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Asymptotic log-concavity of dominant lower Bruhat intervals via Brunn–Minkowski inequality

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Abstract. Björner and Ekedahl [Ann. of Math. (2), 170.2(2009), pp. 799-817] pioneered the study of length-counting sequences associated with parabolic lower Bruhat intervals in crystallographic Coxeter groups. In this extended abstract, we study the asymptotic behavior of these sequences in affine Weyl groups. Let W be an affine Weyl group with corresponding Weyl group W_f, and ${}^fW$ be the set of minimal representatives for the right cosets $W_f\backslash W$. Let t_λ be the translation by a dominant coroot lattice element λ and ${}^fb_i^{t_\lambda}$ be the number of elements of length i below t_λ in the Bruhat order on ${}^fW$. We show that the sequence $\{{}^fb_i^{t_\lambda}\}_i$ is "asymptotically log-concave" in the following sense: the "shape" of the k-fold dilated sequence $\{{}^fb_i^{t_{k\lambda}}\}_i$, as k tends to infinity, converges to a continuous function obtained from a certain polytope P^λ; by the Brunn–Minkowski inequality, this function is log-concave.

Keywords: asymptotic log-concavity, affine Weyl group, dominant Bruhat intervals, dominant lattice formula, Brunn–Minkowski inequality

1 Background

Studying classes of Schubert varieties in the cohomology ring of the generalized flag variety leads to important results in enumerative geometry (the classical "Schubert calculus"), while the study of their intersection cohomology plays a fundamental role in representation theory (the "Kazhdan–Lusztig theory"). Following Björner and Ekedahl [1], we are interested in the behavior of the Betti numbers of Schubert varieties.

More precisely, consider a complex Kac–Moody group G with Borel subgroup B and maximal torus T. The corresponding Weyl group W has the structure of a crystallographic Coxeter system (W, S), where S is the generating set, and we denote by $\ell\colon W \to \mathbb{N}$ the length function. For any $J \subset S$, there is a parabolic subgroup $W_J := \langle s \in J \rangle$ of W and a corresponding subgroup $P_J := BW_JB$ of G.

The quotient $P_J\backslash G$ is a projective (ind-)variety called the *generalized (partial) flag variety*. We have the well-known Bruhat decomposition $P_J\backslash G = \bigsqcup_{w\in {}^JW} P_J\backslash P_JwB$, where

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${}^J W$ is the set of minimal representatives for the right cosets $W_J \backslash W$. The component $C_w := P_J \backslash P_J w B$ is called the *Schubert cell* associated with $w \in {}^J W$. Topologically, C_w is an $\ell(w)$-dimensional affine space $\mathbb{A}^{\ell(w)}$. Its closure $X_w := \overline{C_w}$ is called the *Schubert variety* associated with w. There is a partial order $\leq$ on ${}^J W$ called the *Bruhat–Chevalley order* defined by $v \leq w$ if $C_v \subseteq X_w$. Furthermore, we have the decomposition

$$X_w = \bigsqcup\nolimits_{v \in {}^J W, v \leq w} P_J \backslash P_J v B. \tag{1.1}$$

Question 1. *How many complex i-dimensional cells occur in the decomposition* (1.1) *of* X_w*?*

Let us denote this number by ${}^J b_i^w$. Equation (1.1) gives the equality

$${}^J b_i^w = \mathrm{Card}\{v \in {}^J W \mid v \leq w \text{ and } \ell(v) = i\}, \tag{1.2}$$

which also equals the $2i$-dimensional Betti number of X_w (the odd dimensional Betti numbers of X_w are 0).

Question 1 is difficult to answer in general. If X_w is smooth, the Poincaré duality implies that ${}^J b_i^w = {}^J b_{\ell(w)-i}^w$. While the hard Lefschetz theorem implies that the sequence $\{{}^J b_i^w\}_i$ is *unimodal*, that is, it goes up and then goes down. But X_w is singular in general, hence Poincaré duality and hard Lefschetz theorem usually fail. By means of deep results in Hodge theory, Björner and Ekedahl [1] showed that the sequence $\{{}^J b_i^w\}_i$ satisfies the following two sets of inequalities

$${}^J b_i^w \leq {}^J b_{\ell(w)-i}^w \text{ for } i \leq \frac{\ell(w)}{2}, \quad \text{and} \quad {}^J b_0^w \leq {}^J b_1^w \leq \cdots \leq {}^J b_{\left\lceil \frac{\ell(w)}{2} \right\rceil - 1}^w \leq {}^J b_{\left\lceil \frac{\ell(w)}{2} \right\rceil}^w. \tag{1.3}$$

The first set of inequalities is rephrased as the sequence being *top-heavy*, while the second is the fact that the sequence is weakly increasing in the "lower half part".

Some variants of Question 1 have been studied. By Equation (1.2), one can formulate an analog of Question 1 for general Coxeter groups. Using Soergel bimodules and the Hodge theory established by Elias and Williamson in [11], it is proven that the inequalities in (1.3) hold for a general Coxeter group W in the non-parabolic case (that is, $J = \emptyset$, see [15]). For the parabolic case, we believe that a proof of these inequalities should follow from the Hodge theory of singular Soergel bimodules [18]. On the other hand, in the context of Schubert varieties of hyperplane arrangements, Huh and Wang [14], and Braden, et.al. [3] proved Dowling and Wilson's "Top-Heavy conjecture" for matroids.

Despite of these great achievements, the unimodality of $\{{}^J b_i^w\}_i$ for the "upper half part" remains an interesting open problem. To the best of our knowledge, there is no partial result yet. However, conjectures related to this problem have been made. Before we get into these, let us recall that a sequence $a_0, a_1, \ldots, a_n$ of positive real numbers is said to be *log-concave* if

$$a_{i-1} a_{i+1} \leq a_i^2 \text{ for all } 0 < i < n.$$

This notion is stronger than unimodality: a log-concave sequence is always unimodal. Regarding log-concavity of Bruhat intervals, Brenti conjectured the following:

Conjecture 2 ([4, Conjecture 2.11]). *Let W be a (finite) Weyl group, and $u, v \in W$. The sequence $\{b_i^{[u,v]}\}_i$ is log-concave, where $b_i^{[u,v]} := \mathrm{Card}\,\{w \in W \mid u \leq w \leq v$ and $\ell(w) = i\}$.*

It is known that the parabolic analog of Conjecture 2 does not hold. For example, the Betti numbers of the Schubert variety $X_{(8,8,4,4)}$ inside the Grassmannian $\mathrm{Gr}(4, 12)$ gives a non-unimodal sequence. See [20] for details.

2 Our results

Let $W = \mathbb{Z}\Phi^\vee \rtimes W_f$ be an affine Weyl group with finite Weyl group W_f and root system Φ of rank r. Let $(E, (-|-))$ be the r-dimensional Euclidean space where Φ lives in. Let ${}^f W$ be the set of minimal representatives for the right cosets $W_f \backslash W$. Denote by C_+ the dominant Weyl chamber. Let $\lambda \in \mathbb{Z}\Phi^\vee \cap \overline{C_+}$ be a dominant coroot lattice element, and $t_\lambda \in W$ be the translation by λ. Let ${}^f[e, t_\lambda] := \{w \in {}^f W \mid w \leq t_\lambda\}$ be the dominant lower Bruhat interval. For $0 \leq i \leq \ell(t_\lambda)$, we define

$$ {}^f b_i^{t_\lambda} := \mathrm{Card}\big\{w \in {}^f[e, t_\lambda] \;\big|\; \ell(w) = i\big\}. $$

We prove that sequence $\{{}^f b_i^{t_\lambda}\}_i$ is *asymptotically log-concave* in the following sense:

- The "shape" of the length-counting sequences of the dilated intervals ${}^f[e, t_{k\lambda}]$ converges to a continuous function when k tends to infinity (Theorem 3).
- This continuous function is log-concave (Corollary 7).

2.1 Asymptotic convergence

Let $\lambda \in \mathbb{Z}\Phi^\vee \cap \overline{C_+}$ be a fixed dominant lattice element. We define the convex polytope

$$ P^\lambda := \mathrm{Conv}\{w\lambda \mid w \in W_f\} \cap \overline{C_+} \subset E, $$

where $\mathrm{Conv}\{-\}$ is the convex hull of a set of points. Let $\mathrm{ht}\colon P^\lambda \to \mathbb{R}$ be the *height function* $\mathrm{ht}(x) := (2\rho|x)$, where ρ is the half sum of positive roots. In particular, $\mathrm{ht}(\lambda) = \ell(t_\lambda)$. We denote by Vol_r the Lebesgue measure on E and by $\mathrm{ht}_*\mathrm{Vol}_r$ the corresponding push-forward measure on $\mathbb{R}$. That is, for any Borel set $U \subseteq \mathbb{R}$,

$$ (\mathrm{ht}_*\mathrm{Vol}_r)(U) := \mathrm{Vol}_r(\mathrm{ht}^{-1} U) = \mathrm{Vol}_r\big(\big\{x \in P^\lambda \;\big|\; (2\rho|x) \in U\big\}\big). $$

We also denote by Vol_{r-1} the Lebesgue measure on affine hyperplanes of E. Then, the density function of $\mathrm{ht}_*\mathrm{Vol}_r$ is

$$g(z) = \|2\rho\|^{-1}\,\mathrm{Vol}_{r-1}(\mathrm{ht}^{-1}(z)).$$

Let δ_x denote the Dirac measure (that is, point mass) at the point $x \in \mathbb{R}$. For any positive integer k, we define a discrete measure $\mathfrak{m}_k$ supported on $[0, \ell(t_\lambda)]$ by

$$\mathfrak{m}_k := k^{-r} \textstyle\sum_{0 \le i \le k\ell(t_\lambda)} {}^f b_i^{t_{k\lambda}} \delta_{\frac{i}{k}}. \tag{2.1}$$

Intuitively, we distribute the sequence $\{{}^f b_i^{t_{k\lambda}}\}_i$ evenly on the interval $[0, \ell(t_\lambda)]$. We also define a step function $S_k \colon [0, \ell(t_\lambda)] \to \mathbb{R}$ as follows. For any $x \in [0, \ell(t_\lambda)]$, there exists a unique $i \in \{0, 1, \dots, k\ell(t_\lambda)\}$ such that $x \in [\frac{i}{k}, \frac{i+1}{k})$. We define

$$S_k(x) := k^{-(r-1)} \cdot {}^f b_i^{t_{k\lambda}}.$$

The function S_k records the numbers $\{{}^f b_i^{t_{k\lambda}}\}_i$ and behaves like the "density function" of $\mathfrak{m}_k$. The following is our main theorem.

Theorem 3. *Let $\mathrm{Vol}_r(A_+)$ be the volume of the fundamental alcove A_+.*

(1) (Weak convergence of measures) *The sequence of measures $\{\mathfrak{m}_k\}_k$, as k tends to infinity, converges weakly to $\frac{1}{\mathrm{Vol}_r(A_+)}\mathrm{ht}_*\mathrm{Vol}_r$.*

(2) (Uniform convergence of functions) *The sequence of functions $\{S_k\}_k$, as k tends to infinity, converges uniformly to the function $\frac{1}{\mathrm{Vol}_r(A_+)}g$.*

See Section 3 for an explicit example.

Remark 4. *If λ is strongly dominant, that is, $\lambda \in C_+$, then P^λ is combinatorially equivalent to a hypercube (see [5]).*

2.2 The dominant lattice formula

We define the *Poincaré polynomial* $\pi^{t_\lambda}(q)$ of the sequence $\{{}^f b_i^{t_\lambda}\}_i$ by

$$\pi^{t_\lambda}(q) := \textstyle\sum_{0 \le i \le \ell(t_\lambda)} {}^f b_i^{t_\lambda} q^i = \sum_{w \in {}^f[e, t_\lambda]} q^{\ell(w)}.$$

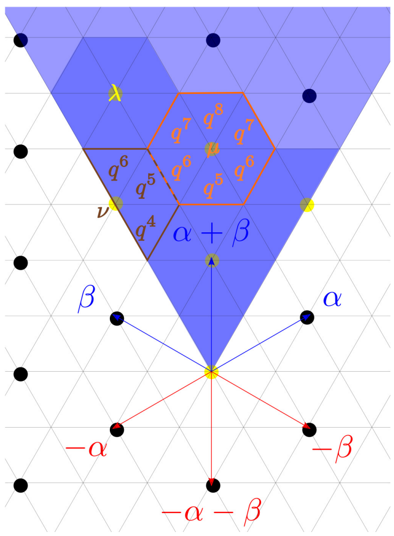
Let $\{\alpha_1, \dots, \alpha_r\}$ be the set of simple roots of Φ, and $\{s_1, \dots, s_r\}$ be the set of corresponding simple reflections. For any $\mu \in \mathbb{Z}\Phi^\vee$, we denote by W_μ the standard parabolic subgroup of W_f generated by $\{s_i \mid 1 \le i \le r, (\mu|\alpha_i) = 0\}$ and by ${}^\mu W_f$ the set of minimal representatives for the right cosets $W_\mu \backslash W_f$. We also define *the Poincaré polynomial* ${}^\mu\pi_f(q)$ of the set ${}^\mu W_f$ by ${}^\mu\pi_f(q) := \sum_{w \in {}^\mu W_f} q^{\ell(w)}$.

The following theorem is one of our most important results, and plays an important role in the proof of Theorem 3:

Theorem 5 (Dominant lattice formula)**.** *Let $\lambda \in \mathbb{Z}\Phi^\vee \cap \overline{C_+}$ as before. Then*

$$\pi^{t_\lambda}(q) = \sum\nolimits_{\mu \in P^\lambda \cap \mathbb{Z}\Phi^\vee} q^{(2\rho|\mu)} \cdot {}^\mu\pi_f(q^{-1}). \tag{2.2}$$

This formula serves as a bridge between the discrete nature of $\{{}^f b_i^{t_\lambda}\}_i$ and the continuous nature of the geometry of P^λ. See Figure 1 for an illustration.



Description of two of the summands of the dominant lattice formula when W is of affine type A_2 and $\lambda = 2\alpha + 3\beta$, where $\alpha := \alpha_1^\vee$ and $\beta := \alpha_2^\vee$. The yellow points are the lattice points inside P^λ. The alcoves of the interval ${}^f[e, t_\lambda]$ are colored with dark blue. There are 6 dominant alcoves arranged around the strongly dominant lattice point $\mu := 2\alpha + 2\beta$, and 3 around $\nu := \alpha + 2\beta$ which is on the wall. The summand corresponding to μ in the formula is given by $q^8 \cdot {}^\mu\pi_f(q^{-1}) = q^5 + 2q^6 + 2q^7 + q^8$. The terms of this polynomial are colored orange and placed inside their corresponding alcoves in the picture. The summand corresponding to ν is given by $q^6 \cdot {}^\nu\pi_f(q^{-1}) = q^4 + q^5 + q^6$, whose terms are colored with brown.

Figure 1: Illustration for the dominant lattice formula.

2.3 Log-concavity and a conjecture on unimodality

The following theorem taken from [17, p. 270] can be deduced from the classical Brunn–Minkowski inequality.

Theorem 6 (Brunn–Minkowski, see [17, p. 270])**.** *Let L_1 be a real vector space and let $M \subset L_1$ be a convex body. Let $p\colon L_1 \to L_2$ be a linear transformation. Then*

$$x \mapsto \big(\mathrm{Vol}\big(p^{-1}(x) \cap M\big)\big)^{1/(\dim M - \dim p(M))}$$

is a concave function on $p(M)$.

Applying the above theorem to the map $\mathrm{ht}\colon P^\lambda \to \mathbb{R}$ and taking logarithm (which is a concave function), we have immediately the following corollary.

Corollary 7. *The density function g of the measure $\mathrm{ht}_*\mathrm{Vol}_r$ is log-concave, that is, $\log g$ is a concave function.*

Remark 8. *The sequence $\{{}^f b_i^{t_\lambda}\}_i$ is not necessarily log-concave. For example, from the step function in Figure 2a, we observe that the sequence contains the consecutive terms $(4, 4, 5)$.*

We propose the following conjecture:

Conjecture 9. *The sequence* $\{{}^f b_i^{t_\lambda}\}_i$ *is unimodal.*

This conjecture holds in rank 2 from basic facts of the Euclidean geometry of the real plane. It has been tested for different choices of λ in affine Weyl groups of rank ≤ 4 (and also type $\widetilde{A_5}$) with the help of SageMath.

3 An example of Theorem 3

Let W be the affine Weyl group associated with the root system Φ of type C_3 and simple roots $\Delta = \{\alpha_1, \alpha_2, \alpha_3\}$. Then, $r = 3$. Following [2, Plate III], we write $\alpha_1 = \epsilon_1 - \epsilon_2$, $\alpha_2 = \epsilon_2 - \epsilon_3$, and $\alpha_3 = 2\epsilon_3$. Let

$$\lambda = 3\alpha_1^\vee + 6\alpha_2^\vee + 7\alpha_3^\vee.$$

We have that $\mathrm{ht}(\lambda) = 32$. For convenience, we define $(a, b, c)_\Phi := a\alpha_1^\vee + b\alpha_2^\vee + c\alpha_3^\vee$. The polytope P^λ is the convex polyhedron with six vertices given by

$$\{(0,0,0)_\Phi, (3,3,3)_\Phi, (3,5,7)_\Phi, (3,6,6)_\Phi, (7/3, 14/3, 7)_\Phi, (3,6,7)_\Phi\},$$

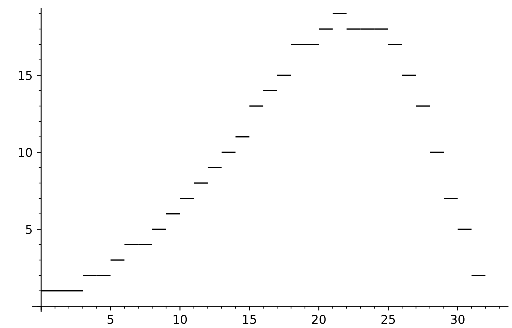
which is an example of a non-lattice polytope. Since $\rho = (3,5,3)_\Phi$, we get $||\rho|| = \sqrt{14}$. From [8, Equation 2.4], or by direct computations, we have that $\mathrm{Vol}_3(A_+) = 1/48$. In view of Theorem 3, the only missing ingredient to compute the limit function is to determine the area function $\mathrm{Vol}_2(\mathrm{ht}^{-1}(x))$. From the theory of convex polytopes, this function is a piece-wise quadratic polynomial and its exact form can be obtained by Lagrange interpolation. We omit the details and just give a graph of the function $g/\,\mathrm{Vol}_3(A_+)$ in Figure 2d.

We can use Theorem 3 to give quick estimates of the terms in the sequence $\{{}^f b_i^{t_{k\lambda}}\}_i$ when k is big enough. For instance, when $k = 8$, the value of ${}^f b_{196}^{t_{8\lambda}}$ is virtually impossible to obtain in a computer directly from definitions. Let us pick $x = 24.5 (= 196/8)$. From our theorem we have

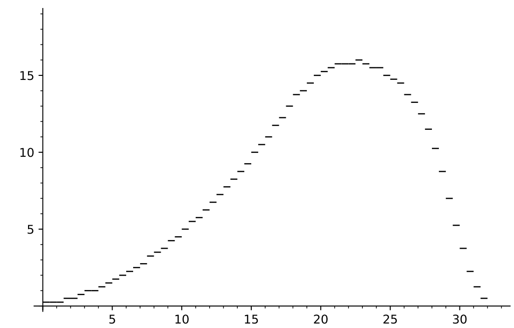
$$S_8(24.5) = {}^f b_{196}^{t_{8\lambda}}/8^2 \sim g(24.5)/48 = 389/30$$

giving ${}^f b_{196}^{t_{8\lambda}} \sim 829.86$.

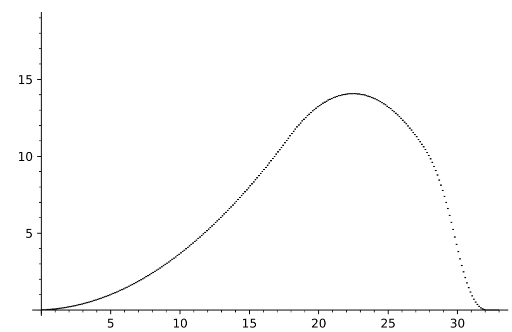
On the other hand, Theorem 5 gives the exact values of the terms in the sequence $\{{}^f b_i^{t_{k\lambda}}\}_i$. We can compute the value of the function S_8 (which takes a considerable time to get in a computer.) In particular, we have ${}^f b_{196}^{t_{8\lambda}} = 863$. Our quick estimate of 829.86 from before was off by 3.84%. In various examples, we observed that the error of the estimation decreases roughly linearly with the growth of k. See Figure 2 for the graphs of the step functions S_1, S_2, and S_8.



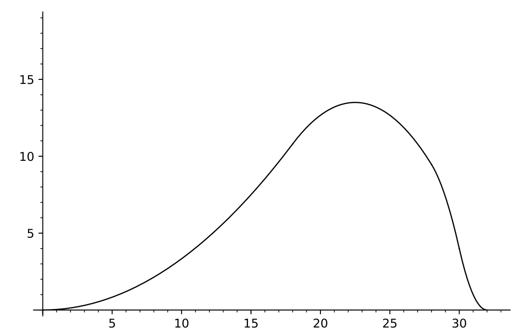
(a) Graph of S_1.



(b) Graph of S_2.



(c) Graph of S_8.



(d) Graph of $g/\operatorname{Vol}_r(A_+)$.

Figure 2: In the affine Weyl group W of affine type C_3, we consider $\lambda = 3\alpha_1^\vee + 6\alpha_2^\vee + 7\alpha_3^\vee$. These pictures illustrate how the sequence of step functions $S_k\colon [0, \ell(t_\lambda)] \to \mathbb{R}$ converges uniformly to the continuous function $g/\operatorname{Vol}_r(A_+)$.

4 Connections with asymptotic representation theory

The formulation of Theorem 3(1) borrows ideas from the construction of the now-called Duistermaat–Heckman measure [13] and Okounkov's work [16] on the asymptotic log-concavity for multiplicities of representations. Let G be a compact connected Lie group and λ be a dominant weight. In [13], Heckman constructed a discrete measure

$$\frac{\sum_\mu \dim V(k\lambda)_\mu \delta_{\frac{\mu}{k}}}{\sum_\mu \dim V(k\lambda)_\mu}$$

supported on the weight polytope $\operatorname{Conv}\{w\lambda \mid w \in W_f\}$, where $\dim V(k\lambda)_\mu$ is the weight multiplicity of the irreducible representation of G with highest weight $k\lambda$. He proved that this sequence of discrete measures, as k tends to infinity, converges weakly to the push-forward of the Lebesgue measure of the coadjoint orbit of λ under the moment map. The density function of the limit measure is a piecewise polynomial function [9] and Graham proved that it is log-concave via Hodge–Riemann inequalities [12]. Later, Okounkov [16] introduced the now-called Newton–Okounkov bodies to prove, in a similar weak limit sense, that for any reductive group G and any representation V of G, the multiplicities of irreducible G-modules in the homogeneous coordinate ring of a G-stable irreducible subvariety of $\mathbb{P}(V)$ are log-concave.

It is not hard to see the similarity between our construction (2.1) and the one of Heckman, and it is indeed similar to the one of Okounkov. However, our proof technique is quite different from theirs. Moreover, it is not obvious that our original cell-counting problem has such a critical relation to the geometry of a convex polytope. Theorem 3(1) is the analog of theirs, while a result like Theorem 3(2) is novel in this kind of setting.

5 Relation with Ehrhart's theory

For an r-dimensional lattice polytope P (that is, all vertices of P are points of a given lattice L), the *Ehrhart polynomial* [10] is a polynomial in k that counts the number of lattice points in the k-fold dilation kP of P. The leading coefficient is equal to the r-dimensional volume $\mathrm{Vol}_r(P)$ of P, divided by the volume $d(L)$ of the fundamental region of the lattice L. This implies that

$$\mathrm{Vol}_r(P) = \lim_{k\to\infty} \frac{d(L)\cdot \mathrm{Card}\{\text{lattice points in } kP\}}{k^r}. \tag{5.1}$$

If X is the toric variety corresponding to the normal fan of P, then P defines an ample line bundle on X. The Ehrhart polynomial of P coincides with the Hilbert polynomial of this line bundle, and the asymptotic result (5.1) is a consequence of the asymptotic Riemann-Roch theorem [19, Tag 0BJ8].

The problem in our work is analogous to the one in Ehrhart's theory, while we count alcoves in each length rather than all lattice points in the polytope P^λ. When the polytope P^λ is not a lattice polytope, it has no Ehrhart polynomial. We want to raise the following question related to Theorem 3(2):

Question 10. *Is ${}^f b_{ki}^{t_{k\lambda}}$ a quasi-polynomial in k of degree $(r-1)$ for k sufficiently large, with*

$$\frac{\mathrm{Vol}_{r-1}(\mathrm{ht}^{-1}(i))}{\mathrm{Vol}_r(A_+)\cdot \|2\rho\|}$$

as the leading coefficient?

6 Main ideas in the proofs of Theorem 3 and Theorem 5

For complete proofs, see [6].

6.1 Theorem 5

Our cell-counting problem can be translated into "counting alcoves" thanks to the natural bijection between the affine Weyl group W and the set of alcoves. In particular, $w \in {}^fW$ if and only if the corresponding alcove A_w is dominant, that is, it is contained in C_+. On the other hand, we have the following well-known result:

Lemma 11. *Suppose $\lambda, \mu \in \mathbb{Z}\Phi^\vee \cap \overline{C_+}$. The following are equivalent:*

(1) *$t_\mu \leq t_\lambda$ in the Bruhat–Chevalley order.*

(2) *$\mu \in \mathrm{Conv}\{w\lambda \mid w \in W_f\}$.*

These facts motivate the definition of the polytope $P^\lambda = \mathrm{Conv}\{w\lambda \mid w \in W_f\} \cap \overline{C_+}$. They also lead to a description of the interval ${}^f[e, t_\lambda]$ in terms of the lattice points in P^λ:

$$ {}^f[e, t_\lambda] = \{t_\mu w \in W \mid \mu \in P^\lambda \cap \mathbb{Z}\Phi^\vee, w \in {}^\mu W_f\}. $$

Then, the dominant lattice formula (Theorem 5) follows from comparing the lengths of the elements on both sides of this equality.

6.2 Theorem 3

The following is the main geometric intuition in our proof of Theorem 3: a dominant Bruhat interval can be realized as a bounded region—a union of finitely many alcoves—inside C_+; after dilating the lattice element λ of the interval ${}^f[e, t_\lambda]$ and re-scaling the region back, the alcoves in the region get smaller and smaller, and the region approaches P^λ. This is illustrated in Figure 3. Other works relating Euclidean geometry and Bruhat intervals in affine Weyl groups can be found in [8, 7].

The following corollary of Theorem 5 is crucial in the proof of Theorem 3:

Corollary 12. *We define*

$$ \pi_f(q) := \sum\nolimits_{w\in W_f} q^{\ell(w)}, \quad \pi^\lambda(q) = \sum\nolimits_{\mu\in P^\lambda\cap\mathbb{Z}\Phi^\vee} q^{(2\rho|\mu)}, \quad \pi^\lambda_+(q) = \sum\nolimits_{\mu\in P^\lambda\cap\mathbb{Z}\Phi^\vee\cap C_+} q^{(2\rho|\mu)}. $$

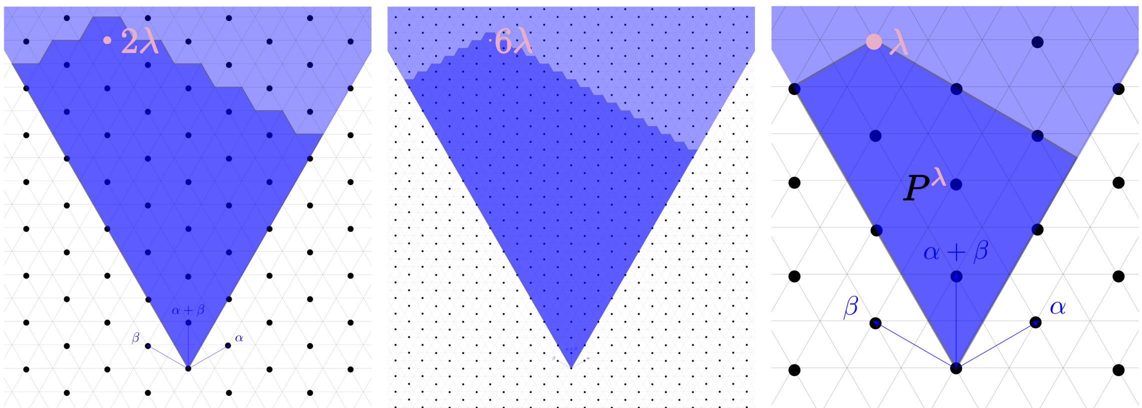
Then we have

$$ \pi^\lambda_+(q) \cdot \pi_f(q^{-1}) \leq \pi^{t_\lambda}(q) \leq \pi^\lambda(q) \cdot \pi_f(q^{-1}), \tag{6.1} $$

where the inequalities between these Laurent polynomials mean to be coefficient-wise.

Considering the coefficients in the inequality (6.1), we are able to approximate ${}^f b_i^{t_{k\lambda}}$ using the numbers

$$ \mathrm{Card}\left(P^\lambda \cap \frac{1}{k}\mathbb{Z}\Phi^\vee \cap \mathrm{ht}^{-1}(y)\right), \tag{6.2} $$



(a) The interval ${}^f[e, t_{2\lambda}]$. (b) The interval ${}^f[e, t_{6\lambda}]$. (c) The polytope P^λ.

Figure 3: Behavior of the intervals ${}^f[e, t_{k\lambda}]$ when W is of affine type A_2 and $\lambda = 3\alpha + 4\beta$, where $\alpha := \alpha_1^\vee$ and $\beta := \alpha_2^\vee$. In each picture, the set of small triangles corresponds to the set of alcoves. The coroot lattice is indicated by black bullets and the dominant Weyl chamber is colored blue. In the first two pictures, the alcoves corresponding to the elements in the intervals are filled with darker blue. So is the polytope P^λ in the third picture.

where y runs over a particular set of numbers near i/k. For these y, it turns out that $(\frac{1}{k}\mathbb{Z}\Phi^\vee) \cap \mathrm{ht}^{-1}(y)$ is a lattice of rank $r-1$ in the affine hyperplane $\mathrm{ht}^{-1}(y)$.

We construct a Riemann sum using the numbers from (6.2). As k tends to infinity, this sum converges to a quantity related to the volume of a part of P^λ. From basic results about weak convergence, this leads to a proof of Theorem 3(1).

The proof of Theorem 3(2) is more technical than the proof of Theorem 3(1). First of all, it suffices to prove that $S_k(x)$ converges uniformly for $x \in [0, \ell(t_\lambda)]$ of the form i/k, because of the definition of S_k and the continuity of g. For this, we need to estimate the value of the step function S_k at those $x = i/k$, which is $k^{1-r} \cdot {}^f b_i^{t_{k\lambda}}$. As before, we switch this estimation to the estimation of the numbers in (6.2). Let y be as before.

We choose a fundamental domain B_k of the lattice $(\frac{1}{k}\mathbb{Z}\Phi^\vee) \cap \mathrm{ht}^{-1}(0)$ containing the origin point of $\frac{1}{k}\mathbb{Z}\Phi^\vee$. If we join all the translations of B_k by points in $P^\lambda \cap (\frac{1}{k}\mathbb{Z}\Phi^\vee) \cap \mathrm{ht}^{-1}(y)$, we obtain the region

$$\mathcal{R} := \bigsqcup \left\{ l + B_k \;\middle|\; l \in P^\lambda \cap \frac{1}{k}\mathbb{Z}\Phi^\vee \cap \mathrm{ht}^{-1}(y) \right\}$$

in the hyperplane $\mathrm{ht}^{-1}(y)$.

Because we can compute the volume of B_k directly from Φ, estimating the value of (6.2) is equivalent to estimating the value of $\mathrm{Vol}_{r-1}(\mathcal{R})$. The proof of the convergence is then achieved by comparing $\mathrm{Vol}_{r-1}(\mathcal{R})$ with $\mathrm{Vol}_{r-1}(P^\lambda \cap \mathrm{ht}^{-1}(x))$. This, as well as the uniformity, requires the use of some "Euclidean geometries" to carefully estimate the volume of some open neighborhood of the boundary of $P^\lambda \cap \mathrm{ht}^{-1}(x)$ (see Figure 4 for an example of such a neighborhood). When k is large enough, for any x and y, the boundary of $\mathcal{R}$ is contained in such a neighborhood. Because the volume of such a neighborhood can be sufficiently small, this implies that $\mathrm{Vol}_{r-1}(\mathcal{R})$ is sufficiently close to $\mathrm{Vol}_{r-1}(P^\lambda \cap \mathrm{ht}^{-1}(x))$. This leads to the proof of the uniform convergence.

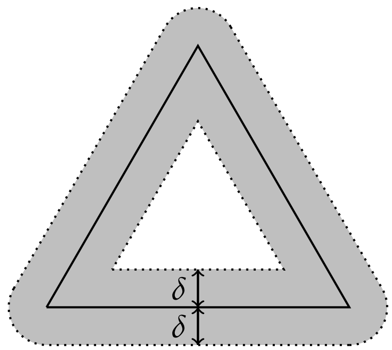


Figure 4: A triangle T and the neighbourhood $\mathcal{N}(\partial T, \delta)$.

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