Séminaire Lotharingien de Combinatoire **91B** (2024)
Article #30, 12 pp.

Proceedings of the 36th Conference on Formal Power Series and Algebraic Combinatorics (Bochum)

Pop-Stack for Cambrian Lattices

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Abstract. The *pop-stack operator* of a finite lattice L is the map that sends each x in L to the meet of x with the set of elements covered by x. Using tools from representation theory, we provide simple Coxeter-theoretic and lattice-theoretic descriptions of the image of the pop-stack operator of a Cambrian lattice of a finite irreducible Coxeter group. When specialized to a bipartite Cambrian lattice of type A, this result settles a conjecture of Choi and Sun. We also settle a related enumerative conjecture of Defant and Williams. When L is an arbitrary lattice quotient of the weak order on W, we prove that the maximum size of a forward orbit under the pop-stack operator of L is at most the Coxeter number of W; when L is a Cambrian lattice, we provide an explicit construction to show that this maximum forward orbit size is actually equal to the Coxeter number.

Keywords: torsion classes, Cambrian lattice, weak order, pop-stack operator

1 Introduction

Let L be a finite lattice with meet operation $\wedge$ and join operation $\vee$. The *pop-stack operator* $\mathrm{pop}_L^{\downarrow}\colon L \to L$ and the *dual pop-stack operator* $\mathrm{pop}_L^{\uparrow}\colon L \to L$ are defined by

$$\mathrm{pop}_L^{\downarrow}(x) = x \wedge \left(\bigwedge\{y \mid y \lessdot x\}\right) \quad \text{and} \quad \mathrm{pop}_L^{\uparrow}(x) = x \vee \left(\bigvee\{y \mid x \lessdot y\}\right),$$

where we write $u \lessdot v$ to mean that u is covered by v in L. These operators have appeared in various contexts; they serve as both useful tools and objects of interest in their own right. When the lattice L is understood, we will omit subscripts and simply denote these operators by $\mathrm{pop}^{\downarrow}$ and $\mathrm{pop}^{\uparrow}$.

Given an element $x \in L$, the *forward orbit* of x under $\mathrm{pop}_L^{\downarrow}$ is the set

$$\mathcal{O}_L(x) = \left\{x, \mathrm{pop}_L^{\downarrow}(x), (\mathrm{pop}_L^{\downarrow})^2(x), \ldots\right\},$$

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where $(\mathrm{pop}_L^{\downarrow})^t$ is the map obtained by composing $\mathrm{pop}_L^{\downarrow}$ with itself t times. If t is sufficiently large, then $(\mathrm{pop}_L^{\downarrow})^t(x)$ is equal to the minimal element $\hat{0}$ of L (which is the unique fixed point of $\mathrm{pop}_L^{\downarrow}$). Thus, $|\mathcal{O}_L(x)| - 1$ is equal to the number of iterations of $\mathrm{pop}_L^{\downarrow}$ needed to send x to $\hat{0}$.

Given an interesting lattice L, one of the primary problems one can consider about its pop-stack operator is that of maximizing $\mathcal{O}_L(x)$. When L is the weak order on a finite irreducible Coxeter group W, Defant [7] proved that $\max_{x\in L} |\mathcal{O}_L(x)|$ is the Coxeter number h of W; in type A, this result was originally proven much earlier by Ungar [17]. Defant also studied this problem for ν-Tamari lattices in [6].

Defant and Williams [8] found that it is fruitful to study the image of the pop-stack operator when L is a semidistributive (or more generally, a semidistrim) lattice; this is because the image of $\mathrm{pop}^{\downarrow}$ has numerous interesting properties, some of which relate to a certain bijective *rowmotion operator* $\mathrm{row}\colon L \to L$. For example, $|\mathrm{pop}^{\downarrow}(L)|$ and $|\mathrm{pop}^{\uparrow}(L)|$ are both equal to the number of elements $x \in L$ such that $\mathrm{row}(x) \leq x$. The images of $\mathrm{pop}^{\downarrow}$ and $\mathrm{pop}^{\uparrow}$ are also naturally in bijection with the set of facets of a certain simplicial complex called the *canonical join complex* of L.

In our full article [3], we take a representation-theoretic perspective and consider a finite-dimensional basic algebra Λ over a field K. The set of torsion classes of finitely-generated (right) Λ-modules forms a lattice, denoted $\mathrm{tors}\,\Lambda$ [13]. While the pop-stack operator of $\mathrm{tors}\,\Lambda$ has already appeared (sometimes under different names) in the theory of lattices of torsion classes (see e.g. [1, 4, 9]; a longer list can be found in introduction of our full article [3]), it has primarily been used as a tool rather than a dynamical operator worthy of its own investigation. Our full article, on the other hand, studies the image and dynamical properties of the pop-stack operator of $\mathrm{tors}\,\Lambda$ in the case when $\mathrm{tors}\,\Lambda$ is finite. We show that applying the pop-stack operator and its dual to a torsion class corresponds to performing certain mutations on associated *2-term simple-minded collections*. We characterize the preimages of a prescribed torsion class under $\mathrm{pop}^{\downarrow}_{\mathrm{tors}\,\Lambda}$ and $\mathrm{pop}^{\uparrow}_{\mathrm{tors}\,\Lambda}$. As corollaries, we obtain descriptions of the elements of $\mathrm{tors}\,\Lambda$ that require exactly 1 or exactly 2 iterations of $\mathrm{pop}^{\downarrow}$ to reach $\hat{0}$.

When Λ is a Dynkin quiver (or more generally, a Dynkin species), the lattice $\mathrm{tors}\,\Lambda$ is isomorphic to a Cambrian lattice [12]. For the sake of remaining explicit and combinatorial, we will devote this extended abstract to the pop-stack operators of Cambrian lattices; we will also consider Cambrian lattices of arbitrary finite irreducible Coxeter groups (not just crystallographic). That said, some of our proofs, which we omit in this extended abstract, are heavily representation-theoretic.

Let c be a (standard) Coxeter element of a finite irreducible Coxeter group W. Let $\mathrm{Weak}(W)$ denote the (right) weak order on W. The set of c-sortable elements of W (see Section 2 for definitions) forms a sublattice Camb_c of $\mathrm{Weak}(W)$ called the *c-Cambrian lattice*. Hong [11] found a description of the image of the pop-stack operator on a Tamari lattice (a particular Cambrian lattice of type A), and Choi and Sun [5] found a similar description for the image of the pop-stack operator on a type B analogue of the Tamari lattice. Choi and Sun also conjectured a description of the image of the pop-stack operator on a type A Cambrian lattice associated to a bipartite Coxeter element.

In Section 3, we provide an explicit description of the image of the pop-stack operator on an arbitrary Cambrian lattice; we were only able to discover this description by thinking representation-theoretically (it involves *projective modules*), but we can state it in purely Coxeter-theoretic and lattice-theoretic terms. This characterization allows us to obtain a surprising dynamical result (see Theorem 2). When W is of type A and $c = c^\times$ is a bipartite Coxeter element, our description of the image of the pop-stack operator allows us to resolve the aforementioned conjecture of Choi and Sun [5]. We then construct a bijection from the image of $\mathrm{pop}^\downarrow_{\mathrm{Camb}_{c^\times}}$ to a certain set of Motzkin paths (Theorem 3); this allows us to resolve an enumerative conjecture of Defant and Williams [8]. This result provides an enumeration of the facets of the canonical join complex of a bipartite type A Cambrian lattice.

When $W_\equiv$ is a lattice quotient of the weak order on a finite irreducible Coxeter group W, we show that $\max_{x\in W_\equiv} |\mathcal{O}_{W_\equiv}(x)| \leq h$, where h is the Coxeter number of W. We prove that this inequality is actually an equality when $W_\equiv$ is the c-Cambrian lattice associated to a Coxeter element c of W.

Section 2 provides background on posets, lattices, Coxeter groups, and Cambrian lattices. Section 3 is devoted to the images of the pop-stack operators of Cambrian lattices, and Section 4 is devoted to studying maximum-sized orbits. In Section 5, we collect several ideas for future work.

2 Background

2.1 Posets and Lattices

Let P be a poset. For $x, y \in P$, we say y *covers* x and write $x \lessdot y$ if $x < y$ and there does not exist $z \in P$ such that $x < z < y$. The *dual* of P is the poset P^* with the same underlying set as P defined so that $x \leq y$ in P^* if and only if $y \leq x$ in P. A *lattice* is a poset L such that any two elements $x, y \in L$ have a greatest lower bound, which is called their *meet* and denoted by $x \wedge y$, and a least upper bound, which is called their *join* and denoted by $x \vee y$. We write $\bigwedge X$ and $\bigvee X$ for the meet and join, respectively, of a finite subset X of a lattice. Given lattices L and L', a *lattice homomorphism* is a map $\phi\colon L \to L'$ such that $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$ and $\phi(x \vee y) = \phi(x) \vee \phi(y)$ for all $x, y \in L$. We say L' is a *lattice quotient* if there is a surjective lattice homomorphism from L to L'.

Assume L is a finite lattice. Then L has a unique minimal element $\hat{0} = \bigwedge L$ and a unique maximal element $\hat{1} = \bigvee L$. An element $j \in L$ is called *join-irreducible* if it covers exactly one element of L. Dually, an element $m \in L$ is *meet-irreducible* if it is covered by exactly one element of L. A set $A \subseteq L$ is *join-irredundant* (resp. *meet-irredundant*) if $\bigvee A' < \bigvee A$ (resp. $\bigwedge A' > \bigwedge A$) for every proper subset A' of A. Let JIrr_L (resp. MIrr_L) be the set of join-irredundant (resp. meet-irredundant) subsets of L. The *canonical join representation* of an element $x \in L$ (if it exists) is the unique set $A \in \mathrm{JIrr}_L$ satisfying $x = \bigvee A$ with the property that for every $B \in \mathrm{JIrr}_L$ satisfying $x = \bigvee B$, there exist $a \in A$ and $b \in B$ such that $a \leq b$. Dually, the *canonical meet representation* of x (if it exists) is the unique set $A \in \mathrm{MIrr}_L$ satisfying $x = \bigwedge A$ with the property that for every $B \in \mathrm{MIrr}_L$ satisfying $x = \bigwedge B$, there exist $a \in A$ and $b \in B$ such that $a \geq b$.

We say L is *semidistributive* if for all $x, y, z \in L$, we have

$$x \wedge y = x \wedge z \implies x \wedge y = x \wedge (y \vee z) \quad \text{and} \quad x \vee y = x \vee z \implies x \vee y = x \vee (y \wedge z).$$

Suppose L is finite and semidistributive. It is known that every element v of L has a canonical join representation $\mathcal{D}(v)$ and a canonical meet representation $\mathcal{U}(v)$; in fact, the existence of both representations for every $v \in L$ is equivalent to semidistributivity. Moreover, the collection of canonical join representations (resp. canonical meet representations) of elements of L forms a simplicial complex called the *canonical join complex* (resp. *canonical meet complex*) of L. The canonical join complex and canonical meet complex of L are isomorphic simplicial complexes by [2, Corollary 5]. Moreover, the number of facets in each of these simplicial complexes is equal to both $|\mathrm{pop}_L^{\downarrow}(L)|$ and $|\mathrm{pop}_L^{\uparrow}(L)|$ by [8, Theorem 9.13]. Indeed, the facets of the canonical join complex (resp. canonical meet complex) of L are precisely the canonical meet representations (resp. canonical join representations) of the elements of $\mathrm{pop}_L^{\downarrow}(L)$ (resp. $\mathrm{pop}_L^{\uparrow}(L)$). Let $\mathbf{P}_L(q)$ be the generating function that counts the facets of the canonical join complex (equivalently, the canonical meet complex) according to their sizes. Then

$$\mathbf{P}_L(q) = \sum_{v \in \mathrm{pop}_L^{\downarrow}(L)} q^{|\mathcal{U}(v)|} = \sum_{v \in \mathrm{pop}_L^{\uparrow}(L)} q^{|\mathcal{D}(v)|}. \tag{2.1}$$

2.2 Coxeter groups

Let (W, S) be a finite Coxeter system. This means that S is a finite set and that W is a finite group with a presentation of the form $\langle S \mid (ss')^{m(s,s')} = e \text{ for all } s, s' \in S \rangle$, where e is the identity element of W and we have $m(s,s) = 1$ and $m(s,s') = m(s',s) \in \{2, 3, \ldots\}$ for all distinct $s, s' \in S$. (We often refer to just the Coxeter group W, tacitly assuming that this refers to the Coxeter system (W, S).)

The elements of S are called the *simple reflections*. A *reflection* is an element of W of the form wsw^{-1} for $s \in S$ and $w \in W$. The *Coxeter graph* of W is the graph Γ_W with vertex set S in which two simple reflections s and s' are connected by an edge whenever $m(s,s') \geq 3$; this edge is labeled with the number $m(s,s')$ if $m(s,s') \geq 4$. We will assume that W is *irreducible*, which means that Γ_W is connected.

A *reduced word* for an element $w \in W$ is a word over S that represents w and is as short as possible. The number of letters in a reduced word for w is called the *length* of w and is denoted $\ell(w)$. A *left inversion* of w is a reflection t such that $\ell(tw) < \ell(w)$. The *(right) weak order* is the partial order $\leq$ on W defined so that $u \leq v$ if and only if there exists a reduced word for v that has a reduced word for u as a prefix. Let $\mathrm{Weak}(W)$ denote the poset $(W, \leq)$. It is well known that $\mathrm{Weak}(W)$ is a lattice. A *descent* of an element $w \in W$ is a simple reflection $s \in S$ such that $ws < w$ in $\mathrm{Weak}(W)$. The *long element* of W, denoted $w_\circ$, is the unique maximal element of $\mathrm{Weak}(W)$.

A *(standard) Coxeter element* of W is an element c obtained by multiplying the simple reflections in some order (with each appearing once in the product). Thus, a reduced word for c is a word in which each simple reflection appears exactly once.

Fix a reduced word c for a Coxeter element c, and consider the infinite word $\mathsf{c}^\infty = \mathsf{c}^{(1)}\mathsf{c}^{(2)}\cdots$, where each $\mathsf{c}^{(k)}$ is a copy of c. Following Reading [15], we define the c*-sorting word* of an element $w \in W$ to be the reduced word $\mathrm{sort}_{\mathsf{c}}(w)$ for w that is lexicographically first as a subword of c^∞. Let $\mathbf{I}_c^{(k)}(w)$ be the set of simple reflections that are taken from $\mathsf{c}^{(k)}$ when we form $\mathrm{sort}_{\mathsf{c}}(w)$ as the lexicographically first subword of c^∞. Although $\mathbf{I}_c^{(k)}(w)$ depends on the Coxeter element c, it does not depend on the choice of the reduced word c. The element w is called *c-sortable* if $\mathbf{I}_c^{(1)}(w) \supseteq \mathbf{I}_c^{(2)}(w) \supseteq \cdots$. The set of c-sortable elements of W forms a sublattice of $\mathrm{Weak}(W)$ called the *c-Cambrian lattice*, which we denote by Camb_c.

For each $w \in W$, the set $\mathrm{Camb}_c \cap \{v \in W \mid v \leq w\}$ has a unique maximal element in the weak order; we denote this element by $\pi_\downarrow^c(w)$. The map $\pi_\downarrow^c$ is a surjective lattice homomorphism from $\mathrm{Weak}(W)$ to Camb_c, so Camb_c is a lattice quotient of $\mathrm{Weak}(W)$ [15]. According to [6, Theorem 3.2], we have $\mathrm{pop}^\downarrow_{\mathrm{Camb}_c} = \pi_\downarrow^c \circ \mathrm{pop}^\downarrow_{\mathrm{Weak}(W)}$.

3 The Image of Pop-Stack on a Cambrian Lattice

Let c be a Coxeter element of a finite irreducible Coxeter group W. Let $s_1, \ldots, s_n$ be the simple reflections of W; these are the elements that cover $\hat{0}$ in Camb_c. For $1 \leq i \leq n$, let

$$p_i = \bigvee \{w \in \mathrm{Camb}_c \mid s_i \leq w \text{ and } s_j \not\leq w \text{ for all } s_j \in S \setminus \{s_i\}\}.$$

Our main result describing the image of $\mathrm{pop}^\downarrow_{\mathrm{Camb}_c}$ is as follows.

Theorem 1 ([3])**.** *For $w \in \mathrm{Camb}_c$, the following are equivalent:*

1. *w is in the image of $\mathrm{pop}^\downarrow_{\mathrm{Camb}_c}$.*
2. *The descents of w all commute, and w has no left inversions in common with c^{-1}.*
3. *The interval $[\mathrm{pop}^\downarrow_{\mathrm{Camb}_c}(w), w]$ is Boolean, and $p_i \not\leq w$ for all $i \in [n]$.*

In [3], we apply Theorem 1 (together with further representation-theoretic arguments) to deduce the following result.

Theorem 2 ([3]). *Let c be a Coxeter element of a finite irreducible crystallographic Coxeter group W. If $w \in \mathrm{Camb}_c$ and $t \geq 0$, then*

$$(\mathrm{pop}^{\downarrow}_{\mathrm{Weak}(W)})^t(\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_c}(w)) = (\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_c})^{t+1}(w).$$

The Coxeter group A_n is the same as the symmetric group whose elements are permutations of the set $[n+1] = \{1, \ldots, n+1\}$. We will frequently represent a permutation $w \in A_n$ in *one-line notation* as the word $w(1) \cdots w(n+1)$. The simple reflections of A_n are $s_1, \ldots, s_n$, where s_i is the transposition that swaps i and $i+1$. The Coxeter graph Γ_{A_n} is a path that contains an (unlabeled) edge $\{s_i, s_{i+1}\}$ for each $i \in [n]$. Let

$$c^{\times}_{(n)} = c_1 c_2, \quad \text{where} \quad c_1 = \prod_{\substack{i \in [n] \\ i \text{ odd}}} s_i \quad \text{and} \quad c_2 = \prod_{\substack{i \in [n] \\ i \text{ even}}} s_i. \tag{3.1}$$

We refer to the Coxeter element $c^{\times}_{(n)}$ as a *bipartite Coxeter element*.

3.1 Arc diagrams

Let c denote an arbitrary Coxeter element of A_n. Define $\nu_c \colon \{2, \ldots, n\} \to \{\mathbf{A}, \mathbf{B}\}$ by

$$\nu_c(i) = \begin{cases} \mathbf{A} & \text{if } s_i \text{ precedes } s_{i-1} \text{ in every reduced word for } c; \\ \mathbf{B} & \text{if } s_{i-1} \text{ precedes } s_i \text{ in every reduced word for } c. \end{cases} \tag{3.2}$$

The map $c \mapsto \nu_c$ is a bijection from the set of Coxeter elements of A_n to the set of functions from $\{2, \ldots, n\}$ to $\{\mathbf{A}, \mathbf{B}\}$. Reading [14, Example 4.9] showed that a permutation $w \in A_n$ is c-sortable if and only if for all $i, j \in [n+1]$ such that $w(j+1) < w(i) < w(j)$, we have $\nu_c(i) = \mathbf{A}$ if and only if $j < i$. Arrange $n+1$ points along a horizontal line, and identify them with the numbers $1, \ldots, n+1$ from left to right. An *arc* is a curve that moves monotonically rightward from a point i to another point j (for some $i < j$), passing above or below each of the points $i+1, \ldots, j-1$. Two arcs are considered to be the same if they have the same endpoints and they pass above the same collection of numbered points. A *noncrossing arc diagram* (of type A_n) is a collection of arcs that can be drawn so that no two arcs have the same left endpoint or have the same right endpoint or cross in their interiors. We write $|\delta|$ for the number of arcs in a noncrossing arc diagram δ. Let AD_n be the set of noncrossing arc diagrams of type A_n.

Given a permutation $w \in A_n$, form a noncrossing arc diagram $\Delta(w) \in \mathrm{AD}_n$ as follows. For each index i such that $w(i) > w(i+1)$, draw an arc from $w(i+1)$ to $w(i)$ such that for each integer k satisfying $w(i+1) < k < w(i)$, the arc passes above (resp.

below) the point k if $i+1 < w^{-1}(k)$ (resp. $w^{-1}(k) < i$). This defines a map $\Delta\colon A_n \to \mathrm{AD}_n$, and it is straightforward to check that Δ is a bijection.

Given a Coxeter element c of A_n, say an arc $\mathfrak{a}$ with left endpoint i and right endpoint j is *c-sortable* if for every $k \in \{i+1, \dots, j-1\}$, $\mathfrak{a}$ passes above (resp. below) k if $\nu_c(k) = \mathbf{A}$ (resp. $\nu_c(k) = \mathbf{B}$). Note that for all $1 \le i < j \le n+1$, there is a unique c-sortable arc from i to j. Let $\mathrm{AD}(c) = \Delta(\mathrm{Camb}_c)$ be the set of noncrossing arc diagrams of c-sortable elements of A_n. It is immediate from Reading's characterization of c-sortable elements that a noncrossing arc diagram is in $\mathrm{AD}(c)$ if and only if all of its arcs are c-sortable. Hence, $\mathrm{AD}(c)$ is a simplicial complex whose vertices are the c-sortable arcs.

Cambrian lattices are semidistributive, so we can consider the canonical join complex and the canonical meet complex of Camb_c (and we know these simplicial complexes are isomorphic by [2, Corollary 5]). An element $v \in \mathrm{Camb}_c$ is join-irreducible if and only if it has exactly one descent, and this occurs if and only if $\Delta(v)$ contains a single arc. Therefore, Δ establishes a one-to-one correspondence between the join-irreducible elements of Camb_c and the c-sortable arcs. Then for each $w \in \mathrm{Camb}_c$, the noncrossing arc diagram $\Delta(w)$ corresponds to the canonical join representation of w. It follows that the simplicial complex $\mathrm{AD}(c)$ is isomorphic to the canonical join complex of Camb_c. Say a noncrossing arc diagram in $\mathrm{AD}(c)$ is *maximal* if it is a facet of $\mathrm{AD}(c)$. In other words, a noncrossing arc diagram in $\mathrm{AD}(c)$ is maximal if it is not properly contained in another noncrossing arc diagram in $\mathrm{AD}(c)$. Let $\mathrm{MAD}(c)$ denote the set of maximal noncrossing arc diagrams in $\mathrm{AD}(c)$.

The preceding discussion yields the identity

$$\mathbf{P}_{\mathrm{Camb}_c}(q) = \sum_{\delta \in \mathrm{MAD}(c)} q^{|\delta|}, \tag{3.3}$$

where $\mathbf{P}_{\mathrm{Camb}_c}(q)$ is the generating function defined in Equation (2.1). Defant and Williams conjectured [8, Conjecture 11.2] that

$$\sum_{n \ge 1} \mathbf{P}_{\mathrm{Camb}_{c^\times_{(n)}}}(q) z^n = \frac{1}{qz}\left(\frac{2}{1 - qz(1-2z) + \sqrt{1 + q^2z^2 - 2qz(1+2z)}} - 1\right) - 1. \tag{3.4}$$

The remainder of this section is devoted to stating the bijection that we use in [3] to prove this conjecture.

3.2 Motzkin paths

A *Motzkin path* is a lattice path in the plane that consists of up (i.e., $(1,1)$) steps, down (i.e., $(1,-1)$) steps, and horizontal (i.e., $(1,0)$) steps, starts at the origin, never passes below the horizontal axis, and ends on the horizontal axis. Let U, D, and H denote up, down, and horizontal steps, respectively. Given a word P over the alphabet $\{\mathtt{U}, \mathtt{D}, \mathtt{H}\}$, let

$\#_{\mathtt{U}}(P)$, $\#_{\mathtt{D}}(P)$, and $\#_{\mathtt{H}}(P)$ denote the number of $\mathtt{U}'s$, the number of $\mathtt{D}$'s, and the number of $\mathtt{H}$'s in P, respectively. We can think of a Moztkin path as a word M over the alphabet $\{\mathtt{U},\mathtt{D},\mathtt{H}\}$ such that $\#_{\mathtt{U}}(M) = \#_{\mathtt{D}}(M)$ and $\#_{\mathtt{U}}(P) \geq \#_{\mathtt{D}}(P)$ for every prefix P of M.

A *peak* of a Motzkin path M is a point (j,k) where an up step in M ends and a down step in M begins; the *height* of this peak is the number k. If we view M as a word over $\{\mathtt{U},\mathtt{D},\mathtt{H}\}$, then a peak corresponds to a consecutive occurrence of $\mathtt{UD}$, and the height of the peak is $\#_{\mathtt{U}}(P) - \#_{\mathtt{D}}(P)$, where P is the prefix of M that ends with the up step involved in the peak. The only peak of the Motzkin path at the bottom of Figure 1 is $(5,2)$.

Let $\overline{\mathcal{M}}_n$ be the set of Motzkin paths of length n that have no peaks of height 1. Let $\overline{\mathbf{M}}(q,z) = \sum_{n\geq 0}\sum_{M\in\overline{\mathcal{M}}_n} q^{\#_{\mathtt{U}}(M)}z^n$. In [3], we use straightforward enumerative techniques to show that

$$\overline{\mathbf{M}}(q,z) = \frac{2}{1-z+2qz^2+\sqrt{1-2z+(1-4q)z^2}}. \tag{3.5}$$

Using Equation (3.5), one can readily check that the expression on the right-hand side of Equation (3.4) is

$$\frac{1}{qz}\left(\overline{\mathbf{M}}(1/q,qz)-1\right)-1 = \sum_{n\geq 1}\sum_{M\in\overline{\mathcal{M}}_{n+1}} q^{n-\#_{\mathtt{U}}(M)}z^n.$$

Therefore, in order to prove Equation (3.4), it suffices (by Equation (3.3)) to exhibit a bijection $\Psi\colon \mathrm{MAD}(c^{\times}_{(n)}) \to \overline{\mathcal{M}}_{n+1}$ such that $|\delta| = n - \#_{\mathtt{U}}(\Psi(\delta))$ for every $\delta \in \mathrm{MAD}(c^{\times}_{(n)})$.

3.3 The bijection

Throughout the remainder of this section, fix a positive integer n, and write $c^{\times} = c^{\times}_{(n)}$. The map $\nu_{c^{\times}}\colon \{2,\ldots,n\} \to \{\mathbf{A},\mathbf{B}\}$ is such that $\nu_{c^{\times}}(i) = \mathbf{A}$ if i is odd and $\nu_{c^{\times}}(i) = \mathbf{B}$ if i is even.

Suppose $\delta \in \mathrm{MAD}(c^{\times})$. Let $\Psi(\delta)$ be the word $\mathtt{M}_1\cdots\mathtt{M}_{n+1}$, where for $1 \leq i \leq n+1$, we define

$$\mathtt{M}_i = \begin{cases} \mathtt{U} & \text{if } i \leq n \text{ and } i+1 \text{ is not the right endpoint of an arc in } \delta; \\ \mathtt{D} & \text{if } i \geq 2 \text{ and } i-1 \text{ is not the left endpoint of an arc in } \delta; \\ \mathtt{H} & \text{otherwise.} \end{cases} \tag{3.6}$$

In [3], we prove that $\Psi(\delta)$ is well defined in the sense that no letter in $\Psi(\delta)$ can be both $\mathtt{U}$ and $\mathtt{D}$. See Figure 1 for an illustration of Ψ.

We can now state the main theorem of this section; as mentioned at the end of Section 3.2, this theorem implies the identity Equation (3.4), thereby settling the conjecture of Defant and Williams.

Theorem 3 ([3])**.** *The map Ψ is a bijection from $\mathrm{MAD}(c^{\times})$ to $\overline{\mathcal{M}}_{n+1}$. For each $\delta \in \mathrm{MAD}(c^{\times})$, we have $\Psi(\delta) \in \overline{\mathcal{M}}_{n+1}$ and $|\delta| = n - \#_{\mathtt{U}}(\Psi(\delta))$.*

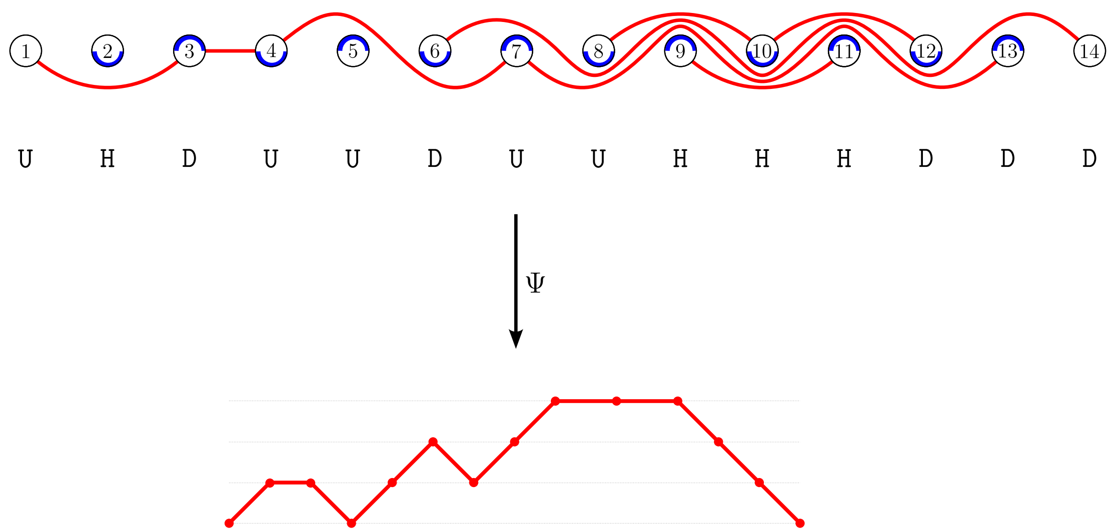

Figure 1: When $n = 13$, the map Ψ sends a noncrossing arc diagram of type A_{13} to a Motzkin path of length 14 with no peaks of height 1. For each $2 \leq i \leq 13$, a blue semicircle appears on the top (resp. bottom) of the circle containing i if $\nu_{c^\times}(i) = \mathbf{A}$ (resp. if $\nu_{c^\times}(i) = \mathbf{B}$). (The letters drawn below the noncrossing arc diagram represent the Moztkin path; they are not part of the noncrossing arc diagram.)

4 Maximum-Size Pop-Stack Orbits

As above, let (W, S) be a finite irreducible Coxeter system. The *Coxeter number* of W is the quantity $h = 2|T|/|S|$, where T is the set of reflections in W.

Theorem 4 ([3])**.** *If $W_\equiv$ is a lattice quotient of* $\mathrm{Weak}(W)$*, then*

$$\max_{x \in W_\equiv} |\mathcal{O}_{W_\equiv}(x)| \leq h.$$

The next theorem states that the inequality in Theorem 4 is tight for Cambrian lattices.

Theorem 5 ([3])**.** *For each Coxeter element c of W, we have*

$$\max_{x \in W_\equiv} |\mathcal{O}_{\mathrm{Camb}_c}(x)| = h.$$

The *spine* of Camb_c, denoted $\mathrm{spine}(\mathrm{Camb}_c)$, is the union of the maximum-length chains of Camb_c. Hohlweg, Lange, and Thomas [10] proved that $\mathrm{spine}(\mathrm{Camb}_c)$ is a distributive sublattice of Camb_c. Let us define $\mathbf{z}_c = (\mathrm{pop}^{\uparrow}_{\mathrm{spine}(\mathrm{Camb}_c)})^{h-1}(e)$ (where $e = \hat{0}$ is the identity element). In our full article [3], we prove Theorem 5 by showing that $|\mathcal{O}_{\mathrm{Camb}_c}(\mathbf{z}_c)| = h$. To do so, we make use of *combinatorial AR quivers* and the combinatorial aspects of the *c-sorting word* for the long element of W (we omit this proof here).

Example 1. Let W be the hyperoctahedral group B_3. Then $S = \{s_0, s_1, s_2\}$, and we have $m(s_0, s_1) = 4$, $m(s_1, s_2) = 3$, and $m(s_0, s_2) = 2$. Let $c = s_0s_2s_1$ and $c' = s_0s_1s_2$. The lattices Camb_c and $\mathrm{Camb}_{c'}$ are shown on the left and right, respectively, in Figure 2. The spine of each lattice has been colored in red. The Coxeter number of B_3 is $h = 6$. In the lattice on the left, we obtain the element $\mathbf{z}_c$, which is marked by a blue circle, by starting at the bottom element and applying the dual pop-stack operator *in the spine* $h - 1 = 5$ times. This amounts to traveling up the blue dotted curves. If we start at $\mathbf{z}_c$ and iteratively apply $\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_c}$, then we just travel down the same blue dotted curves (the fact that this happens for arbitrary Cambrian lattices is not obvious). This shows that $\mathcal{O}_{\mathrm{Camb}_c}(\mathbf{z}_c)$ is contained in the spine of Camb_c and that $|\mathcal{O}_{\mathrm{Camb}_c}(\mathbf{z}_c)| = h$. Similarly, $\mathbf{z}_{c'}$ is obtained by traveling up the blue dotted curves in the lattice on the right, and $\mathcal{O}_{\mathrm{Camb}_{c'}}(\mathbf{z}_{c'})$ has size h and is contained in the spine of $\mathrm{Camb}_{c'}$.

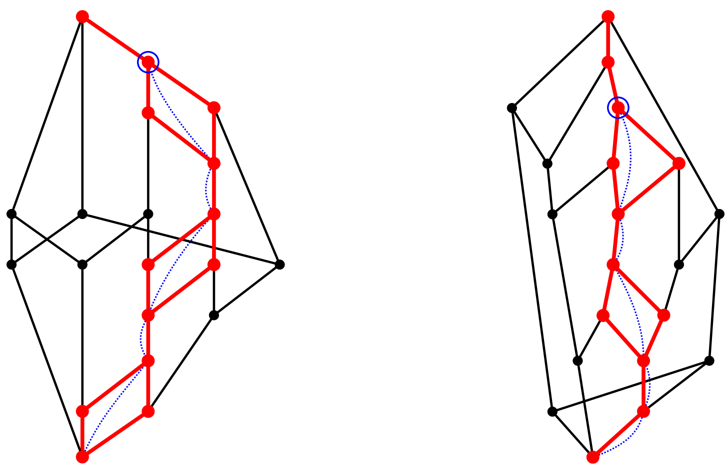

Figure 2: Two Cambrian lattices of type B_3. The spine of each lattice is in thick red. In each lattice, we have circled in blue an element whose forward orbit under the pop-stack operator has size $h = 6$.

5 Future Directions

Consider the *linear Coxeter element* $c^{\rightarrow} = s_1s_2 \cdots s_n$ of A_n. The Cambrian lattice $\mathrm{Camb}_{c^{\rightarrow}}$ is the $(n+1)$-st *Tamari lattice*. Hong [11] proved that the size of the image of $\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_{c^{\rightarrow}}}$ is the n-th *Motzkin number* (i.e., the number of Motzkin paths of length n). In Section 3, we determined the size of the image of $\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_{c^{\times}}}$, where $c^{\times} = c^{\times}_{(n)}$ is the bipartite Coxeter element of A_n defined in Equation (3.1). Using these formulas, one can verify that $|\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_{c^{\rightarrow}}}(\mathrm{Camb}_{c^{\rightarrow}})| \leq |\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_{c^{\times}}}(\mathrm{Camb}_{c^{\times}})|$. Numerical evidence has led us to conjecture that the linear and bipartite Coxeter elements are, in some sense, extremal with regard to the sizes of the images of pop-stack operators.

Conjecture 1. *For every Coxeter element c of A_n, we have*

$$|\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_{c^{\rightarrow}}}(\mathrm{Camb}_{c^{\rightarrow}})| \leq |\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_{c}}(\mathrm{Camb}_{c})| \leq |\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_{c^{\times}}}(\mathrm{Camb}_{c^{\times}})|.$$

Suppose L is a finite lattice, and let $v_L = \max_{x \in L} |\mathcal{O}_L(x)|$. Let

$$\mathrm{Y}_L = \{x \in L \mid |\mathcal{O}_L(x)| = v_L\}$$

be the set of elements of L whose forward orbits under $\mathrm{pop}^{\downarrow}_L$ attain the maximum possible size.

Let c be a Coxeter element of a finite irreducible Coxeter group W. We saw in Theorem 5 that the maximum possible size of the forward orbit of an element of Camb_c under $\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_c}$ is the Coxeter number h; however, we said nothing about the number of elements that actually attain this maximum. In the case when $W = A_n$ and c is the linear Coxeter element $c^{\rightarrow}$ (i.e., Camb_c is the $(n+1)$-st Tamari lattice), it is known that $|\mathrm{Y}_{\mathrm{Camb}_c}|$ is the $(n-1)$-st Catalan number [6]. It would be interesting to understand $|\mathrm{Y}_{\mathrm{Camb}_c}|$ for other Cambrian lattices Camb_c. In particular, we have the following conjecture.

Conjecture 2. *The number of elements of* $\mathrm{Camb}_{c^{\times}}$ *whose forward orbits under the pop-stack operator have size h is* 1 *if n is even and is* 2 *if n is odd.*

The original use of the term *pop-stack* comes from the setting where L is the weak order on A_n; in this case, Ungar proved that $\max_{x \in \mathrm{Weak}(A_n)} |\mathcal{O}_{\mathrm{Weak}(A_n)}(x)|$ is $n+1$ (which is the Coxeter number of A_n).

Question 1. *What can be said about $|\mathrm{Y}_{\mathrm{Weak}(A_n)}|$?*

Defant [7] proved that if W is a finite irreducible Coxeter group with Coxeter number h, then $\max_{x \in W} |\mathcal{O}_{\mathrm{Weak}(W)}(x)| = h$. In Theorem 4, we found that $\max_{x \in L} |\mathcal{O}_L(x)| \leq h$ whenever L is a lattice quotient of $\mathrm{Weak}(W)$, and we saw in Theorem 5 that this inequality is an equality whenever L is a Cambrian lattice. We are naturally led to ask the following questions.

Question 2. *Let W be a finite irreducible Coxeter group with Coxeter number h. For which lattice quotients L of* $\mathrm{Weak}(W)$ *is it the case that* $\max_{x \in L} |\mathcal{O}_L(x)| = h$*?*

Question 3. *Let L' be a lattice quotient of a finite lattice L. Is it necessarily the case that*

$$\max_{x' \in L'} |\mathcal{O}_{L'}(x')| \leq \max_{x \in L} |\mathcal{O}_L(x)|?$$

It would be interesting to see how much of our work on Cambrian lattices can be extended to more general families of lattices. For example, it could be interesting to study the pop-stack operators on *m-Cambrian lattices*, which were introduced by Stump, Thomas, and Williams [16].

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