcal{A}}$, so it is enough to verify bijectivity on each stratum separately. Further, it is enough to check the open stratum $Y^E_{\mathcal{A}} \cong V$, since each $Y^F_{\mathcal{A}}$ is the open stratum in $Y_{\mathcal{A}^F}$.

For injectivity, let $v, w \in V$ and consider $(d_e) = (f(\alpha_e(v)) - f(\alpha_e(w))) \in (-1,1)^E$. Observe that f must be (strictly) increasing or decreasing; without loss of generality, assume that f is increasing. The sign of d_e is then the same as the sign of $\alpha_e(v-w)$. So $\sum_{e\in E} d_e\alpha_e(v-w)$ is non-negative, and it is zero if and only if $d_e = \alpha_e(v-w) = 0$ for every $e \in E$. But if v and w map to the same point in $\mathcal{Z}^E_{\mathcal{A}}$, then $\sum_{e\in E} d_e\alpha_e = 0$. It follows that $\alpha_e(v-w) = 0$ for every $e \in E$, and thus $v = w$ as the α_e span V^*.

To show surjectivity, consider the quotients of $Y_{\mathcal{A}}$ and $\widetilde{\mathcal{Z}}_{\mathcal{A}}$ identifying all boundary strata to a point ∞. Both quotients are homeomorphic to spheres, with induced cell decompositions $V \sqcup \{\infty\}$ and $\mathcal{Z}^E_{\mathcal{A}} \sqcup \{\infty\}$ respectively. Since ϕ sends strata to strata, it descends to a continuous cellular map $\overline{\phi}$ between the quotients. If ϕ were not surjective, then the image of $\overline{\phi}$ would be contained in the sphere minus one point and hence be homeomorphic to (a subset of) V. By the Borsuk–Ulam theorem $\overline{\phi}$ would not be injective. In particular, cellularity of $\overline{\phi}$ implies that $\overline{\phi}|_V = \phi|_V$ would not be injective, contradicting what was shown above. □

Remark 7. In the case of Coxeter arrangements, Theorem 6 was proved in [6, Appendix A] using somewhat involved root system combinatorics.

Example 8. If $\dim V = \dim V^* = 2$, then $\mathcal{Z}_\mathcal{A}$ is a $2n$-gon (where $n \geqslant 2$ is the number of rank 1 flats). Identifying parallel edges of $\mathcal{Z}_\mathcal{A}$ gives a connected compact orientable surface without boundary. The resulting cell structure on the surface has one 0-cell if n is even and two 0-cells if n is odd. By an Euler characteristic computation and the classification of surfaces, it follows that $Y_\mathcal{A}$ is homeomorphic to Σ_g (if $n = 2g$ is even) or Σ_g with two (distinct) points identified (if $n = 2g+1$ is odd). For example, the matroid Schubert variety corresponding to the rank 2 braid arrangement of Example 5 is homeomorphic to the torus with two points identified.

4 Computations of invariants

The combinatorial model of Theorem 6 allows for easy computation of some topological invariants of $Y_\mathcal{A}$.

Homology

There is a cellular chain complex for $\widetilde{\mathcal{Z}}_\mathcal{A}$ with cells given by the strata $\mathcal{Z}_\mathcal{A}^F$. The boundary map in this complex is necessarily zero, since in the computation for any cell $\mathcal{Z}_\mathcal{A}^F$ opposite faces of $\mathcal{Z}_\mathcal{A}^F$ both occur and with opposite sign. Their contributions cancel, as the opposite faces are identified in $\widetilde{\mathcal{Z}}_\mathcal{A}$. Hence the homology of $Y_\mathcal{A}$ is easy to compute.

Proposition 9. *$H_\bullet(Y_\mathcal{A}, \mathbb{Z}) \cong H_\bullet(\widetilde{\mathcal{Z}}_\mathcal{A}, \mathbb{Z}) \cong \bigoplus_{F \in \mathcal{L}(M)} \mathbb{Z} x_F$, where $\deg x_F = \operatorname{rk} F$.*

Remark 10. It is interesting to note that the cellular boundary maps for the analogous cell structures on complex matroid Schubert varieties are also zero, but for the different reason that the cells are concentrated in even (real) dimension.

Fundamental group

The fundamental group of a cell complex depends only on the 2-skeleton. For $\widetilde{\mathcal{Z}}_\mathcal{A}$, the cells of dimension k correspond to the flats of rank k. We take the 0-cell corresponding to the unique rank 0 flat to be the basepoint. There is then a presentation of $\pi_1(Y_\mathcal{A})$ with generators x_F indexed by rank 1 flats and relations indexed by rank 2 flats.

To compute the relations, it is helpful to work with an acyclic reorientation of $\mathcal{A}$ (this does not change the zonotope $\mathcal{Z}_\mathcal{A}$). Let G be a rank 2 flat. The rank 1 flats contained in G can be ordered as follows. If G contains n rank 1 flats, the zonotope $\mathcal{Z}_{\mathcal{A}^G}$ is a $2n$-gon. One vertex of this $2n$-gon has a covector without any $+$ coordinates (this follows from the choice of acyclic orientation). A length n sequence of edges from this vertex to its opposite vertex defines a total order $F_1 < \ldots < F_n$ on the rank 1 flats contained in G. There are two such sequences, giving opposite orders. The relation corresponding to G then says that the two paths $x_{F_1} \cdots x_{F_n}$ and $x_{F_n} \cdots x_{F_1}$ determined by these sequences are equal.

Theorem 11. *The fundamental group $\pi_1(Y_{\mathcal{A}})$ has a presentation with generators $\{x_F : F \in \mathcal{L}(M), \operatorname{rk} F = 1\}$ and relations $x_{F_1} \cdots x_{F_n} x_{F_1}^{-1} \cdots x_{F_n}^{-1}$ for every rank 2 flat G, where $F_1, \ldots, F_n$ are the rank 1 flats contained in G ordered as above.*

Example 12. Continuing Example 5, the fundamental group of the matroid Schubert variety in this case has a presentation $\langle x_1, x_2, x_{12} \mid x_1 x_{12} x_2 x_1^{-1} x_{12}^{-1} x_2^{-1} \rangle$.

Remark 13. When $\mathcal{Z}_{\mathcal{A}}$ is the (type A) permutohedron of dimension n, the homology and fundamental group were computed in this way in [2, Proposition 8.3] and [2, Theorem 8.1], generalising the $n = 2$ computation in Example 12. In particular, the fundamental group was shown to be isomorphic to the triangular group $\mathbf{Tr}_{n+1}$, also known as the pure flat braid group.

Remark 14. Every rank 2 oriented matroid is representable over $\mathbb{R}$ [4, Corollary 8.3.3], so the above presentation of $\pi_1(Y_{\mathcal{A}})$ can be used to define a group for any oriented matroid.

Homotopy groups

In certain cases, the higher homotopy groups $\pi_n(Y_{\mathcal{A}})$ are also known.

Theorem 15 ([2, Theorem 8.1])**.** *If $\mathcal{Z}_{\mathcal{A}}$ is the (type A) permutohedron, then $\widetilde{\mathcal{Z}_{\mathcal{A}}}$ is a classifying space and hence $\pi_n(Y_{\mathcal{A}}) = \pi_n(\widetilde{\mathcal{Z}_{\mathcal{A}}})$ is trivial for all $n > 1$.*

The proof uses the theory of non-positively curved polyhedral complexes. We expect that the same result holds more generally, at least for nice enough choices of $\mathcal{A}$.

5 Coxeter arrangements

We first fix some notation. Let Φ be a root system with simple roots Π and positive roots Φ^+. Each (positive) root defines a hyperplane in the dual space, and the corresponding hyperplane arrangement is called a *Coxeter arrangement*. By abuse of notation, we also use Φ^+ for the matroid representation with coordinates given by the positive roots.

Further, let $(m_{\alpha,\beta})_{\alpha,\beta\in\Pi}$ be the Coxeter matrix associated to the root system Φ, and let $\Sigma = \{\sigma_\alpha : \alpha \in \Pi\}$ and $S = \{s_\alpha : \alpha \in \Pi\}$ be abstract sets indexed by Π. The Artin group A has a presentation with generators Σ and relations $\operatorname{Prod}(\sigma_\alpha, \sigma_\beta, m_{\alpha,\beta}) = \operatorname{Prod}(\sigma_\beta, \sigma_\alpha, m_{\alpha,\beta})$ for all $\alpha, \beta \in \Pi$ with $\alpha \neq \beta$ and $m_{\alpha,\beta} \neq \infty$. Here $\operatorname{Prod}(a, b, m)$ is the word $aba\ldots$ of length m. Similarly, the Weyl group W has a presentation with generators S and relations $s_\alpha^2 = 1$

for all $\alpha \in \Pi$ and $\mathrm{Prod}(s_\alpha, s_\beta, m_{\alpha,\beta}) = \mathrm{Prod}(s_\beta, s_\alpha, m_{\alpha,\beta})$ for all $\alpha, \beta \in \Pi$ with $\alpha \neq \beta$ and $m_{\alpha,\beta} \neq \infty$.

Bellingeri–Paris–Thiel [3] have recently defined the *virtual Artin group* VA as the free product of W and A modulo some "mixed relations" coming from the action of W on Φ. Their definition unifies the Coxeter-theoretic and knot-theoretic generalisations of the classical braid group to Artin groups and virtual braid groups respectively. We are interested in a quotient of their group that can be considered as a virtual analogue of the corresponding Weyl group.

Definition 16 ([3])**.** The *virtual Artin group* VA is the free product of W and A modulo relations $\mathrm{Prod}(s_\alpha, s_\beta, m_{\alpha,\beta} - 1)\sigma_\alpha = \sigma_\gamma \mathrm{Prod}(s_\alpha, s_\beta, m_{\alpha,\beta} - 1)$ for all $\alpha, \beta \in \Pi$ with $\alpha \neq \beta$ and $m_{\alpha,\beta} \neq \infty$. In these relations, the positive root γ is defined as α if $m_{\alpha,\beta}$ is even and β if $m_{\alpha,\beta}$ is odd.

Definition 17. The *virtual Weyl group* VW is the quotient of VA by the relations $\sigma_\alpha^2 = 1$ for all $\alpha \in \Pi$.

There is a surjective group homomorphism VW $\to W$ defined on generators by $\sigma_\alpha, s_\alpha \mapsto s_\alpha$ for all $\alpha \in \Pi$, and we call its kernel the *pure virtual Weyl group* PVW. The map $\pi_P \colon \mathrm{VA} \to W$ [3, Section 2] is the composition of this map with the quotient VA $\to$ VW, and its kernel is the *pure virtual Artin group* PVA.

Proposition 18. *The fundamental group $\pi_1(Y_{\Phi^+})$ is isomorphic to* PVW.

Proof sketch. There is a presentation of the pure virtual Artin group with generators $\{\zeta_\beta \colon \beta \in \Phi\}$ and certain relations [3, Theorem 2.6]. As PVW is the image of PVA under the quotient map VA $\to$ VW, we can obtain a presentation of PVW by imposing (consequences of) the relations $\sigma_\alpha^2 = 1$ to the presentation of PVA.

In fact, the generator ζ_β is the element $ws_\alpha\sigma_\alpha w^{-1} \in \mathrm{VA}$ for some $w \in W$ and $\alpha \in \Pi$ such that $w(\alpha) = \beta$ [3, p. 197]. This definition is independent of the choices of w and α [3, Lemma 2.2]. Then $-\beta = ws_\alpha(\alpha)$, so $\zeta_{-\beta} = (ws_\alpha)s_\alpha\sigma_\alpha(s_\alpha w^{-1}) = w\sigma_\alpha s_\alpha w^{-1}$ and $\zeta_\beta\zeta_{-\beta} = ws_\alpha\sigma_\alpha^2 s_\alpha w^{-1}$. But this is the identity if and only if $\sigma_\alpha^2 = 1$, so PVW has a presentation with generators $\{\zeta_\beta \colon \beta \in \Phi^+\}$ and the same relations as PVA.

A root subsystem $\Phi' \subset \Phi$ is *parabolic* if $\Phi' \cap \Phi^+$ corresponds to a flat of the Coxeter arrangement. The relations in the above presentation of PVA correspond to choices of simple roots for rank 2 parabolic root subsystems of Φ. One can compute the relations and show that, after accounting for the extra relations $\zeta_\beta\zeta_{-\beta} = 1$, the relations for pairs of simple roots depend only on the parabolic root subsystem, and that they are the same as the relations in Theorem 11 coming from the rank 2 flats. Hence the presentations of PVW and $\pi_1(Y_{\Phi^+})$ define the same group. □

The Weyl group W acts on V, and hence on Y_{Φ^+} and $\pi_1(Y_{\Phi^+})$. We can therefore consider the W-equivariant fundamental group $\pi_1^W(Y_{\Phi^+})$ [6, Definition 11.1]. As the unique

0-cell is fixed by the action of W, taking it as the basepoint gives a semidirect product decomposition $\pi_1^W(Y_{\Phi^+}) \cong W \ltimes \pi_1(Y_{\Phi^+})$. The homomorphism $W \to \mathrm{Aut}(\pi_1(Y_{\Phi^+}))$ defining the semidirect product is exactly the W-action indicated above. Explicitly, an element $w \in W$ acts on generators of $\pi_1(Y_{\Phi^+})$ by $\zeta_\beta \mapsto \zeta_{w(\beta)}$.

Theorem 19. *The W-equivariant fundamental group $\pi_1^W(Y_{\Phi^+})$ is isomorphic to the virtual Weyl group* VW*.*

Proof sketch. The virtual Weyl group also has a semidirect product decomposition $W \ltimes$ PVW descending from the semidirect product decomposition of the virtual Artin group [3, Proposition 2.1]. As $\pi_1(Y_{\Phi^+}) \cong$ PVW (Proposition 18) and the action of W on PVW [3, p. 203] agrees with the action of W on $\pi_1(Y_{\Phi^+})$, the semidirect products $\pi_1^W(Y_{\Phi^+})$ and VW must be isomorphic. □

Remark 20. In type A, the virtual Weyl group is known as the *flat (virtual) braid group*. It was called the virtual symmetric group in [6], where it was realised as the equivariant fundamental group $\pi_1^{S_n}(Y_{\Phi^+})$ of the corresponding matroid Schubert variety [6, Lemma 11.6].

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