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Triangular (q,t)-Schröder Polynomials and Khovanov-Rozansky Homology

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Abstract. We define generalized Schröder polynomials $S_\lambda(q,t,a)$ for triangular partitions and prove that these polynomials recover the triangular (q,t)-Catalan polynomials of [2] at $a = 0$. Moreover, we show that the Poincaré polynomials of the reduced Khovanov-Rozansky homology of Coxeter knots of these partitions are given by $S_\lambda(q,t,a)$. Finally, combined with recent results in [8], we compute the Poincaré polynomial of the $(d, dnm+1)$-cable of the (n,m)-torus knot, thus proving a special case of the Oblomkov-Rassmusen-Shende conjecture [16, 18] for generic unibranched planar curves with two Puiseux pairs.

1 Introduction

A fundamental pursuit in knot theory for the last century has been the classification of knots and links. One particularly effective method has been the study of certain homology theories that realize the knot as certain chain complexes, whose Poincaré polynomials, which compute the graded dimensions of the homology groups, are then used as knot invariants. One such especially celebrated homology theory is (reduced) *Khovanov-Rozansky homology*. This tri-graded theory associates to each link L a polynomial in three variable $P_L^{KR}(q,t,a)$. It turns out that computing these polynomials explicitly is very difficult, and the pursuit of a closed form for them has spurned a remarkable volume of deep and surprising results bridging combinatorics with low dimensional topology and algebraic geometry. One of the only cases where these polynomials are explicitly known is the case of *torus knots and links*, where by transforming these knots to certain binary sequences and defining a family of recursions, Hogancamp and Mellit were able to compute explicit solutions [11, 15]. The connections between these polynomials and (q,t)-Catalan combinatorics have been deeply established [5, 12, 10], with the $a = 0$

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specialization of $P_L^{KR}(q,t,a)$ for a torus knot recovering the *(q,t)-Catalan polynomial* and higher a powers equaling *(q,t)-Schröder polynomials*.

One of our first main results is the generalization of (q,t)-Schröder polynomials to the context of *triangular partitions*, which are defined as maximal partitions that fit under a line of arbitrary slope (i.e. certain Dyck paths under lines with non-integer intercepts.). These partitions were thoroughly studied by Bergeron and Mazin [1]. In particular, our triangular Schröder polynomial recovers at $a = 0$ the triangular (q,t)-Catalan polynomials studied in [1] that appeared in the generalized shuffle theorem under any line [2]. Our construction relies on certain recursions introduced by Gorsky-Mazin-Vazirani [9] using so called (m,n)-*invariant subsets*, which allow us to produce certain binary sequences from the triangular partitions and compute them using the recursions.

Recently, Oblomkov and Rozansky considered *Coxeter links*, which contain torus knots and links as special cases, and identified their homology with certain sections on the flag Hilbert scheme [17]. Even more recently, Galashin and Lam introduced a family of knots that arise directly from certain monotone paths on an $m \times n$ grid and proved that all *monotone links* are Coxeter. Thus, since the Gorsky-Mazin-Vazirani recursions agree with the Hogancamp-Mellit recursions, using the results above we prove that the Poincaré polynomial of the reduced Khovanov-Rozansky homology of Coxeter knots arising from triangular partitions is precisely our triangular Schröder polynomial.

The next natural family of links to try to understand are cabled torus knots. Informally, a *cabled knot* is a knot within a knot, so that the string that makes up the knot locally carries a smaller knot on it. A highly nontrivial and celebrated conjecture due to Oblomkov-Rasmussen-Shende, relates the Khovanov-Rozansky homology of algebraic links to certain plane curve singularities on the Hilbert scheme of points. Combining results in [8] with our previous results above, we compute the Poincaré polynomial for a certain family of cabled knots, proving a special case of the *Oblomkov-Rasmussen-Shende conjecture* for unibranched planar curves with two Puiseux pairs.

2 Background and Definitions

2.1 Recursions for the Poincaré Series of Link Homology

In [11] the third author and Anton Mellit introduced a recursive method for computing the Poincaré series of the reduced Khovanov-Rozansky homology of torus links. Given two finite binary sequences $\mathbf{u}$ and $\mathbf{v}$ with the same number of 1's, they introduced the power series $R_{\mathbf{u},\mathbf{v}}(q,t,a)$ via the following recursive relations:

$$R_{0\mathbf{u},0\mathbf{v}} = t^{-|\mathbf{u}|}R_{\mathbf{u}1,\mathbf{v}1} + qt^{-|\mathbf{u}|}R_{\mathbf{u}0,\mathbf{v}0}, \qquad R_{1\mathbf{u},0\mathbf{v}} = R_{\mathbf{u}1,\mathbf{v}}, \qquad R_{\emptyset,0^n} = \left(\frac{1+a}{1-q}\right)^n,$$

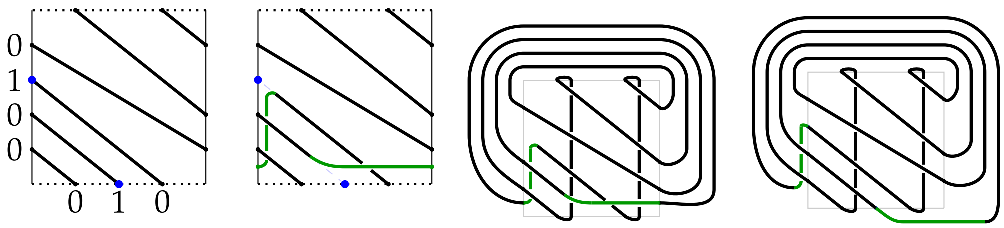


Figure 1: (Left) Steps 1-4 in constructing the link $L_{\mathbf{u},\mathbf{v}}$ for $\mathbf{u} = \mathbf{0010}$ and $\mathbf{v} = \mathbf{010}$. (Right) Two diagrams for the knot $L_{0010,010}$. Middle right: the closure of the diagram to its left (step 5). Far right: an equivalent closure considered in [11].

$$R_{1\mathbf{u},1\mathbf{v}} = (t^{|\mathbf{u}|} + a)R_{\mathbf{u},\mathbf{v}}, \qquad R_{0\mathbf{u},1\mathbf{v}} = R_{\mathbf{u},\mathbf{v}1}, \qquad R_{0^m,\varnothing} = \left(\frac{1+a}{1-q}\right)^m,$$

where $|\mathbf{u}|$ is the number of 1's in $\mathbf{u}$ and $R_{\varnothing,\varnothing} = 1$. Let $l(\mathbf{u})$ denote the length of $\mathbf{u}$.

Theorem 1 ([11])**.** *Let (n,m) be any positive integers. The Poincaré series of the reduced Khovanov-Rozansky homology of the (n,m)-torus link is given by*

$$P^{KR}_{L_{n,m}}(q,t,a) = (1-q)R_{0^n,0^m} = R_{0^{n-1}1,0^{m-1}1}.$$

Furthermore, it follows from their construction that for $|\mathbf{u}| = |\mathbf{v}| = 1$, the recurrence applied to $R_{\mathbf{u},\mathbf{v}}(q,t,a)$ will terminate and compute the Poincaré series of the reduced Khovanov-Rozansky homology for the link $L_{\mathbf{u},\mathbf{v}}$constructed as follows:
Step 1: Mark $\ell(\mathbf{v})$ points on the bottom edge and $\ell(\mathbf{u})$ points on the left edge of $[0,1]^2$, labeled with the sequences starting from the bottom left corner. Mark also the points on the top and right edges directly across from marked points labeled by 0.
Step 2: Starting with the lowest point on the left edge and leftmost point on the bottom edge, connect the dots with diagonal non-intersecting lines until all points are matched.
Step 3: Erase the tail of the line connected to the point labeled 1 on the bottom wall and connect it to right side of $[0,1]^2$, going *above* all other strands in the process.
Step 4: Erase the tail of the line connected to the point labeled 1 on the left wall, pass it *underneath* all other strands beneath it, and connect it once again to the left wall, but now in the first position, directly across the new marked point created in Step 3.
Step 5: Close the diagram by identifying the edges in the usual way for a torus.

Example 2. Consider $\mathbf{u} = \mathbf{0010}$ and $\mathbf{v} = \mathbf{010}$ with lengths $\ell(\mathbf{u}) = \mathbf{4}$, $\ell(\mathbf{v}) = \mathbf{3}$, and with $|\mathbf{u}| = |\mathbf{v}| = \mathbf{1}$. Steps 1 and 2 followed by 3 and 4 will yield the left two diagrams in Figure 1. It's closure, Step 5, is the third diagram. Iteratively applying the recurrence we see that the Poincare series of the KhR-homology of $L_{\mathbf{u},\mathbf{v}}$ is then equal to:

$$\begin{aligned} R_{0010,010} &= t^{-1}R_{0101,101} + qt^{-1}R_{0100,100} = \dots \\ &= t^{-1}(t+a)(a)R_{\emptyset,\emptyset} + (qt^{-1})^2(a)R_{\emptyset,\emptyset} + (qt^{-1})(t^{-1})(t+a)(a)R_{0,\emptyset} \\ &= a(t^{-1}(t+a) + (qt^{-1})^2) + (qt^{-1})(t^{-1})(t+a)(a)\frac{(1+a)}{1-q}. \end{aligned}$$

2.2 Invariant Subsets and Dyck Paths

Given positive integers m and n, an (n,m)-***Dyck path*** is a lattice path from $(m,0)$ to $(0,n)$ consisting exclusively of north and west steps that stays weakly below the diagonal line $y = n - \frac{n}{m}x$. Indexing each cell by its top right lattice point, for any such choice of (n,m) we define an ***Anderson filling*** on each of the cells of the lattice via the function $\gamma : \mathbb{Z}^2 \to \mathbb{Z}$ by $\gamma(x,y) = mn - nx - my$.

Definition 3. A subset $\Delta \subset \mathbb{Z}_{\geq 0}$ is called (n,m)-***invariant*** if $\Delta + n \subset \Delta$ and $\Delta + m \subset \Delta$. Let $I_{n,m}$ denote the set of all (n,m)-invariant subsets. In addition, an (n,m)-invariant subset Δ is called 0-***normalized*** if $0 \in \Delta$. We will use the notation $I^0_{n,m}$ for the set of 0-normalized (n,m)-invariant subsets.

If n and m are relatively prime then the set of (n,m)-Dyck paths is in a natural bijection with the set of 0-normalized (n,m)-invariant subsets. Namely, given an (n,m)-Dyck path D let $\mathrm{Gaps}(D)$ be the set of positive Anderson labels corresponding to the cells above D (positivity of a label is equivalent to the cell fitting under the diagonal). The corresponding 0-normalized (n,m)-invariant subset is given by $\Delta(D) = \mathbb{Z}_{\geq 0} \setminus \mathrm{Gaps}(D)$. It is not hard to see that this defines a bijection.

The rational (q,t)-Catalan polynomials and the rational Schröder polynomials are usually defined in terms of the (n,m)-Dyck and Schröder paths. However, it is more suitable for us to follow [6, 7] and define these polynomials in terms of the invariant subsets. The two approaches are equivalent due to the bijection described above. To do so, we need the following statistics. Let $\Delta \in I_{n,m}$ and define:

- the ***area*** to be the number of gaps in Δ, $\mathrm{area}(\Delta) := \sharp(\mathbb{Z}_{\geq 0} \setminus \Delta)$.
- the n-***generators*** of Δ as the set $\mathtt{ngen}(\Delta) := \Delta \setminus (\Delta + n) = \{a \in \Delta : a - n \notin \Delta\}$.
- the ***codinv*** as the number of gaps in length m intervals beginning at n-generators:

$$\mathrm{codinv}(\Delta) := \sum_{a \in \mathtt{ngen}(\Delta)} \sharp\{a \leq g < a + m : g \notin \Delta\}, \tag{2.1}$$

$$\mathrm{dinv}(\Delta) := \delta(n,m) - \mathrm{codinv}(\Delta), \quad \delta(n,m) := \frac{(n-1)(m-1)}{2}.$$

Definition 4. For each coprime pair (m,n), the ***rational*** (q,t)***-Catalan polynomial***, denote $C_{n,m}(q,t)$, is given by:

$$C_{n,m}(q,t) := \sum_{\Delta \in I^0_{n,m}} q^{\mathrm{area}(\Delta)} t^{\mathrm{dinv}(\Delta)} = (1-q) \sum_{\Delta \in I_{n,m}} q^{\mathrm{area}(\Delta)} t^{\mathrm{dinv}(\Delta)}.$$

In order to define Schröder polynomials, we will need a couple more ingredients.

- Let $\mathtt{Cogen}(\Delta) := \{a \in \mathbb{Z} : a \notin \Delta, a+n \in \Delta, a+m \in \Delta\}$ be the set of ***double cogenerators*** of Δ.
- Let $k \in \mathbb{Z}$. Set $\lambda_k(\Delta) := \sharp\{a \in \mathtt{ngen}(\Delta) : k+n+1 \leq a \leq k+n+m\}$.

Definition 5. For each coprime pair (m,n), the ***Schröder polynomial*** $S_{n,m}(q,t,a)$ is:

$$S_{n,m}(q,t,a) := \sum_{\Delta \in I^0_{n,m}} q^{\mathrm{area}(\Delta)} t^{\mathrm{dinv}(\Delta)} \prod_{k \in \mathtt{Cogen}(\Delta)} \left(1 + at^{-\lambda_k(\Delta)}\right).$$

Example 6. In Figure 2, the bijection between $(3,4)$-Dyck paths and the 0-normalized $(3,4)$-invariant subsets is illustrated, complemented with a computation of the area and dinv statistics, as well as the factors necessary for the Schröder polynomial, for two out of five invariant subsets in $I^0_{3,4}$. The rest are computed similarly. Summing all together, one obtains the Schröder polynomial:

$$\begin{aligned} S_{3,4}(q,t,a) =& t^3(1+a)(1+at^{-1})(1+at^{-2}) + qt^2(1+a)(1+at^{-1}) + qt(1+a)(1+at^{-1}) \\ &+ q^2t(1+a)(1+at^{-1}) + q^3(1+a) \\ =& q^3 + q^2t + qt^2 + t^3 + qt + a(q^3 + q^2t + qt^2 + t^3 + q^2 + 2qt + t^2 + q + t) \\ &+ a^2(q^2 + qt + t^2 + q + t + 1) + a^3. \end{aligned}$$

2.3 Recursions for Invariant Subsets

In [9] the fourth author together with Gorsky and Vazirani introduced a recursion computing the rational (q,t)-Catalan series and showed that their recursion is equivalent to the Hogancamp-Mellit recursion in the case of the torus link. Hence, in the relatively prime case the Gorsky-Mazin-Vazirani recursion recovers the (q,t)-Catalan polynomials.

Let (m,n) be a pair of positive relatively prime integers. In order to define the recursion, one needs to consider subfamilies in the set of invariant subsets $I_{n,m}$ given by fixing the intersection of the subsets with the interval $[0, n+m-1]$. Let $\mathbf{w} \in \{0,1\}^{n+m}$ be a binary sequence of length $n+m$.

Definition 7. Set $I_{\mathbf{w}} := \{\Delta \in I_{n,m} : \forall 0 \leq i < n+m, i \in \Delta \Leftrightarrow w_i = 1\}$ and define:

$$P_{\mathbf{w}}(q,t,a) := \sum_{\Delta \in I_{\mathbf{w}}} q^{\mathrm{area}(\Delta)} t^{\mathrm{codinv}(\Delta)} \prod_{k \in \mathtt{Cogen}(\Delta) \cap \mathbb{Z}_{\geq 0}} (1 + at^{\lambda_k(\Delta)}). \tag{2.2}$$

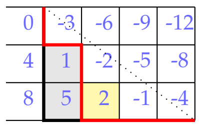
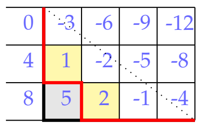
	Gaps $= \{2\}$ $\Delta = \mathbb{Z}_{\geq 0} \setminus \{2\}$ $\mathrm{area}(\Delta) = 1$	3-gen $= \{0,1,5\}$ $\mathrm{codinv}(\Delta) = 2$ $\mathrm{dinv}(\Delta) = 1$	$\texttt{Cogen} = \{-3,2\}$ $(1+a)(1+at^{-1})$
	Gaps $= \{1,2\}$ $\Delta = \mathbb{Z}_{\geq 0} \setminus \{1,2\}$ $\mathrm{area}(\Delta) = 2$	3-gen $= \{0,4,5\}$ $\mathrm{codinv}(\Delta) = 2$ $\mathrm{dinv}(\Delta) = 1$	$\texttt{Cogen} = \{1,2\}$ $(1+a)(1+at^{-1})$

Figure 2: Three out of five $(3,4)$-Dyck paths are on the left, with the cells corresponding to the gaps in yellow. The corresponding $(3,4)$-invariant subsets are in the second column, together with the area, 3-generators, codinv, and dinv in the third column, and the corresponding Schröder factor is in the fourth.

Then, the Schröder polynomial can be obtained from (2.2),

$$S_{n,m}(q,t,a) = \frac{(1-q)t^{\delta(n,m)}}{q^{n+m}} P_{0^{n+m}}(q,t^{-1},a) = \frac{t^{\delta(n,m)}}{q^{n+m-1}} P_{0^{n+m-1}1}(q,t^{-1},a).$$

The polynomials $P_{\mathbf{w}}$ satisfy a recursion, however, in order to match this recursion to the Hogancamp-Mellit recursion, certain adjustments are required.

First, we need to replace the sequence $\mathbf{w}$ of length $n+m$ by two sequences $(\mathbf{x}, \mathbf{y})$ in the alphabet $\{0, \bullet, 1\}$ of lengths m and n respectively. The sequence $\mathbf{x}$ records gaps (encoded by 0), n-generators (encoded by 1), and the rest of the elements of Δ (encoded by $\bullet$) on the interval $[n, n+m-1]$. Similarly, the sequence $\mathbf{y}$ records gaps, m-generators, and the rest of the elements of Δ on the interval $[m, n+m-1]$. In other words,

Definition 8. Let $\mathbf{x}, \mathbf{y}$ be sequences as above. Let $I_{\mathbf{x},\mathbf{y}}$ be the set of $\Delta \in I_{n,m}$ such that:

$$\forall 0 \leq k < m \begin{cases} x_k = 0 \Leftrightarrow k+n \notin \Delta, \\ x_k = 1 \Leftrightarrow k+n \in \mathtt{ngen}(\Delta), \\ x_k = \bullet \Leftrightarrow k \in \Delta, \end{cases} \quad \forall 0 \leq k < n \begin{cases} y_k = 0 \Leftrightarrow k+m \notin \Delta, \\ y_k = 1 \Leftrightarrow k+m \in \mathtt{mgen}(\Delta), \\ y_k = \bullet \Leftrightarrow k \in \Delta. \end{cases}$$

Example 9. Let $(n,m) = (4,7)$ and $\Delta = \mathbb{Z}_{\geq 0} \setminus \{0,1,2,3,4,6,7,8,10\}$. Then, the associated binary sequence $w = 00000100010$ yields the ternary sequences $\mathbf{x} = (01000 \bullet 0)$ and $\mathbf{y} = (0010)$, since the only 4-generator in $[4,10]$ is 5 and the only 7-generator in $[7,10]$ is 9. In particular, 9 is not a 4-generator since $9-4=5 \in \Delta$. Thus, $\Delta \in I_{00000100010} = I_{01000\bullet 0,0010}$.

The statistics on $I_{n,m}$ are modified as follows. Set:

$$\mathrm{area}'(\Delta) = \sharp\{k \in \mathbb{Z}_{\geq n+m} : k \notin \Delta\} = \sharp\left(\mathrm{Gaps}(\Delta) \cap \mathbb{Z}_{\geq n+m}\right),$$

$$\mathrm{codinv}'(\Delta) = \sum_{a \in \mathtt{ngen}(\Delta)} \sharp\{k \in \mathbb{Z}_{\geq n+m} : a \leq k < a+m,\ k \notin \Delta\} - \frac{\lambda(\Delta)(\lambda(\Delta)-1)}{2},$$

where $\lambda(\Delta) := \lambda_{-1}(\Delta) = \sharp\{a \in \mathtt{ngen}(\Delta) : n \le a < n+m\}$.

Definition 10. Given sequences $\mathbf{x}$ and $\mathbf{y}$ as above, let

$$Q_{\mathbf{x},\mathbf{y}}(q,t,a) := \sum_{\Delta \in I_{\mathbf{x},\mathbf{y}}} q^{\mathrm{area}'(\Delta)} t^{-\mathrm{codinv}'(\Delta)} \prod_{k \in \mathtt{Cogen}(\Delta) \cap \mathbb{Z}_{\ge 0}} (1 + a t^{-\lambda_k(\Delta)}).$$

Note that for any $\Delta \in I_{0^{n+m}}$ one gets $\mathrm{area}'(\Delta) = -n - m + \mathrm{area}(\Delta)$, $\mathrm{codinv}'(\Delta) = \mathrm{codinv}(\Delta)$, and all the double co-generators are non-negative. Therefore,

$$Q_{0^m,0^n}(q,t,a) = q^{-n-m} P_{0^{n+m}}(q,t^{-1},a).$$

Theorem 11 ([9])**.** *The following recursions hold:*

$$Q_{0\mathbf{u},0\mathbf{v}} = t^{-|\mathbf{u}|} Q_{\mathbf{u}1,\mathbf{v}1} + q t^{-|\mathbf{u}|} Q_{\mathbf{u}0,\mathbf{v}0}, \qquad Q_{1\mathbf{u},0\mathbf{v}} = Q_{\mathbf{u}1,\mathbf{v}\bullet}, \qquad Q_{\bullet\mathbf{u},\bullet\mathbf{v}} = Q_{\mathbf{u}\bullet,\mathbf{v}\bullet},$$
$$Q_{1\mathbf{u},1\mathbf{v}} = (t^{|\mathbf{u}|} + a) Q_{\mathbf{u}\bullet,\mathbf{v}\bullet}, \qquad Q_{0\mathbf{u},1\mathbf{v}} = Q_{\mathbf{u}\bullet,\mathbf{v}1}, \qquad Q_{\emptyset,\emptyset} = 1.$$

Finally, notice that in the recursion for Q one can completely ignore all the $\bullet$'s. Also, it follows that the recursion always terminates in $Q_{\emptyset,\emptyset}$, so one doesn't need the normalization conditions for $Q_{\emptyset,0^n}$ and $Q_{0^m,\emptyset}$.

Theorem 12 ([9])**.** *Let $(\mathbf{u},\mathbf{v})$ be the sequences obtained from the sequences $(\mathbf{x},\mathbf{y})$ by ignoring all $\bullet$'s. Then*

$$R_{\mathbf{u},\mathbf{v}}(q,t,a) = Q_{\mathbf{x},\mathbf{y}}(q,t,a).$$

Corollary 13 ([15, 9])**.** *The Poincaré polynomial of the reduced Khovanov-Rozansky homology of the (n,m)-torus knot is given by*

$$R_{0^{m-1}1,0^{n-1}1}(q,t,a) = Q_{0^{m-1}1,0^{n-1}1}(q,t,a) = \frac{P_{0^{n+m-1}1}(q,t^{-1},a)}{q^{n+m-1}} = \frac{S_{n,m}(q,t,a)}{t^{\delta(n,m)}}.$$

Remark 14. *The last formula was first proven by Anton Mellit in [15]. We follow notations from [11] and [9], where the result was generalized to torus links.*

2.4 Monotone and Coxeter Links

In [4] Galashin-Lam study a family of links, called *monotone* that arise from certain curves on the plane. They define a new invariant, the *elliptic Hall algebra superpolynomial*, which they prove recovers the HOMFLY polynomial of L_C and conjecture agrees with the Poincaré series of the Khovanov-Rozansky homology of any algebraic link L_C.

Definition 15. Let C denote a curve from $(0,n)$ to $(m,0)$. A ***monotone link*** L_C is a projection onto $\mathbb{R}^2/\mathbb{Z}^2$ of a curve C such that the x- and y-coordinates of C are monotone increasing and decreasing, respectively[1]. Trace the projection of C starting from the left top corner, crossing the earlier strand on top.

[1] This differs slightly from the definition in [4] by a flip sending $x \mapsto -x$.

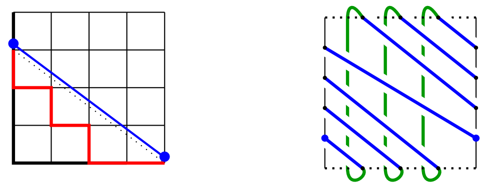


Figure 3: The curve C on the left and its projection onto $[0,1]^2$ and annular closure β_C, on the right. The associated triangular partition is displayed in red, so that $\mu = (3,2,1,0,0)$ with $\mathbf{b} = (0,1,1,1)$ and $\mathbf{e} = (1,1,1)$. It is straightforward to verify that the braid $\beta^{cox}_{\mathbf{b},\mathbf{e}} = JM_2^1 JM_3^1 JM_4^1 \sigma_1\sigma_2\sigma_3$ is isotopic to β_C.

It is well known [11] that if C is the straight diagonal line, then L_C is the (m,n) torus link, with the special case of m,n relatively prime yielding a knot.

Let $\mathbb{A}$ denote the annulus. Given a curve C, we can construct its annular closure $\beta_C \in \mathbb{A} \times [0,1]$ as follows. Consider the projection of C onto $[0,1]^2$. Now, identify the top and bottom boundaries so that for each point $x \in (0,1)$ for which $(x,0)$ and $(x,1)$ are in C, the line connecting them lies *underneath* all other strands. Denote the resulting braid in $\mathbb{A} \times [0,1]$ by β_C (see Figure 3).

Denote by $\sigma_i \in B_n$, with B_n the braid group, the positive crossing of the i^{th} and $i+1^{st}$ strands, i.e. the i^{th} strand is above the $i+1^{st}$ strand.

Definition 16. Given sequences $\mathbf{b} = (b_1, \ldots, b_m) \in \mathbb{Z}^m_{\geq 0}$ and $\mathbf{e} = (\epsilon_1, \ldots, \epsilon_{m-1}) \in \{0,1\}^{m-1}$, the ***Coxeter braid*** $\beta^{cox}_{\mathbf{b},\mathbf{e}}$ is given by:

$$\beta^{cox}_{\mathbf{b},\mathbf{e}} := JM_1^{b_1} \ldots JM_m^{b_m} \sigma_1^{\epsilon_1} \ldots \sigma_{m-1}^{\epsilon_{m-1}}, \tag{2.3}$$

where $JM_i := \sigma_i \ldots \sigma_{m-1}\sigma_{m-1} \ldots \sigma_i$ for each $1 \leq i \leq m$.

To any curve C, we can assign a Coxeter braid β^{cox}_C in the following way. Let $\mu = (n, \mu_1, \ldots, \mu_m)$ be such that $(\mu_1, \ldots, \mu_m)$ is the transpose of the triangular partition corresponding to the curve C. Set $b_{m-i+1} = \mu_{i-1} - \mu_i$ (with $\mu_0 = n$) and for each $1 \leq i \leq m-1$, set $\epsilon_i = 0$ if C passes through the lattice point (i,j) for some $j \in \mathbb{Z}$, with $\epsilon_i = 1$ otherwise. Then for $\mathbf{e}_C = (\epsilon_1, \ldots, \epsilon_{m-1})$ and $\mathbf{b_C} = (b_1 \ldots, b_m)$, let $\beta^{cox}_C := \beta^{cox}_{\mathbf{b}_C,\mathbf{e}_C}$.

Theorem 17 ([4])**.** *The braid β^{cox}_C is conjugate to the annular closure β_C of C. In particular, all monotone links in $\mathbb{A} \times [0,1]$ are Coxeter links, and all Coxeter links arise in this way.*

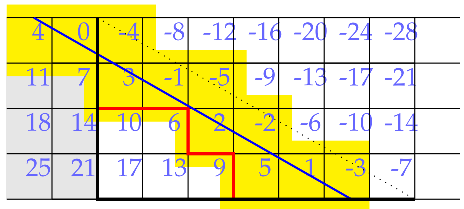


Figure 4: The triangular partition $\lambda = (2,3) = \tau_{3.2,5.6}$ with $(m,n) = (4,7)$ and the Anderson labels denoted in blue.

3 Main Results

3.1 Schröder Polynomials for Triangular Partitions

Definition 18. A partition λ is called ***triangular*** if there exist two not necessarily integral points $(0,s)$ and $(r,0)$ such that λ consists of all the boxes below the line connecting these points, in which case we denote λ by $\tau_{r,s}$.

Evidently, for any given triangular partition λ, there exist many choices for positive real numbers r and s such that $\lambda = \tau_{r,s}$ (see [1] for details). In particular, one can always choose r and s in such a way that $\lambda = \tau_{r,s}$, and $r/s = n/m$, where (n,m) are positive relatively prime integers with $r \le n$ (equivalently, the line connecting $(r,0)$ to $(0,s)$ is below the line connecting $(n,0)$ to $(0,m)$). Generalized (q,t)-Catalan polynomials corresponding to triangular partitions appeared in the generalized shuffle theorem [2].

We claim that to any triangular partition $\tau_{r,s}$ one can associate a pair of binary sequences $\mathbf{u}(s,t)$ and $\mathbf{v}(s,t)$ [3], which we explain how to construct in Example 20 below. With this in hand, we extend the Schröder polynomial to the triangular setting.

Definition 19. Let $\lambda = \tau_{r,s}$ and (n,m) be as above with associated sequences $\mathbf{u}(r,s)$ and $\mathbf{v}(r,s)$ as in Example 20. The ***(q,t)-Schröder polynomial*** for triangular partitions is defined as $S_\lambda(q,t,a) := t^{|\lambda|}R_{\mathbf{u}(r,s),\mathbf{v}(r,s)}(q,t,a)$.

Example 20. Let $\lambda = \tau_{3.2,5.6} = (2,3)$ as in Figure 4 and observe that the line connecting $(0,3.2)$ and $(5.6,0)$ has the same slope as the diagonal line connecting $(0,4)$ and $(7,0)$. We call the line connecting $(0,3.2)$ and $(5.6,0)$ the ***shifted diagonal***, with $(n,m) = (4,7)$ denoting the closest line above it with equal slope and integer x and y -intercepts.

Let $W = \{-5,-4\ldots,5\} = [-5,5]$ be the window of labels of all cells intersected by the shifted diagonal (shaded yellow in Figure 4). The subdiagrams of λ are in bijection with the subfamily $I^0_{3.2,5.6} \subset I^0_{4,7}$ consisting of subsets Δ, such that $\{1,2,3,5\} \cap \Delta = \varnothing$, where $\{1,2,3,5\} = \mathrm{Gaps}(\lambda)$. This is equivalent to saying that $\Delta \cap W = \{0,4\}$. Hence,

$$\begin{aligned} I^0_{3.2,5.6} =& \{\Delta \in I^0_{4,7} : \{1,2,3,5\} \cap \Delta = \varnothing\} \\ =& \{\Delta \in I^0_{4,7} : \Delta \cap W = \{0,4\}\} = \{\Delta \in I^0_{4,7} : \Delta + 5 \in I_{00000100010}\}. \end{aligned}$$

$\text{Gaps} \cap \mathbb{Z}_{\geq 6} = \{6, 9, 13\}$ area $= 3$	4-gen $= \{0, 7, 10, 17\}$ codinv $= 4$, dinv $= 1$	Cogen $= \{3, 13\}$ $(1 + a)(1 + at^{-1})$
$\text{Gaps} \cap \mathbb{Z}_{\geq 6} = \{6, 9\}$ area $= 2$	4-gen $= \{0, 7, 10, 13\}$ codinv $= 2$, dinv $= 3$	Cogen $= \{3, 6, 9\}$ $(1 + a)(1 + at^{-1})(1 + at^{-2})$
$\text{Gaps} \cap \mathbb{Z}_{\geq 6} = \{6, 10\}$ area $= 2$	4-gen $= \{0, 7, 9, 14\}$ codinv $= 3$, dinv $= 2$	Cogen $= \{5, 10\}$ $(1 + a)(1 + at^{-1})$

Figure 5: For three of the nine $\Delta \in I^0_{3.2,5.6}$ we record the gaps that are greater than 5, since only those contribute to area and codinv (respectively, area′ and codinv′ on $I_{01000\bullet 0,0010}$). There cannot be any double co-generators below the interval $W = [0, n + m - 1] - 5$, therefore all co-generators are used for the Schröder factors.

That is, the family $I^0_{3.2,5.6}$ is simply $I_{00000100010} = I_{01000\bullet 0,0010}$ from Example 9 shifted down by 5. So setting $\mathbf{u}(3.2, 5.6) = 010000$ and $\mathbf{v}(3.2, 5.6) = 0010$, we obtain $S_\lambda(q, t, a) = t^5 R_{010000,0010}(q, t, a)$. In Figure 5 we illustrate the computation of the contributions towards $S_\lambda(q, t, a)$ of three of the nine invariant subsets in $I^0_{3.2,5.6}$. All together:

$$\begin{aligned} S_\lambda(q,t,a) = & q^5(1+a) + q^4t(1+a)(1+at^{-1}) + q^3t^2(1+a)(1+at^{-1}) \\ & + q^3t(1+a)(1+at^{-1}) + q^2t^3(1+a)(1+at^{-1})(1+at^{-2}) \\ & + q^2t^2(1+a)(1+at^{-1}) + qt^3(1+a)(1+at^{-1}) \\ & + qt^4(1+a)(1+at^{-1})(1+at^{-2}) + t^5(1+a)(1+at^{-1})(1+at^{-2}). \end{aligned}$$

Note that plugging in $a = 0$ we recover the corresponding (q, t)-Catalan polynomial:

$$C_\lambda(q,t) = S_\lambda(q,t,0) = q^5 + q^4t + q^3t^2 + q^2t^3 + qt^4 + t^5 + q^3t + q^2t^2 + qt^3.$$

With the definition of the Schröder polynomial established for any triangular partition, we can now state our first main theorem.

Theorem 21. *[3] The triangular Schröder polynomial $S_\lambda(q, t, a)$ at $a = 0$ specializes to the triangular (q, t)-Catalan polynomial of [1] and [2]. Hence, for $\lambda = \tau_{r,s}$ as before, we obtain that*

$$C_\lambda(q,t) = S_\lambda(q,t,0) = t^{|\lambda|} R_{\mathbf{u}(r,s),\mathbf{v}(r,s)}(q,t,0).$$

By construction, the polynomial $S_\lambda(q, t, a)$ depends on a choice of a shifted diagonal. At $a = 0$ this corresponds to choosing an appropriate slope in the definition of the dinv statistic [1, 2]. The Catalan polynomial doesn't depend on that choice (this follows from the shuffle theorem of [2], see also [13, 14]). The shuffle theorem argument can be generalized to show that the full Schröder polynomial also depends only on the partition. Nonetheless, this will also follow from our results below.

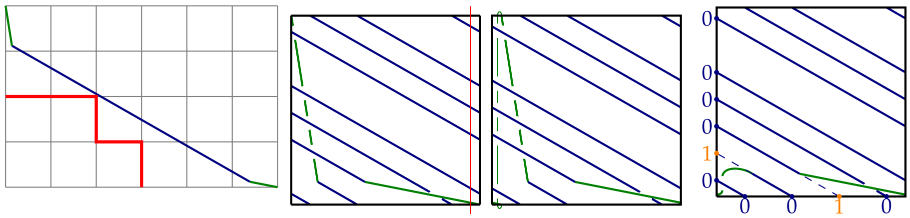


Figure 6: Left: $\lambda = (3,2)$ together with the monotone curve C obtained by augmenting the shifted diagonal. Second: the monotone knot $K_{r,s}$ drawn on a torus. We cut a vertical strip on the right (the red line) and reattach it on the left for the third picture. We also close the vertical green strand. Finally, we pull the green strand from under the blue ones to obtain the picture on the right, which is the knot $L_{010000,0010}$.

3.2 The Monotone Knot of a Triangular Partition

Let $\lambda = \tau_{r,s}$ be a triangular partition, and (r,s) be as in the previous section: $r/s = n/m$, where $n, m \in \mathbf{Z}_{>0}$ are relatively prime and $r \leq n$. The monotone curve C from $(0, \lceil r \rceil)$ to $(\lceil s \rceil, 0)$ is constructed by augmenting the shifted diagonal connecting $(0,r)$ to $(s,0)$ by adding an almost vertical segment at the top and an almost horizontal segment at the bottom (see Figure 6). Let $K_{r,s}$ be the closure of the corresponding monotone braid β_C (see Section 2.4). It follows from [4, Prop. 7.5] that $K_{r,s}$ is isotopic to the closure of the Coxeter braid β_C^{cox}, which only depends on the partition λ and not on the choice of the shifted diagonal. We will call it the ***Coxeter knot*** of the partition λ and denote it K_λ.

Theorem 22 ([3])**.** *The monotone knot $K_{r,s}$ is isotopic to the knot $L_{\mathbf{u}(r,s),\mathbf{v}(r,s)}$ (see Section 2.1 for a definition). In particular, the Poincaré polynomial of the reduced Khovanov-Rozansky homology of the Coxeter knot K_λ is given by $P^{KR}_{K_\lambda}(q,t,a) = R_{\mathbf{u}(r,s),\mathbf{v}(r,s)}(q,t,a) = t^{-|\lambda|} S_\lambda(q,t,a)$.*

Remark 23. *Theorem 22 implies that the Schröder polynomial $S_\lambda(q,t,a)$ does not depend on the choice of the shifted diagonal $(0,r) - (s,0)$, but only on the triangular partition $\lambda = \tau_{r,s}$.*

Example 24. In Figure 6 we illustrate the construction of the monotone knot $K_{r,s}$ and its isotopy to $L_{\mathbf{u}(r,s),\mathbf{v}(r,s)}$ for $(r,s) = (3.2, 5.6)$, continuing Example 20.

In the case when $(r,s) = (dn, dm)$, where d, n, m are integers and n and m are relatively prime, Galashin and Lam in [4, Lem. 8.1] proved that the monotone knot $K_{\tau_{dn,dm}}$ is the $(d, dnm+1)$-cable of the (n,m)-torus knot, which is an algebraic knot: it can be obtained as the intersection of the planar curve $(x = t^{dn}, y = t^{dm} + t^{dm+1})$ with a small 3D sphere around the origin in $\mathbb{C}^2$. Such curves were studied by the fourth author, Gorsky, and Oblomkov in [8], were they showed that the Poincaré polynomial $P_{\overline{JC}}(t)$ of the Compactified Jacobian of such a curve is a specialization of the (dn, dm) (q,t)-Catalan polynomial. Combining this with our Theorem 22, we obtain

$$P_{\overline{JC}}(t) = t^{2\delta} C_{nd,md}(1, t^{-2}) = t^{2\delta} S_{\tau_{dn,dm}}(1, t^{-2}, 0) = P^{KR}_{K_{\tau_{dn,dm}}}(1, t^{-2}, 0),$$

which is a special case of the Oblomkov-Rasmussen-Shende conjecture for such curves.

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