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Antichains in the representation theory of finite Lattices

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Abstract. The interface between the combinatorics of a partially ordered set (poset) and the representation theory of its incidence algebra has been studied for a long time. Antichains naturally arise as encoding certain representations of combinatorial nature. In this paper, we study antichains with extra properties motivated by the search for good bases for the Coxeter matrix of a poset and the hope of categorifying its properties. We then turn to a concrete example where our methods apply nicely and solve a conjecture on the poset of cominuscule roots.

Résumé. Les interactions entre la combinatoire d'un ensemble ordonné et la théorie des représentations de son algèbre d'incidence forment un sujet bien étudié. Les antichaines apparaissent comme décrivant certaines représentations de nature combinatoire. Dans cet article nous étudions des antichaines satisfaisant des hypothèses de rigidité, motivé par la recherche de bonnes bases pour la matrice de Coxeter d'un poset et l'espoir de catégorifier ses propriétés. On traite ensuite un exemple concret où nos méthodes s'appliquent élégamment et nous permettent de résoudre une conjecture sur des ensembles ordonnés de racines cominuscules.

Keywords: antichain, Calabi-Yau category, Coxeter matrix, distributive lattice

1 Introduction

Fractionally Calabi-Yau posets are fascinating objects in part due to a hypothetical relation to product formulas due to Chapoton [3]. In combinatorics, many families of sets $(E_n)_{n\in\mathbb{N}}$ can be counted by product formulas $|E_n| = \Pi_{i=1}^{n}\frac{D-d_i}{d_i}$ where the sum of the numerator and denominator is constant and equal to D. Such families include the Catalan numbers, the number of alternating sign matrices, the West family and the Tamari intervals family. Chapoton's conjectural explanation is that there should exist a partial order on E_n whose derived category is equivalent to a triangulated Calabi-Yau category constructed from the data of D and the d_i coefficients. That category should be geometric in nature, a type of Fukaya category. This explanation provides with predictions about the Calabi-Yau dimension of the incidence algebra of the poset as well as its Coxeter polynomial that can be tested, *e.g.* with a computer.

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Observe that the binomial $\binom{m+n}{m}$ can be written as

$$\frac{m+n}{1}\frac{m+n-1}{2}\cdot\ldots\cdot\frac{m+1}{n} \tag{1.1}$$

where $D = m+n+1$. This is probably the most natural example of product formula as discussed above. The poset of order ideals of a product of total orders of length m and n has cardinality $\binom{n+m}{m}$. We write it $J_{m,n}$. Using our results on *boolean antichains* we are able to confirm Chapoton's prediction about the Calabi-Yau dimension of these posets.

Theorem 1. The bounded derived category of $J_{m,n}$ is $\frac{mn}{m+n+1}$-Calabi-Yau.

Moreover we provide a link to a type of Fukaya category.

Theorem 2. The bounded derived category of the algebra of the poset $J_{m,n}$ is equivalent to the partially wrapped Fukaya category of the $m-1^{th}$ symmetric power of the disc with $n+1$ marked points on the boundary, $\mathcal{W}(\mathrm{Sym}^{m-1}\mathbb{D}, \lambda_{n+1}^{(m-1)})$.

We prove this equivalence through an intermediate category, the derived category of the higher Auslander algebra A_{m+1}^{n-1} [5] which appears at the intersection of several hot topics in contemporary representation theory [6], [9]. This algebra is known to be equivalent to the Fukaya category which appears in Theorem 2 [4]. As a corollary to Theorem 1 we give a positive answer to the Chapoton-Yıldırım conjecture [11] on cominuscule root posets of type A and B.

Corollary 1. *The bounded derived category of cominuscule posets of type A, B are fractionally Calabi-Yau.*

2 Representations of partially ordered sets

Let $\mathbb{k}$ be a field and X a finite poset. Define its incidence algebra $\mathcal{A} = \mathcal{A}_{\mathbb{k}}(X)$ over $\mathbb{k}$ to be the $\mathbb{k}$ vector space with basis pairs (x,y) such that $x \leq y$ with multiplication defined by

$$(x,y)(z,t) = \begin{cases} (x,t) & \text{if } y = z, \\ 0 & \text{otherwise.} \end{cases}$$

For $x \in X$ we write $e_x = (x,x)$ the usual primitive idempotent. Then we have $1_{\mathcal{A}} = \sum_{x\in X} e_x$. Throughout this work we consider left modules over $\mathcal{A}$. For every element $x \in X$ the associated simple module is $S_x \cong \mathbb{k}$ with action $(y,t)\cdot 1_{\mathbb{k}} = 0$ unless $y = t = x$ in which case $e_x \cdot 1_{\mathbb{k}} = 1_{\mathbb{k}}$. Its projective cover $P_x = \mathcal{A}\cdot e_x$ has basis $\{(y,x)|y \leq x\}$. Its injective hull is the injective indecomposable $I_x = (e_x\cdot\mathcal{A})^*$ and has basis $\{(x,y)^*|x \leq y\}$. Recall that morphisms between the projective indecomposables are characterised by

$$\mathrm{Hom}_{\mathcal{A}}(P_x, P_y) = \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}e_x, \mathcal{A}e_y) \cong \begin{cases} e_x\mathcal{A}e_y \cong \mathbb{k} & \text{if } x \leq y, \\ 0 & \text{otherwise.} \end{cases}$$

We denote the canonical inclusion as $\iota_x^y : P_x \hookrightarrow P_y$ whenever $x \leq y$: this inclusion is nothing more than right multiplication by (x, y). More generally for any left $\mathcal{A}$-module M, we have $\mathrm{Hom}_{\mathcal{A}}(P_x, M) \cong e_x M$. This isomorphism makes the following diagram commute

$$\begin{tikzcd} f \arrow[d, mapsto] & \in & \mathrm{Hom}_{\mathcal{A}}(P_x, M) \arrow[d] & \mathrm{Hom}_{\mathcal{A}}(P_y, M) \arrow[l, "\circ\iota_x^y"'] \arrow[d] & \ni & g \arrow[d, mapsto] \\ f(e_x) & \in & e_x M & e_y M \arrow[l, "(x,y)\cdot"] & \ni & g(e_y) \end{tikzcd} \tag{2.1}$$

The total hom complex $\mathrm{Hom}^{\bullet}_{\mathcal{A}}(C, M)$ where C is a chain complex $C = ((C_n)_n, (\partial_n))$ of $\mathcal{A}$ modules and M is an $\mathcal{A}$-module, is the complex

$$\cdots \to \mathrm{Hom}_{\mathcal{A}}(C_n, M) \xrightarrow{\partial^*_{n+1}} \mathrm{Hom}_{\mathcal{A}}(C_{n+1}, M) \to \ldots$$

Assuming that $C_n = \bigoplus_{x \in S_n} P_x$ with $S_n \subseteq X$ and taking its cohomology gives shifted morphisms in the derived category $\mathrm{D^b}(\mathcal{A})$ [12, Lemma 3.7.10]:

$$\mathrm{H}^i(\mathrm{Hom}^{\bullet}_{\mathcal{A}}(C, M)) \cong \mathrm{Hom}_{\mathrm{D^b}(\mathcal{A})}(C, M[i]) \tag{2.2}$$

Moreover, using equation (2.1) we have an isomorphism of cochain complexes

$$\begin{tikzcd} \ldots \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(\bigoplus_{x \in S_n} P_x, M) \arrow[r, "\partial^*_{n+1}"] \arrow[d] & \mathrm{Hom}_{\mathcal{A}}(\bigoplus_{x \in S_{n+1}} P_x, M) \arrow[r] \arrow[d] & \ldots \\ \ldots \arrow[r] & \bigoplus_{x \in S_n} e_x M \arrow[r] & \bigoplus_{x \in S_{n+1}} e_x M \arrow[r] & \ldots \end{tikzcd} \tag{2.3}$$

The boundary maps of the bottom complex are linear combinations of left multiplication by elements (x, y) of the algebra with coefficients inherited from the top complex.

3 Doing homological algebra with antichains

3.1 Antichains

Let $(L, \wedge, \vee)$ be a finite lattice. We write $\hat{1}$ its maximum and $\hat{0}$ its minimum. Let C be an *antichain* in L *i.e.* a subset C of L that consists of pairwise incomparable elements of L. We say an antichain C is below an element α of L if for all $c \in C$, we have $c \leq \alpha$, and when needed we record this information in the notation C_α. Following [7, Proposition 2.1] we associate to an antichain $C = \{c_1, \dots, c_r\}$ the submodule $N_C = \sum_{i=1}^{r} A \cdot (c_i, \hat{1})$ of the projective $P_{\hat{1}}$ generated by the antichain. It follows directly from the same proposition that there is a one to one correspondence between antichains and submodules of $P_{\hat{1}}$. The *antichain module* associated to C is defined by $M_C := P_{\hat{1}}/N_C$. We will talk of antichain modules below $\alpha \in L$ by restricting to the sublattice $[\hat{0}, \alpha]$ of L. Then α is the greatest element of this lattice and there is a bijection between submodules of P_α and antichains below α. The corresponding modules will be denoted N_C^α and M_C^α. As our main example consider $a \leq b$ in L. The maxima of the set of elements of L which are strictly less than b but not above a for an antichain C and the antichain module below b associated to C has support the interval $[a, b]$. The corresponding antichain module is usually called an *interval module*. In the rest of the paper we identify intervals with their interval modules.

Lemma 1. Intervals are antichain modules.

With the conventions of the previous paragraph, morphisms between interval modules follow a simple rule

$$\mathrm{Hom}_{\mathcal{A}}([a,b],[c,d]) = \begin{cases} \Bbbk & \text{if } a \leq c \leq b \leq d, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

By [7, Theorem 2.2], for every antichain C of cardinal r of a lattice L the associated antichain module M_C has a projective resolution $\mathcal{P}_C$ of the form

$$0 \to P_r \to \cdots \to P_0 \to M_C \text{ where } P_0 = P_{\hat{1}} \text{ and } P_l = \bigoplus_{\substack{S \subseteq C \\ |S| = l}} P_{\wedge S} \text{ for } 1 \leq l \leq r.$$

Similarly, we obtain a projective resolution $\mathcal{P}_C^\alpha$ for the antichain module M_C^α below α.

The boundary maps are defined by fixing an arbitrary total ordering of elements in C and, in each degree, setting the following maps between the indecomposable summands of the source and target in each degree:

$$\begin{array}{ccc} P_{\wedge S} & \to & P_{\wedge T} \\ (x, \wedge S) & \mapsto & \begin{cases} (-1)^{|i|_S}(x, \wedge T) & \text{if } T \sqcup \{i\} = S, \\ 0 & \text{otherwise} \end{cases} \end{array} \tag{3.2}$$

for each $S = \{i_1, \dots, i_k\}$ and $(\wedge T, \wedge S) \in P_{\wedge S}$ where $|i|_S = |\{j \in S | j \leq i\}|$.

3.2 Boolean antichains

Note that in degree i of the projective resolution $\mathcal{P}_C^\alpha$ of M_C^α there are $\binom{r}{i}$ indecomposable components in the direct sum. If one forgets the modules, the complex has the shape of

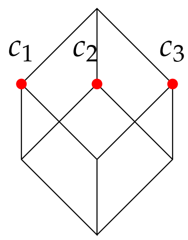


Figure 1: Boolean antichain

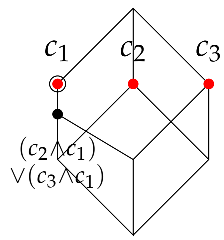


Figure 2: Strong not intersective

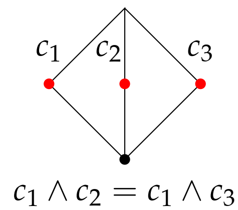


Figure 3: Intersective not strong

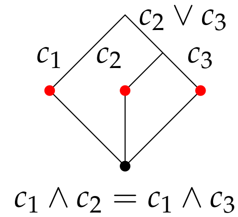


Figure 4: An antichain that is neither

the power set of C, however the indices of the modules in each degree are not necessarily in bijection with the lattice $(\mathcal{P}(C), \subseteq, \cup, \cap)$ (see Figures (1) to (4)).

To make this statement more precise, let us introduce four definitions regarding an antichain C.

Inclusive antichain. For all subsets S and S' of C, if $\wedge S \leq \wedge S'$ then $S' \subseteq S$.

Intersective antichain. For all subsets S and S' of C, we have $(\wedge S) \vee (\wedge S') = \wedge(S \cap S')$.

Strong antichain. For all S, S' subsets of C of the same cardinal, $\wedge S$ and $\wedge S'$ are incomparable *i.e.* if $\wedge S \leq \wedge S'$ then $S = S'$.

Boolean antichain C is both inclusive and intersective.

If C is below α, we say that C_α satisfies one of these properties if it satisfies it in the lattice $[0, \alpha]$. Note that intersectivity depends on the choice of a top element α whereas inclusivity and strength do not. Note also the following lemma.

Lemma 2. An antichain is **inclusive** if and only if it is **strong**.

Proof. The inclusion condition gives the strong antichain condition when the subsets S and S' have the same cardinal. To see the converse, assume that the antichain C is a strong antichain and let S and S' be two subsets of C such that $\wedge S' \leq \wedge S$. Suppose at first that $|S| + n = |S'|$ with $n > 0$. Then there exists $s_1, \ldots, s_n \in S' \setminus S$. Set $S'' = S \sqcup \{s_1, \ldots, s_n\}$. Because the inequalities $\wedge S' \leq \wedge S$ and $\wedge S' \leq \wedge\{s_1, \ldots, s_n\}$ hold, we have

$$\wedge S' \leq (\wedge S) \wedge (\wedge\{s_1, \ldots, s_n\}) = \wedge S''.$$

Because $|S'| = |S''|$, the strong incomparability condition yields $S' = S''$ hence $S \subseteq S'$. Next if $|S| = |S'| + n$, again with $n > 0$, then take $s_1, \ldots, s_n$ in $S \setminus S'$ and set $S'' = S \cup \{s_1, \ldots, s_n\}$. Then we have

$$\wedge S'' = (\wedge S') \wedge (\wedge\{s_1, \ldots, s_n\}) \leq \wedge S.$$

By the strong incomparability condition $S = S''$. Then $S' \subseteq S$ so $\wedge S' \geq \wedge S$ and thus $\wedge S' = \wedge S$. Using the first part of the proof we get $S = S'$. This contradicts the assumption on the integer n. □

Remark 1. If an antichain is strong, then it follows that for each n, the set $\{\wedge S | S \subseteq C \text{ with } |S| = n\}$ is an antichain.

Denote $\langle C \rangle^{\alpha}_{\vee,\wedge}$ the sublattice of $[0,\alpha]$ generated by the elements of C and α.

Lemma 3. An antichain is boolean if and only if the map

$$\begin{array}{ccc} (\mathcal{P}(C), \cap, \cup) & \xrightarrow{\phi} & (\langle C \rangle^{\alpha}_{\vee,\wedge}, \wedge, \vee) \\ S & \mapsto & \wedge S \end{array}$$

is a lattice anti-isomorphism.

Proof. Assume that the map ϕ is a lattice anti-isomorphism. Then C_α is intersective because ϕ sends $\cap$ to $\vee$. Now consider $S, S' \subseteq C$ such that $\wedge S \leq \wedge S'$. This is equivalent to the following equality

$$\wedge S = (\wedge S) \wedge (\wedge S').$$

The left hand side is equal to $\wedge(S \cup S')$ and because ϕ is a bijection, $S = S \cup S'$ meaning that $S' \subseteq S$. Thus C has the inclusion property as well. Conversely assume C has both the inclusion and the intersection property below α. The fact that ϕ sends $\cup$ to $\wedge$ is true for any subset of a lattice. The intersection property makes ϕ send $\cap$ to $\vee$. To see that ϕ is injective, note that if $\wedge S = \wedge S'$ then the inclusion property forces $S = S'$. To see that the map is surjective, notice that the image of ϕ, $\mathrm{Im}(\phi) = \{\wedge S | S \subseteq C\}$ is a lattice, using the properties we just exhibited. Moreover, any sublattice of L containing C contains $\mathrm{Im}(\phi)$. It is thus the sublattice of L generated by C, *i.e.*, $\langle C \rangle^{\alpha}_{\vee,\wedge} = \{\wedge S | S \subseteq C\}$ and ϕ is surjective. □

Recall that finite lattices are boolean if and only if they are isomorphic to the powerset of a finite set; this is the real reason for our terminology.

3.3 Morphisms

It is a recurring theme in algebra (and mathematics that use categories) that morphisms are more important than objects. For antichains with the properties we just introduced, we can compute morphisms between their corresponding objects in the derived category more easily as per the following two results.

Proposition 1. *Let C be an antichain of a lattice L and let $I \subseteq L$ be an interval. Suppose the set $E = \{S \subseteq C | \wedge S \in I\}$ is an interval of the lattice $\mathcal{P}(C)$. Then there exists at most one integer p such that $\mathrm{Hom}_{\mathrm{D^b}(\mathcal{A})}(M_C, I[p])$ is non zero. When such an integer exists, the hom space is one dimensional.*

Sketch of proof. Because the set E is an interval, we can show that the total hom complex has the shape:

$$0 \leftarrow \Bbbk \leftarrow \cdots \leftarrow \Bbbk^{\binom{|E|}{j}} \leftarrow \cdots \leftarrow \Bbbk \leftarrow 0. \tag{3.3}$$

Where $\Bbbk^{\binom{|E|}{j}}$ is the term in degree j. This is the shape of the simplicial resolution associated to the standard simplex. By reindexing the components of the boundary maps we show that these match the standard simplex resolution as well. In other words this is the Koszul complex $\bigotimes(\Bbbk \xrightarrow{id} \Bbbk)$. By the Künneth's formula [1, Chapter 6.3], it is thus either acyclic or concentrated in one degree when E contains only one element. □

According to the proof, there exists a non trivial morphism if and only if the set E is a singleton *i.e.* there exists a unique $S \subseteq C$ such that $\wedge S \in I$. In this case, the morphism is concentrated in degree $p = |S|$. When the antichain is boolean, we can show that the set E is always an interval which leads to a proof of the following theorem.

Theorem 3. Let C be a *boolean* antichain of a lattice L. Let $I \subseteq L$ be an interval. There exists at most one integer p such that $\mathrm{Hom}_{\mathrm{D^b}(\mathcal{A})}(M_C, I[p])$ is non zero. Moreover in this case it is of dimension 1.

Example 1. Consider the lattice in Figure 2 and the strong antichain $C = \{c_1, c_2, c_3\}$ below $\hat{1}$. Its associated module is the simple $S_{\hat{1}}$. Consider moreover the interval $I = [c_1 \wedge c_2, (c_1 \wedge c_2) \vee (c_1 \wedge c_3)]$. The set $E = \{S \subseteq C | \wedge S \in I\}$ is the singleton $\{\{c_1, c_2\}\}$ which is an interval. So Proposition 1 applies and $\dim \mathrm{Hom}_{\mathrm{D^b}}(\mathcal{P}_C, I[2]) = 1$ while for any other shift of the interval it is 0.

Example 2. Consider the lattice in Figure 1 and the antichain $C = \{c_1, c_2\}$ below $\hat{1}$. Its associated antichain module is the interval $[c_3, \hat{1}]$. Consider moreover the interval $I = [\hat{0}, c_1]$. Because the antichain is boolean, Theorem 3 applies. We can check that the set $E = \{S \subseteq C | \wedge S \in I\}$ is the interval $[c_1 \wedge c_2, c_1]$. In that case, $\dim \mathrm{Hom}_{\mathrm{D^b}}(\mathcal{P}_C, I[p]) = 0$ for all integer p as E is not a singleton.

4 Fractionally Calabi-Yau Lattices

For a finite poset with incidence matrix I, the Coxeter matrix is defined as $C = -I \times (I^{-1})^t$. If the poset is a lattice, then it is closely related to its rowmotion bijection [8]. The notion of Calabi-Yau categories was introduced by Kontsevich in the late nineties. A triangulated category $\mathcal{T}$ with a Serre functor $\mathbb{S}$ is said to be fractionally Calabi-Yau if there exists ℓ and d such that $\mathbb{S}^\ell$ is isomorphic as a functor to the suspension functor applied d times. We say that $\mathcal{T}$ is $\frac{d}{\ell}$-Calabi-Yau. When the triangulated category is the derived category of the incidence algebra of a poset, the action of the Serre functor on the Grothendieck group of the category $\mathcal{T}$ coincides with the opposite of the Coxeter matrix. We can safely say that the Serre functor categorifies the opposite of the Coxeter matrix. When the category $\mathcal{T}$ is fractionally Calabi-Yau, its Coxeter matrix has finite order and its characteristic polynomial is a product of cyclotomic polynomials. Both properties can be checked using a computer. In the case of posets with a unique maximal element or a unique minimal element, [10, Theorem 3.1] enables one to prove the fractional Calabi-Yau property by looking at the action of the Serre functor on the projective indecomposable modules only. However, for a given poset the computation itself is still in general very hard in general, see [2] and [11] for example. Using **strong antichains** as described previously, we are able to provide a relaxation of [10, Theorem 3.1] to help overcome that difficulty.

Theorem 4. Let L be a finite lattice, let m and n be integers and let $(C_\alpha)_{\alpha\in L}$ be a family of antichains in L. For all $\alpha \in L$, consider the following assumptions.

1. The antichain C_α is below α.
2. The module M_C^α is non zero and there is an isomorphism $\mathbb{S}^n\, M_{C_\alpha}^\alpha \cong M_{C_\alpha}^\alpha[m]$ in $\mathrm{D^b}(\mathcal{A})$.
3. The antichain C_α is strong.

If there exists a family of antichains $(C_\alpha)_{\alpha\in L}$ satisfying these assumptions then $D^b(\mathcal{A})$ is $\frac{m}{n}$-Calabi-Yau.

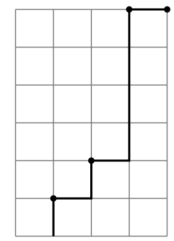
Remark 2. When $C_\alpha = \varnothing$ for all α, we recover [10, Theorem 3.1]. If C_α is the set of all the elements covered by α then $M_C^\alpha = S_\alpha$. Such antichains will often be strong in the examples that we consider. When it is the case the theorem can also be applied to a family of modules which combines some projective indecomposables and some simples. As we will see, the theorem can be applied to less obvious candidates as well.

5 Application: The lattice of order ideals of a grid and its enhancements

In this section we discuss the lattice $J_{m,n}$ in more details and we apply Theorems 3 and 4 on non trivial families of antichain modules in order to prove Theorems 1 and 2 of the introduction.

5.1 Families of antichains

Recall that an order ideal I of a poset P is a subset $I \subseteq P$ which is downward closed, *i.e.* if $x \in I$ and $y \leq x$ then $y \in I$. Order ideals of a poset can be ordered by inclusion and form a distributive lattice when equipped with the union and the intersection of subsets.



We now consider an element I of $J_{m,n}$ *i.e.* an order ideal of the product of total orders $[m] \times [n]$. We can draw the ideal I as a path in an $m \times n$ grid as in the figure on the right. The elements of the order ideal are the points of the grid which lie below the path in the picture. Because I is closed downward, counting the number of points in each column that belong to I, say with increasing first value, gives a monotone sequence which completely determines the ideal. We thus obtain a bijection

$$J_{m,n} \cong \{(a_1, \ldots, a_m) | a_i \in \{0, \ldots, n\} \text{ and } a_1 \leq \ldots, \leq a_n\} \tag{5.1}$$

with non decreasing sequences. If the second set is equipped with term wise comparison this is an isomorphism of posets. We call these non decreasing sequences *partitions*. They can also be written as $(\lambda_1^{\mu_1}, \ldots, \lambda_r^{\mu_r})$ with $\sum_i \mu_i = m$, where μ_i encodes the multiplicity of the value λ_i and $\lambda_i \neq \lambda_j$ if $i \neq j$.

To apply Theorems 3 and 4, we consider several antichains below x for each $x \in J_{m,n}$. These antichains can be encoded as or *enhancements* of x.

Definition 1. An *enhanced partition* is a sequence $(\lambda_1^{\mu_1}, \ldots, \lambda_r^{\mu_r} | n^{\mu_{r+1}})$ where multiplicities sum to m. We allow $\mu_{r+1} = 0$. If $\mu_{r+1} \neq 0$ we say the partition is strictly enhanced. A partition with $\mu_{r+1} = 0$ is called plain. Call $E_{m,n}$ the set of enhanced partitions. We easily count $\binom{m+n+1}{m}$ enhanced partitions.

For an enhanced partition $\alpha = (\lambda_1^{\mu_1}, \ldots, \lambda_r^{\mu_r} | n^{\mu_{r+1}})$ define the *mutable coefficients* to be $S_\alpha = \{\epsilon, \ldots, r\}$ the indices corresponding to nonzero coefficients. The number ϵ is either 1 or 2. Please remark that this excludes the coefficients beyond the enhancement bar. Similarly, we can define a *left enhanced partition* $(0^{\mu_0} | \lambda_1^{\mu_1}, \ldots, \lambda_r^{\mu_r})$.

Definition 2. Take an enhanced partition $\alpha = (\lambda_1^{\mu_1}, \ldots, \lambda_r^{\mu_r} | n^{\mu_{r+1}})$. For any subset J of S_α define a new partition $q_J(\alpha) = ((\lambda'_1)^{\mu_1}, \ldots, (\lambda'_r)^{\mu_r} | n^{\mu_{r+1}})$ by

$$\lambda'_i = \begin{cases} \lambda_i - 1 & \text{if } i \in J \\ \lambda_i & \text{otherwise.} \end{cases}$$

Consider now the set $C_\alpha = \{q_i(\alpha) | i \in S_\alpha\}$. Because $q_i(\alpha)$ and $q_j(\alpha)$ differ from α at different indices, their associated plain partitions form an antichain. We denote $\mathcal{P}_\alpha$ the resolution associated to it. Here is another family of transformations which leads to antichains.

Definition 3. Let $\alpha = (\lambda_1^{\mu_1}, \ldots, \lambda_r^{\mu_r} | n^{\mu_{r+1}})$ and take $i \in S_\alpha - \{r\}$. We set

$$p_i(\alpha) = \begin{cases} (0^1, \lambda_1^{\mu_1 - 1}, \lambda_2^{\mu_2} \ldots | n^{\mu_{n+1}}) & \text{if } i = 0, \\ (\lambda_1^{\mu_1} \ldots \lambda_i^{\mu_i + 1}, \lambda_{i+1}^{\mu_{i+1} - 1} \ldots | n^{\mu_{n+1}}) & \text{otherwise.} \end{cases} \tag{5.2}$$

For $J = (j_1, \ldots, j_k)$ a sequence of elements $S_\alpha - \{r\}$, set $p_J = p_{j_1} \circ \cdots \circ p_{j_k}$.

For all enhanced partition α consider the set $C_\alpha^\vee = \{p_i(\alpha) | i \in \{0, \dots, r-1\}$ or $i = r$ if $\mu_{r+1>0}\}$. The corresponding plain partitions form an antichain below α. It is easy to prove that C_α and $C_\alpha^\vee$ are boolean antichains below α. We recover these two results of [11]. The second one is an easy categorification of [11, Proposition 4.2].

Proposition 2 ([11, Proposition 2.13])**.** *Let α be a right enhanced partition. Then $\mathcal{P}_\alpha$ is a projective resolution of the interval $[f(\alpha), \alpha]$ where the function f is defined by*

$$f : (\lambda_1^{\mu_1}, \dots, \lambda_r^{\mu_r} | n^{\mu_{r+1}}) \mapsto (0^{\mu_1 - 1} | \lambda_1^{\mu_2}, \dots, \lambda_r^{\mu_{r+1}}). \tag{5.3}$$

Proposition 3. *Let α be a right enhanced partition. Then $\mathbb{S}^{m+n+1}(\mathcal{P}_\alpha) \cong \mathcal{P}_\alpha[mn]$.*

Combining this with Theorem 4 we obtain a proof of Theorem 1. What remains of this section is dedicated to the description of the full subcategory of the derived category $D^b(J_{m,n})$ whose objects are the $\mathcal{P}_\alpha$ with $\alpha \in E_{m,n}$. We write it $\mathcal{Y}_{m,n}$.

5.2 Elementary morphisms

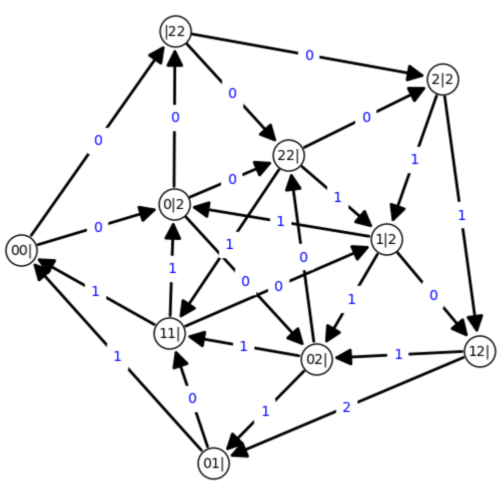


Figure 5: Graph of the category $\mathcal{Y}_{2,2}$

We first describe the morphisms $\phi : \mathcal{P}_\alpha \to \mathcal{P}_\beta$ using the combinatorics of the partitions introduced in the previous subsection combined with Theorem 3.

Proposition 4. *Let $\phi : \mathcal{P}_\alpha \to \mathcal{P}_\beta[i]$ be a non zero morphism in $\mathcal{Y}_{m,n}$. Then there exists a unique subset J of S_α such that ϕ factors through $\mathcal{P}_{q_J(\alpha)}[i]$ and $|J| = i$ completing the following diagram.*

$$\begin{tikzcd} \mathcal{P}_\alpha \arrow[rr, "\phi"] \arrow[dr, "u"'] & & \mathcal{P}_\beta[|J|] \\ & \mathcal{P}_{q_J(\alpha)}[|J|] \arrow[ur, "\mu"'] & \end{tikzcd}$$

If we order $J = \{j_1, \dots j_k\}$ such that $j_t < j_{t+1}$, then the extension $u : \mathcal{P}_\alpha \to \mathcal{P}_{q_J(\alpha)}$ decomposes as $\mathcal{P}_\alpha \to \mathcal{P}_{q_{\{j_1\}}(\alpha)}[1] \to \cdots \to \mathcal{P}_{q_J(\alpha)}[|J|]$ up to signs. Similarly there exists a sequence $d_0, \dots, d_s$ such that for all $0 \le j < s$, $0 \le d_j < m_{j+1}$, for $j = s$, $-m_s < d_s \le m_{s+1}$, $\alpha = p_s^{d_s} \circ \dots \circ p_0^{d_0}(\beta)$ and the degree 0 morphism $\mu : \mathcal{P}_{q_J(\alpha)} \to \mathcal{P}_\beta$ factors through each of the objects associated to the intermediate partitions.

Theorem 2 follows from this proposition. More precisely, we construct a tilting complex with the set of plain partitions and show that its endomorphism algebra is the higher Auslander algebra A_{n+1}^{m-1}.

5.3 Through the lens of configurations

Using a clever bijection of Yıldırım we can give a more satisfying description of the category $\mathcal{Y}_{m,n}$. Let $\mathcal{Z} = \{-m, \dots, -1, 0, 1, \dots, n\}$ be a set of representatives of $\mathbb{Z}/(m+n)\mathbb{Z}$. A *configuration* $C = \{c_1 < \cdots < c_m\}$ is a strictly increasing sequence of m elements in $\mathcal{Z}$. We write $C_{m,n}$ the set of configurations of length m on $\mathcal{Z}$. It is easy to see that $|C_{m,n}| = \binom{m+n+1}{m}$. Given a partition α we can construct a configuration containing α's coefficients in its *nonnegative side* and encoding the *multiplicities* of α in its *negative side*. Write x_i to record the index of the last occurrence of the i^{th} coefficient. It will be called the *ending index*. The negative side is thought of as the indices of the elements of the sequence α but with a minus sign. Remove the opposite of the ending index of each coefficient. Call the resulting configuration the *right configuration* associated to α, write it R_α. We denote ϕ the map sending α to R_α.

Example 3. Take $n = 7$, $m = 5$ and consider the partitions $a = (0, 2, 3, 7|7)$. For this partition, $r = 4$ and coefficients end at indices 1, 2, 3 and 4. The associated right configuration is $\{-5 < 0 < 2 < 3 < 7\}$, containing the values $0, 2, 3, 7$ and omitting the opposite of the ending coefficients $-1, -2, -3$ and -4. It is represented in an abacus as follows:

-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
•					•		•	•				•

Note that an enhanced partition is plain if and only if the column $-m$ of its abacus is empty.

Proposition 5 ([11, Proposition 3.3])**.** *The map ϕ is a bijection between $E_{m,n}$ and $C_{m,n}$.*

Indeed, partitions are entirely determined by their coefficients and multiplicities which can be recovered from the positive elements of a configurations and its negative gaps.

Definition 4. Consider two sequences $(x_1, \dots, x_k)$ and $(y_1, \dots, y_k)$ of length k in $[\![0, N]\!]$. Define a sequence on $\mathbb{Z}$ by setting $y_l = y_r + q \cdot N$ where $l = g \times k + r$ is the euclidian division. We say that they *interpolate circularly* if for all integers h, f and l such that $h \equiv l \equiv f - 1 \mod [k]$ we have $x_h \leq y_l < x_f$.

Example 4. The configurations $\{-5 < 0 < 2 < 3 < 7\}$ and $\{-5 < 0 < 2 < 3 < 6\}$ interpolate circularly.

Definition 5. Let n and m be integers. Define the category $\mathcal{I}_{m,n}$ as follows :

- set $\mathrm{Ob}(\mathcal{I}_{m,n}) = \{$increasing sequences of length m in $[\![0, m+n]\!]\}$;
- given two increasing sequences a and b in $\mathrm{Ob}(\mathcal{I}_{m,n})$, set

$$\mathcal{I}_{m,n}(a, b) = \begin{cases} \Bbbk & \text{if } a \text{ and } b \text{ interpolate circularly,} \\ 0 & \text{otherwise.} \end{cases}$$

Conjecture 1. *The categories $\mathcal{Y}_{m,n}$ and $\mathcal{I}_{m,n}$ are equivalent.*

Theorem 2 also follows from this conjecture, however its proof only involves part of $\mathcal{Y}_{m,n}$ and A_{n+1}^{m-1} is Morita equivalent to the corresponding subcategory of $\mathcal{I}_{m,n}$.

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