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A Whitney polynomial for hypermaps

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Abstract. We introduce a Whitney polynomial for hypermaps and use it to generalize the results connecting the circuit partition polynomial to the Martin polynomial and the results on several graph invariants.

Résumé. Nous introduisons un polynôme de Whitney pour les hypercartes et nous l'utilisons pour généraliser les résultats liant le polynôme des partitions de circuit aux polynômes de Martin et les résultats de plusieurs invariants de graphes.

Keywords: set partitions, noncrossing partitions, genus of a hypermap, Tutte polynomial, Whitney polynomial, medial map, circuit partition, characteristic polynomial, chromatic polynomial, flow polynomial

Introduction

The Tutte polynomial is a key invariant of graph theory, which has been generalized to matroids, polymatroids, signed graphs and hypergraphs in many ways. A partial list of recent topological graph and hypergraph generalizations includes [2, 3, 4, 19, 21].

The work we present [8] generalizes a variant of the Tutte polynomial, the Whitney rank generating function to *hypermaps* which encode hypergraphs topologically embedded in a surface. This polynomial may be recursively computed using a generalized deletion-contraction formula, and many of the famous special substitutions (for instance, counting spanning subsets of edges, or trees contained in the a graph) may be easily generalized to this setting. Our approach seems to be most amenable to generalize results on the Eulerian circuit partition polynomials, but we also have a promising generalization of the characteristic polynomial. This last generalization (involving the Möbius function of the noncrossing partition lattice) also indicates that, for hypermaps many invariants cannot be obtained by a simple negative substitution into some generalized Tutte polynomial. For similar reasons, a generalized Whitney polynomial seems to work better than a generalized Tutte polynomial, and the difference between the two should not be thought of as a mere linear shift.

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Our work is organized as follows. After the Preliminaries, the key definition of our Whitney polynomial is contained in Section 2. We may generalize the well-known deletion-contraction recurrence formulas from graphs to this setting. This section also contains several important specializations and the proof of the fact that taking the dual of a planar hypermap amounts to swapping the two variables in its Whitney polynomial.

We introduce the directed medial map of a hypermap in Section 3. Every directed Eulerian graph arises as the directed medial map of a collection of hypermaps. In this section and in Section 5, we present analogues and generalizations of several results of Arratia, Bollobás, Ellis-Monaghan, Martin and Sorkin [1, 5, 12, 13, 15, 14, 25, 26] on the circuit partition polynomials of Eulerian digraphs and the medial graph of a plane graph.

The visually most appealing part of our work is in Section 4. Here we extend the circuit partition approach to a refined count which keeps track of circuits bounding external ("wet") and internal ("dry") faces and define a process that allows the computation of the Whitney polynomial of a planar hypermap using paper and a scissor.

Finally, in Section 6 we introduce a characteristic polynomial for hypermaps which generalizes the characteristic polynomial of a map, as well as of a graded poset. We show that for hypermaps whose hyperedges have length at most three, this variant of the characteristic polynomial is still a chromatic polynomial, counting the admissible colorings of the vertices.

1 Preliminaries

A *hypermap* is a pair of permutations (σ, α) acting on the same finite set of labels, generating a transitive permutation group. It encodes a hypergraph, topologically embedded in a surface. Fig. 1 represents the planar hypermap (σ, α) for $\sigma = (1,4)(2,5)(3)$ and $\alpha = (1,2,3)(4,5)$. The cycles of σ are the *vertices*, the cycles of α are the *hyperedges* and the cycles of $\alpha^{-1}\sigma = (1,5)(2,4,3)$ are the *faces*. A hypermap is a *map* if the length of each cycle in α is at most 2. In terms of the function $z(\pi)$, counting the cycles of the

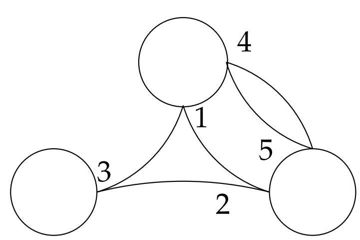

Figure 1: The hypermap (σ, α)

permutation π, the *genus* $g(\sigma, \alpha)$ of a hypermap may be computed using the equation

$n+2-2g(\sigma,\alpha) = z(\sigma)+z(\alpha)+z(\alpha^{-1}\sigma)$ due to Jacques [17]. The present work was motivated by the study of the *spanning hypertrees of a hypermap*, initiated in [6, 9, 10, 24] and continued in [7]. A hypermap (σ,α) is *unicellular* if it has only one face, and it is a *hypertree* if it also has genus zero. A permutation β is a *refinement* of a permutation α if β is obtained by replacing each cycle α_i of α by a permutation β_i acting on the same set of points in such a way that $g(\alpha_i,\beta_i)=0$. We will use the notation $\beta \leq \alpha$ to denote that β is a refinement of α. A hypermap (σ,β) *spans* the hypermap (σ,α) if β is a refinement of α. A *hyperdeletion* is the operation of replacing a hypermap (σ,α) with the hypermap $(\sigma,\alpha\delta)$ where $\delta=(i,j)$ is a transposition *disconnecting* α, that is, i and j must belong to the same cycle. This time we will work with *collections of hypermaps* (defined in Section 2) hence we may perform hyperdeletions even if the permutation group generated by the pair $(\sigma,\alpha\delta)$ has more orbits than the one generated by the pair (σ,α). For maps the deletion operation corresponds to deleting an edge (i,j). A *hypercontraction* is the operation of replacing a hypermap (σ,α) with the hypermap $(\gamma\sigma,\gamma\alpha)$ where $\gamma=(i,j)$ is a transposition disconnecting α. All hypercontractions considered in this work will be *topological*: i and j have to belong to different cycles of σ, for maps this operation corresponds to contracting an edge that is not a loop.

2 A Whitney polynomial of a collection of hypermaps

Definition 1. *A* collection of hypermaps (σ,α) *is an ordered pair of permutations acting on the same set of points. We call the orbits of the permutation group generated by σ and α the* connected components of (σ,α) *and denote their number by $\kappa(\sigma,\alpha)$.*

Definition 2. *The* Whitney polynomial $R(\sigma,\alpha;u,v)$ *of a collection of hypermaps (σ,α) on a set of n points is defined by the formula*

$$R(\sigma,\alpha;u,v) = \sum_{\beta\leq\alpha} u^{\kappa(\sigma,\beta)-\kappa(\sigma,\alpha)} \cdot v^{\kappa(\sigma,\beta)+n-z(\beta)-z(\sigma)}$$

Here the summation is over all permutations β refining α.

For maps we recover the usual definition of the Whitney polynomial of the underlying graph. This invariant is multiplicative for a pair of collections of hypermaps on disjoint sets of points. The function $R(\sigma,\alpha;u,v)$ may be computed recursively using the following generalization of the the well-known deletion-contraction recurrence for the Whitney polynomial $R(G;u,v)$ of a graph G.

Theorem 3. *Let $H=(\sigma,\alpha)$ be a collection of hypermaps on the set $\{1,2,\ldots,n\}$ and assume that $(1,2,\ldots,m)$ is a cycle of α of length at least 2. Then the Whitney polynomial $R(H;u,v)$ is given by the sum*

$$R(H;u,v) = \sum_{k=1}^{m} R(\phi_k(H);u,v)\cdot w_k,$$

where each $\phi_k(H)$ is a collection of hypermaps and each w_k is a monomial from the set $\{1,u,v,uv\}$, according to the following rules:

$$\phi_k(H) = \begin{cases} ((1,k)\sigma,(1,k)\alpha(1,k-1)) & \text{if } z((1,k)\sigma \leq z(\sigma), \\ (\sigma,(1,k)\alpha(1,k-1)) & \text{otherwise.} \end{cases} \tag{2.1}$$

$$w_k = \begin{cases} u^{\kappa(\phi_k(H))-\kappa(H)} & \text{if } z((1,k)\sigma \leq z(\sigma), \\ u^{\kappa(\phi_k(H))-\kappa(H)}v & \text{otherwise.} \end{cases} \tag{2.2}$$

In rule (2.1) we count modulo m, that is, we replace $k-1$ with m if $k=1$, and we read $(1,1)$ as a shorthand for the identity permutation.

In analogy to the case of maps, Theorem 3 may be modified in such a way that for a hypermap (σ,α) the recurrence only involves hypermaps. This recurrence uses hyperdeletions and hypercontractions.

Example 4. For the hypermap shown in Fig 1, repeated use of Theorem 3 gives

$$R((1,4)(2,5)(3),(1,2,3)(4,5);u,v) = u^2+uv+4u+v+3.$$

Certain substitutions into the Tutte polynomial yield famous graph theoretic invariants. Some of these results carry over easily to the Whitney polynomial of a collection of hypermaps. We define a *hyperforest* as a collection of genus zero unicellular hypermaps and we call a collection of hypermaps (σ,β) associated to some refinement β of α (σ,α) *a spanning collection of hypermaps* if the subgroup generated by σ and β has the same orbits as the permutation group generated by σ and α. Then

1. $R(\sigma,\alpha;0,0)$ is the number of spanning hyperforests of (σ,α).
2. $R(\sigma,\alpha;0,1)$ is the number of spanning collections of hypermaps of (σ,α).

The Tutte polynomial $T(G;x,y)$ of a graph G (or of a map) is given by $T(G;x,y) = R(G;x-1,y-1)$. Extending this definition to collections of hypermaps the obvious way does not seem to be a good idea because of the following example.

Example 5. Consider the hypermap (σ,α) given by $\sigma=(1)(2)\cdots(n)$ and $\alpha=(1,2,\ldots,n)$. For this, $R(\sigma,\alpha;u,v)$ and $R(\alpha^{-1}\sigma,\alpha^{-1};u,v)$ are the *Narayana polynomials* of u and and v, respectively, associated to the noncrossing partitions of $\{1,2,\ldots,n\}$. For $n=2$ these are $1+u$ and $1+v$ respectively, but they become much more complicated for larger values of n. For $n=3$ we get $R(\sigma,\alpha;u,v)=u^2+3u+1$, substituting $u=x-1$ yields x^2+x-1, a polynomial with a negative coefficient.

On the other hand, we have the following generalized duality result.

Theorem 6. *A collection of hypermaps (σ,α) of genus zero and its dual collection $(\alpha^{-1}\sigma,\alpha^{-1})$ satisfy $R(\sigma,\alpha;u,v) = R(\alpha^{-1}\sigma,\alpha^{-1};v,u)$.*

3 Medial maps

Definition 7. *Let (σ, α) be a collection of hypermaps on the set of points $\{1, 2, \ldots, n\}$. We define its* medial map $M(\sigma, \alpha)$ *as the following map (σ', α') on $\{1^-, 1^+, 2^-, 2^+, \ldots, n^-, n^+\}$:*

1. *the cycles of σ' are all cycles of the form $(i_1^-, i_1^+, i_2^-, i_2^+, \ldots, i_k^-, i_k^+)$ where $(i_1, i_2, \ldots, i_k)$ is a cycle of α;*
2. *the cycles of α' are all cycles of the form $(i^+, \sigma(i)^-)$.*

We obtain a collection of maps (σ', α') satisfying $\sigma'(i^-) = i^+$ for all i, such that the endpoints of each edge have opposite signs. We call each such collection of maps *Eulerian* and define its *underlying Eulerian digraph* by directing each edge (i^+, j^-) from its positive endpoint toward its negative endpoint. The process of creating the medial map

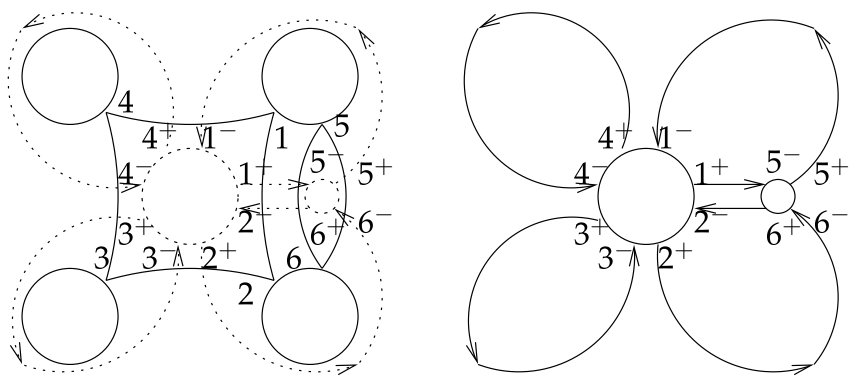

Figure 2: A planar hypermap and its medial map

of $((1,5)(2,6)(3)(4), (1,2,3,4)(5,6))$ is shown in Figure 2.

Proposition 8. *Every directed Eulerian graph arises as the directed medial graph of a collection of hypermaps.*

A hypermap of any genus has a medial map of the same genus. A planar map (σ, α) encodes a plane graph G. In this case $M(\sigma, \alpha)$ is essentially the *directed medial graph* $\overrightarrow{G_m}$ of G, as defined by Martin [25, 26]. Next we generalize the circuit partition polynomials appearing in the works of Ellis-Monaghan [12, 13] and Arratia, Bollobás and Sorkin [1] (see also the Introduction of [5]).

Definition 9. *Let (σ, α) be a collection of Eulerian maps. A* noncrossing Eulerian state *is a partitioning of the edges of the underlying directed medial graph into closed paths in such a way that these paths do not cross at any of the vertices.*

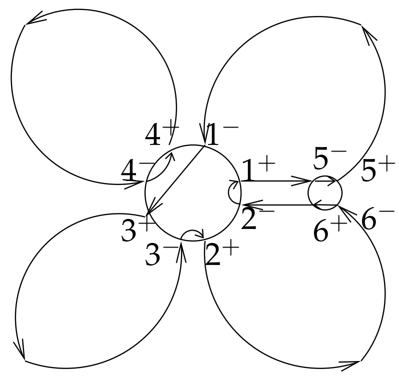

Figure 3: A noncrossing Eulerian state

Figure 3 represents a noncrossing Eulerian state of the Eulerian map shown in the right hand side of Figure 2. We partition the set of edges into closed paths by matching each negative point on a vertex to a positive point on the same vertex. The arrows inside the vertices are the ones pointing from the negative points towards the positive points. A noncrossing Eulerian state is uniquely defined by a *coherent matching* that refines the vertex permutation of the Eulerian map, and matches positive points to negative points.

Definition 10. *We define the* noncrossing circuit partition polynomial *of an Eulerian map* (σ, α) *as*

$$j((\sigma, \alpha); x) = \sum_{k \geq 0} f_k(\sigma, \alpha) x^k.$$

Here $f_k(\sigma, \alpha)$ *is the number of noncrossing Eulerian states with k cycles.*

The following result generalizes Ellis-Monaghan's generalization of Martin's formula [15, Eq. (15)] from planar maps to hypermaps.

Theorem 11. *Let* (σ, α) *be a genus zero collection of hypermaps and* $M(\sigma, \alpha)$ *the collection of its medial maps. Then* $j(M(\sigma, \alpha); x) = x^{\kappa(\sigma,\alpha)} R(\sigma, \alpha; x, x)$ *holds.*

4 A visual computation of $R(\sigma, \alpha; u, v)$ in the planar case

Consider the hypermap (σ, α) given by $\sigma = (1,5,12)(4,11,10)(3,9,8)(2,7,6)$ and $\alpha = (1,2,3,4)(5,6)(7,8)(9,10)(11,12)$, shown in Figure 4. For each point i we add the points i^- and i^+ of the medial map. The vertices of $M(\sigma, \alpha)$ are identified with the hyperedges of (σ, α), and we "shrink" the edges $(i^+, \sigma(i)^-)$ of $M(\sigma, \alpha)$ to follow the outline of the original vertices and the hyperedges. Thus we fatten the outline of the diagram of (σ, α), except for the counterclockwise arcs from i^- to i^+ along the vertices (unless $\sigma(i) = i$) and for he arcs from i^+ to $\alpha(i)^-$ along the hyperedges (unless $\alpha(i) = i$). Next we select

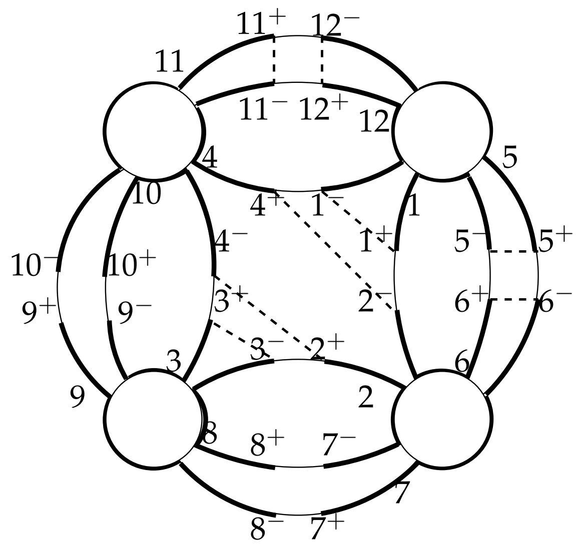

Figure 4: A planar hypermap with the edges of its medial map shrunk to its outline

a coherent matching on the signed points. The *nontrivial choices* are the ones when i^+ is not matched to $\alpha(i)^-$, the remaining choices are trivial. We can think of the diagram of (σ, α) as a paper cutout, with the vertices and the hyperedges being solid and the faces missing. Each nontrivial pair of matched points corresponds then to a cut into the object using a scissor, subject to the following rules:

(R1) Each cut is a simple curve connecting a point i^+ with a point j^-, inside a hyperedge.
(R2) Each point i^+ and j^- may be used at most once.
(R3) The remaining points not used in the cuts must come in pairs $(i^+, \alpha(i)^-)$.
(R4) A new cut cannot cut into the cut-line of a previous cut.

At the end of the process the curves of the outline correspond to the faces $\beta^{-1}\sigma$ of (σ, β) for some $\beta \leq \alpha$, and they are also the circuits of the corresponding circuit partition. Let us think of the unbounded face of the hypermap (σ, α) as "the ocean" with a "wet coastline". After a few nontrivial cuts, we may have several connected components, each has one coastline. Figure 5 illustrates this situation after performing the nontrivial cuts indicated in Figure 4. The shaded regions indicate the faces of σ, α whose border has been merged with a (thickened) wet coastline.

Theorem 12. *Given a planar hypermap (σ, α), we may visually compute its Whitney polynomial by making its model in paper, and performing the above cutting procedure in all possible ways and associating to each outcome u raised to the power of the wet coastlines and v raised to the power of the dry faces. The sum of all weights is $u \cdot R(\sigma, \alpha; u, v)$.*

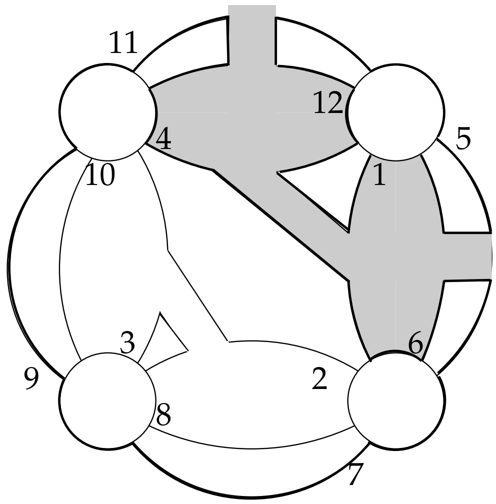

Figure 5: Wet coastlines after a few cuts

5 Counting noncrossing Eulerian colorings

In this section we extend the formula counting the Eulerian colorings of the medial graph of a plane graph [14, Evaluation 6.9] to medial maps of planar hypermaps. The following definition may be found in [14, Definition 4.3].

Definition 13. *An* Eulerian m-coloring *of an Eulerian directed graph $\overrightarrow{G}$ is an edge coloring of $\overrightarrow{G}$ with m colors so that for each color the (possibly empty) set of all edges of the given color forms an Eulerian subdigraph.*

Consider now a planar hypermap (σ, α) and its directed Eulerian medial map $M(\sigma, \alpha)$. Given an Eulerian m-coloring of the edges, let us color the endpoints of $(i^+, \sigma(i)^-)$ with the color of the edge. We call this coloring of the points the *coloring of the points induced by the Eulerian m-coloring*. In order to relate the count of the Eulerian m-colorings to our Whitney polynomial, we must restrict our attention to *noncrossing Eulerian m-colorings*, defined as follows.

Definition 14. *Let (σ, α) be a planar hypermap and let $M(\sigma, \alpha)$ be its directed medial map. We call an Eulerian m-coloring* noncrossing *if there is a noncrossing Eulerian state such that all edges of the same connected circuit have the same color.*

Remark 15. If (σ, α) is a map then the above noncrossing condition is automatically satisfied by each Eulerian m-coloring of $M(\sigma, \alpha)$ as all Eulerian states of $M(\sigma, \alpha)$ are noncrossing. The more general case of partitioning the edge set of an Eulerian digraph into Eulerian subdigraphs was addressed in [1, 5, 12].

Using the induced coloring of the points we can verify vertex by vertex whether an Eulerian m-coloring is non-crossing: it is necessary and sufficient to be able to find a coherent matching at each vertex such that only points of the same color are matched. This observation motivates the following definition.

Definition 16. *Let (σ, α) be a planar hypermap on the set of points $\{1, 2 \ldots, n\}$ and $M(\sigma, \alpha) = (\sigma', \alpha')$ its directed medial map on the set of points $\{1^-, 1^+, 2^-, 2^+, \ldots, n^-, n^+\}$. We call an m-coloring of the points $\{1^-, 1^+, 2^-, 2^+, \ldots, n^-, n^+\}$ a* legal coloring *if it satisfies the following conditions:*

1. *The endpoints of each edge $(i^+, \sigma(i)^-) \in \alpha'$ of $M(\sigma, \alpha)$ have the same color.*
2. *There is a coherent matching of the set $\{1^-, 1^+, 2^-, 2^+, \ldots, n^-, n^+\}$ such that each point is matched to a point of the same color.*

Definition 16 is motivated by the following observation.

Proposition 17. *Given a planar hypermap (σ, α) and its directed medial map $M(\sigma, \alpha)$ a coloring of the set of points of $M(\sigma, \alpha)$ is legal if and only if it is induced by a noncrossing Eulerian m-coloring of the edges of $M(\sigma, \alpha)$.*

Note that for a given m-coloring of the signed points, induced by a coloring of the edges in α', condition (2) may be independently verified at each vertex of σ'. This observation motivates the following definition.

Definition 18. *Let $(i_1^-, i_1^+, i_2^-, i_2^+, \ldots, i_k^-, i_k^+)$ be a cyclic signed permutation and let is fix an m-coloring of its points. We say that the* valence *of this colored cycle is number of coherent matchings of its points that match each point the a point of the same color.*

Now we are able to state the generalization of [14, Evaluation 6.9].

Theorem 19. *Let (σ, α) be a planar hypermap. Then, for a fixed positive integer m, we have*

$$m^{\kappa(\sigma,\alpha)} R(\sigma, \alpha; m, m) = \sum_{\lambda} \prod_{v \in \sigma'} \nu(v, \lambda).$$

Here the summation runs over all Eulerian m-colorings λ of the directed medial map $M(\sigma, \alpha) = (\sigma', \alpha')$, and for each vertex $v \in \sigma'$ the symbol $\nu(v, \lambda)$ represents the valence of v colored by the restriction of the point coloring induced by λ to the points of v.

Example 20. For maps (σ, α), there are essentially two types of vertices in the directed medial map $M(\sigma, \alpha)$: monochromatic vertices and vertices colored with two colors. We can recover the formula [14, Evaluation 6.9]:

$$m^{\kappa(\sigma,\alpha)} R(\sigma, \alpha; m, m) = \sum_{\lambda} 2^{m(\lambda)}$$

where $m(\lambda)$ is the number of monochromatic vertices. Evaluating this formula at $m = 2$ was used by Las Vergnas [23] to describe the exact power of 2 that divides $R(\sigma, \alpha; 2, 2)$ for a map (σ, α), or equivalently the evaluation of its Tutte polynomial at $(3, 3)$.

6 The characteristic polynomial of a hypermap

Proposition 21. *Let α be a permutation of $\{1, 2, \ldots, n\}$ with k cycles of lengths $c_1, \ldots, c_k$. Then the partially ordered set of all refinements of α, ordered by the refinement operation, is a direct product $[\mathrm{id}, \alpha] = \prod_{i=1}^{k} \mathrm{NC}(c_i)$. Here* id *is the identity permutation $(1)(2)\cdots(n)$ and $\mathrm{NC}(c_i)$ is the lattice of noncrossing partitions on c_i elements.*

Using the results in [27], the Möbius function $\mu(\beta, \alpha)$ of any interval $[\beta, \alpha]$ may be expressed in terms of Catalan numbers.

Definition 22. *Given a collection of hypermaps (σ, α) on the set of points $\{1, 2, \ldots, n\}$, we define its* characteristic polynomial *$\chi(\sigma, \alpha; t)$ by*

$$\chi(\sigma, \alpha; t) = \sum_{\beta \le \alpha} \mu(\mathrm{id}, \beta) \cdot t^{\kappa(\sigma,\beta) - \kappa(\sigma,\alpha)}.$$

When (σ, α) is a collection of maps on the set $\{1, 2, \ldots, n\}$, we get that $\chi(\sigma, \alpha; t)$ is the characteristic polynomial of its underlying graph. For a fixed collection of hypermaps (σ, α), let us define the function $X([\alpha_1, \alpha_2]; t)$ on the intervals of the partially ordered set of the refinements of α by

$$X([\alpha_1, \alpha_2]; t) = \sum_{\beta \in [\alpha_1, \alpha_2]} \mu(\alpha_1, \beta) \cdot t^{\kappa(\sigma,\beta)} \tag{6.1}$$

A Möbius inversion formula computation yields $\sum_{\beta \le \alpha} X([\beta, \alpha]; t) = t^{z(\sigma)}$. For maps, this computation implies that the chromatic polynomial $\kappa(\sigma, \alpha)\chi(\sigma, \alpha; t) = X([\mathrm{id}, \alpha]; t)$ is the number of ways to color the vertices using t colors such that adjacent vertices have the same color. This reasoning cannot be extended to arbitrary hypermaps, but it is possible to generalize it to hypermaps with hyperedges containing at most 3 points.

Theorem 23. *Let (σ, α) be a collection of hypermaps such that each cycle of α has length at most 3. Then for any positive integer n, the number $n^{\kappa(\sigma,\alpha)} \cdot \chi(\sigma, \alpha, n)$ is the number of ways to n-color the vertices of (σ, α) in such a way that no two vertices of the same color are incident to the same cycle of α.*

A completely analogous dual reasoning may be developed for the *flow polynomial* $C(\sigma, \alpha, ; t)$ of a collection of hypermaps (σ, α) on the set of points $\{1, 2, \ldots, n\}$, defined by

$$C(\sigma, \alpha; t) = \sum_{\beta \le \alpha} \mu(\beta, \alpha) t^{n + \kappa(\sigma,\beta) - z(\beta) - z(\sigma)}.$$

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