Séminaire Lotharingien de Combinatoire **91B** (2024)
Article #7, 12 pp.

Proceedings of the 36th Conference on Formal Power Series and Algebraic Combinatorics (Bochum)

Pattern heights and the minimal power of q in a Kazhdan–Lusztig polynomial

Christian Gaetz*[1] and Yibo Gao[2]

[1]*Department of Mathematics, University of California, Berkeley, CA, USA*
[2]*Beijing International Center for Mathematical Research, Peking University, Beijing, CN*

Abstract. For w in the symmetric group, we use permutation patterns to provide an exact formula for the smallest positive power $q^{h(w)}$ appearing in the Kazhdan–Lusztig polynomial $P_{e,w}(q)$. We also use Weyl group patterns to provide a tight upper bound on $h(w)$ in simply-laced types, resolving a conjecture of Billey–Postnikov from 2002.

Keywords: Kazhdan–Lusztig polynomial, permutation pattern, Bruhat order

1 Introduction

Let G be a complex semisimple Lie group, with Borel subgroup B containing maximal torus T and corresponding Weyl group W. The Bruhat decomposition $G = \bigsqcup_{w \in W} BwB$ gives rise to the *Schubert varieties* $X_w := \overline{BwB/B}$ inside the flag variety G/B, whose containments determine the Bruhat order on W: $y \leq w$ if $X_y \subset X_w$. The *Kazhdan–Lusztig polynomials* $P_{y,w}(q) \in \mathbb{Z}[q]$ have since their discovery [14] proven to underlie deep connections between canonical bases of Hecke algebras, singularities of Schubert varieties, and representations of Lie algebras.

Theorem 1 (Kazhdan and Lusztig [15])**.** *For $y \leq w$, let $IH^*(X_w)_y$ denote the local intersection cohomology of X_w at the T-fixed point yB, then*

$$P_{y,w}(q) = \sum_i \dim(IH^{2i}(X_w)_y)q^i.$$

Theorem 1 implies that $P_{y,w}(q)$ has nonnegative coefficients, a property which is completely obscured by their recursive definition (Definition 7). It is known that for all $y \leq w$ one has $P_{y,w}(0) = 1$.

Theorem 2 (Deodhar [11]; Peterson (see [9]))**.** *If G is simply-laced and $y \leq w$, then X_w is smooth at yB if and only if $P_{y,w}(q) = 1$. In particular, X_w is smooth if and only if $P_{e,w}(q) = 1$.*

*gaetz@berkeley.edu

In light of Theorem 1, one wants to understand $P_{y,w}(q)$ explicitly enough to determine which coefficients vanish. Indeed, the view of the $P_{y,w}$ as a measure of the failure of local Poincaré duality in X_w was among the original motivations in [14]. Unfortunately, $P_{y,w}$ may be arbitrarily complicated [18] and the formulae [8] which exist involve cancellation, and are thus not well-suited to this problem. If X_w is singular (as is typically true) one can at least ask for the smallest nontrivial coefficient, the first degree in which Poincaré duality fails. Writing $[q^i]P_{y,w}$ for the coefficient of q^i in $P_{y,w}(q)$, define:

$$h(w) := \min\{i > 0 \mid [q^i]P_{e,w} \neq 0\} = \min_{y \leq w} \min\{i > 0 \mid [q^i]P_{y,w} \neq 0\}.$$

The second equality follows from the monotonicity property of the $P_{y,w}$ [7]. We make the convention that $h(w) = +\infty$ when X_w is smooth.

Conjecture 3 (Billey and Postnikov [2])**.** *Let G be simply-laced of rank r, and suppose X_w is singular. Then $h(w) \leq r$.*

Billey and Postnikov's conjecture is somewhat surprising, since $\deg(P_{y,w})$ may be as large as $\frac{1}{2}(\ell(w) - \ell(y) - 1)$ which is of the order of r^2, where ℓ denotes length. An upper bound on $h(w)$ in certain special infinite Coxeter groups was given in [19].

The decomposition $X_w = \bigsqcup_{y \leq w} ByB/B$ is an affine paving, with the cell ByB/B having complex dimension $\ell(y)$. We thus have

$$L(w) := \sum_{y \leq w} q^{\ell(y)} = \sum_{j \geq 0} \dim(H^j(X_w))q^{j/2},$$

the Poincaré polynomial of X_w. Björner–Ekedahl [6] gave a precise interpretation of $h(w)$ in terms of $L(w)$, as the smallest homological degree in which Poincaré duality fails.

Theorem 4 (Björner and Ekedahl [6])**.** *For $0 \leq i \leq \ell(w)/2$ we have $[q^i]L(w) \leq [q^{\ell(w)-i}]L(w)$, and*

$$h(w) = \min\{i > 0 \mid [q^i]L(w) < [q^{\ell(w)-i}]L(w)\}.$$

Theorem 4 will be a useful tool in this work, but cannot be directly used to resolve Conjecture 3 since it is difficult to compute $[q^i]L(w)$ in general.

Our first main theorem[1] is a refinement and proof of Conjecture 3.

Theorem 5. *Let G be simply-laced of rank r, and suppose X_w is singular. Then $h(w) \leq r - 2$.*

The bound of $r - 2$ is tight when G is a member of the infinite families SL_{r+1} or SO_{2r}. When G is one of the exceptional simply-laced groups of type E_6, E_7, or E_8, Theorem 5 follows from the computations made by Billey–Postnikov [2]. In the case $G = \mathrm{SL}_{n+1}$, the theorem can be derived from the classification of the singular locus of X_w [5, 17]. However, in this case we provide a new exact formula for $h(w)$ for any permutation w. This theorem is phrased in terms of *pattern containment* (see Section 2.5.2).

[1] A full version of this work is available at [13]

Theorem 6. *Let $G = \mathrm{SL}_{n+1}$, and suppose X_w is singular. Then*

$$h(w) = \begin{cases} 1 & \text{if } w \text{ contains } 4231, \\ \mathrm{mHeight}(w) & \text{otherwise,} \end{cases}$$

where $\mathrm{mHeight}(w)$ *denotes the minimum height of a* 3412 *pattern in* w.

In the case $P_{e,w}(1) = 2$, Theorem 6 follows from the work of Woo [21]. Our theorem adds to the deep [22] and ubiquitous [1] links between singularities of Schubert varieties and pattern containment.

2 Preliminaries

2.1 Bruhat order and Kazhdan–Lusztig polynomials

Let W be a Weyl group with simple reflections $S = \{s_1, s_2, \ldots\}$ and length function ℓ. Write $\mathcal{R}$ for the set of reflections (conjugates of simple reflections), then *Bruhat order* $\leq$ on W is defined as the transitive closure of the relation $y < yr$ if $r \in \mathcal{R}$ and $\ell(y) < \ell(yr)$.

The left (respectively, right) *descents* $D_L(w)$ (resp. $D_R(w)$) are those $s \in S$ such that $sw < w$ (resp. $ws < w$).

Definition 7 (Kazhdan and Lusztig [14])**.** Define polynomials $R_{y,w}(q) \in \mathbb{Z}[q]$ by setting $R_{y,w}(q) = 0$ if $y \not\leq w$, $R_{y,w}(q) = 1$ if $y = w$, and requiring:

$$R_{y,w}(q) = \begin{cases} R_{ys,ws}(q), & \text{if } s \in D_R(y) \cap D_R(w), \text{ and} \\ qR_{ys,ws}(q) + (q-1)R_{y,ws}, & \text{if } s \in D_R(w) \setminus D_R(y). \end{cases}$$

Then there is a unique family of polynomials $P_{y,w}(q) \in \mathbb{Z}[q]$, the *Kazhdan–Lusztig polynomials* satisfying $P_{y,w}(q) = 0$ if $y \not\leq w$, $P_{w,w}(q) = 1$, and such that if $y < w$ then $P_{y,w}$ has degree at most $\frac{1}{2}(\ell(w) - \ell(y) - 1)$ and

$$q^{\ell(w)-\ell(y)} P_{y,w}(q^{-1}) = \sum_{a \in [y,w]} R_{y,a}(q) P_{a,w}(q).$$

Although not apparent from Definition 7, the $P_{y,w}$ satisfy an inversion symmetry:

Proposition 8. *Let $y, w \in W$, then $P_{y,w}(q) = P_{y^{-1},w^{-1}}(q)$. In particular, $h(w) = h(w^{-1})$.*

2.2 Fiber bundles of Schubert varieties

For $J \subset S$, we write W_J for the subgroup generated by J, P_J for the parabolic subgroup of G generated by B and J, and W^J for the set of minimal length representatives of the left cosets W/W_J. We have $W^J = \{w \in W \mid D_R(w) \cap J = \emptyset\}$. Each $w \in W$ decomposes uniquely as $w^J w_J$ with $w^J \in W^J$ and $w_J \in W_J$. Using right cosets instead gives decompositions $w = {}_J w\, {}^J w$ with ${}_J w \in W_J$ and ${}^J w \in {}^J W = (W^J)^{-1}$. Notice that $(w^{-1})_J = ({}_J w)^{-1}$.

We write $w_0(J)$ for the unique element of W_J of maximum length and write $[u,v]^J$ for the set $[u,v] \cap W^J$. Since parabolic decompositions are unique, we have an injection $[e,w^J]^J \times [e,w_J] \hookrightarrow [e,w]$ given by multiplication.

Schubert varieties $X^J_{w^J} := \overline{Bw^J P_J/P_J}$ in the partial flag variety G/P_J have an affine paving by ByP_J/P_J for $y \in W^J$ and $y \leq w^J$, and so

$$L^J(w^J) := \sum_{\substack{y \in W^J \\ y \leq w^J}} q^{\ell(y)} = \sum_{j \geq 0} \dim(H^j(X^J_{w^J}))q^{j/2}.$$

Definition 9 (Richmond and Slofstra [20])**.** The parabolic decomposition $w = w^J w_J$ is called a *Billey–Postnikov decomposition* or *BP-decomposition* of w if $\mathrm{supp}(w^J) \cap J \subset D_L(w_J)$.

Theorem 10 (Richmond and Slofstra [20])**.** *The map $X_w \twoheadrightarrow X^J_{w^J}$ induced by the map $G/B \to G/P_J$ is a bundle projection if and only if J is a BP-decomposition of w, and in this case the fiber is isomorphic to X_{w_J}. Taking Poincaré polynomials, we have $L^J(w^J)L(w_J) = L(w)$ in this case.*

2.3 Patterns in Weyl groups

Let Φ denote the root system for G, with positive roots Φ^+ and simple roots Δ. For $w \in W$, the *inversion set* is $\mathrm{Inv}(w) := \{\alpha \in \Phi^+ \mid w\alpha \in \Phi^-\}$.

A subgroup W' of W generated by reflections is called a *reflection subgroup*, and is itself a Coxeter group with reflections $\mathcal{R}' = \mathcal{R} \cap W'$. We write $\leq'$ for the intrinsic Bruhat order on W', Φ' for the root system, and Inv' for inversion sets.

Proposition 11 (Billey and Braden [4]; Billey and Postnikov [2])**.** *Let $W' \subset W$ be a reflection subgroup, there is a unique function* $\mathrm{fl} : W \to W'$, *the* flattening map *satisfying:*

(1) fl *is W'-equivariant, and*

(2) *if* $\mathrm{fl}(x) \leq' \mathrm{fl}(wx)$ *for some* $w \in W'$*, then* $x \leq wx$*.*

Furthermore, fl *has the following explicit description:* $\mathrm{fl}(w)$ *is the unique element $w' \in W'$ with* $\mathrm{Inv}'(w') = \mathrm{Inv}(w) \cap \Phi'$*. If $W' = W_J$ is a parabolic subgroup, then* $\mathrm{fl}(w) = w_J$*.*

Definition 12. We say that $w \in W$ *contains the pattern* $w'' \in W''$, if W has some reflection subgroup W', with an isomorphism $W' \xrightarrow{\varphi} W''$ as Coxeter systems, such that $\varphi(\mathrm{fl}(w)) = w''$. Otherwise, w is said to *avoid* w''.

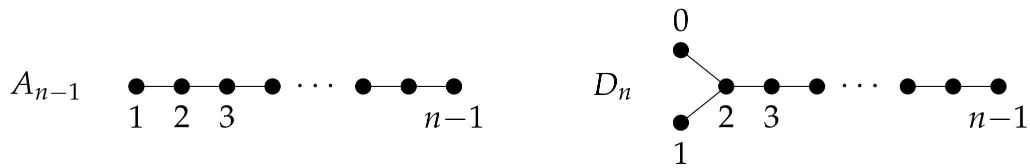

Figure 1: The Dynkin diagrams for Types A_{n-1} and D_n.

We will make use of the following result, which is proven using patterns.

Theorem 13 (Billey and Braden [4]). *Let $J \subset S$, then $h(w) \leq h(w_J)$.*

Billey and Postnikov gave the following characterization of smooth Schubert varieties, generalizing the work of Lakshmibai–Sandhya [16]. We write $W(Z)$ to denote the Weyl group of Type Z, where Z is one of the types in the Cartan–Killing classification.

Theorem 14 (Billey and Postnikov [2]). *Let G be simply-laced, then the Schubert variety $X_w \subset G/B$ is smooth if and only if w avoids the following patterns: $s_2s_1s_3s_2 \in W(A_3)$, $s_1s_2s_3s_2s_1 \in W(A_3)$, and $s_2s_0s_1s_3s_2 \in W(D_4)$.*

2.4 Conventions for simply-laced groups

2.4.1 $G = \mathrm{SL}_n$ (Type A_{n-1})

We let B be the set of lower triangular matrices in G, and $T \subset B$ the diagonal matrices in G. We have $\Phi(A_{n-1}) = \{e_j - e_i \mid 1 \leq i \neq j \leq n\}, \Phi^+(A_{n-1}) = \{e_j - e_i \mid 1 \leq i < j \leq n\}$, and $\Delta(A_{n-1}) = \{e_{i+1} - e_i \mid 1 \leq i \leq n-1\}$.

Under these conventions, the Weyl group $W(A_{n-1})$ acts on $\mathrm{Lie}_{\mathbb{R}}(T)^* = \mathbb{R}^n/(1,\dots,1)$ by permutation of the coordinates, yielding an isomorphism $W(A_{n-1})$ with the symmetric group $\mathfrak{S}_n$. Letting $\alpha_i := e_{i+1} - e_i$, the corresponding simple reflection s_i is identified with the transposition $(i\, i+1)$. It will often be convenient for us to write permutations w in one-line notation as $w(1)\dots w(n)$. The Dynkin diagram is shown in Figure 1.

2.4.2 $G = \mathrm{SO}_{2n}$ (Type D_n)

We let B be the set of lower triangular matrices in G, and $T \subset B$ the diagonal matrices in G. We have $\Phi(D_n) = \{e_j \pm e_i \mid 1 \leq i \neq j \leq n\}, \Phi^+(D_n) = \{e_j \pm e_i \mid 1 \leq i < j \leq n\}$, and $\Delta(D_n) = \{e_2 + e_1\} \cup \{e_{i+1} - e_i \mid 1 \leq i \leq n-1\}$.

Under these conventions, the Weyl group $W(D_n)$ acts on $\mathrm{Lie}_{\mathbb{R}}(T)^* = \mathbb{R}^n$ by permuting coordinates and negating pairs of coordinates. This identifies $W(D_n)$ with the permutations w of $\{-n,\dots,-1,1,\dots,n\}$ satisfying $w(i) = -w(-i)$ for all i, and such that $|\{w(1),\dots,w(n)\} \cap \{-n,\dots,-1\}|$ is even. We write $\mathfrak{D}_n$ for this realization of $W(D_n)$. Such a permutation can be uniquely specified by its *window notation* $[w(1)\dots w(n)]$.

Write $\delta_0 = e_2 + e_1$ and $\delta_i = e_{i+1} - e_i$, $i = 1, 2, \ldots, n-1$ for the simple roots. It will often be convenient for us to write $\bar{i}$ for $-i$, and we use these interchangeably. We also make the convention that $e_{\bar{i}} = e_{-i} := -e_i$ for $i > 0$. We have simple reflections $s_0 = (1\,\bar{2})(\bar{1}\,2)$ and $s_i = (i\,i+1)(\bar{i}\,\overline{i+1})$ for $i = 1, \ldots, n-1$.

2.5 Reflection subgroups and diagram automorphisms

See Figure 1 for our labeling of the Dynkin diagrams. The following is clear:

Proposition 15. *The diagram of the Type A_{n-1} has an automorphism ε_A sending $\alpha_i \mapsto \alpha_{n-i}$ for $i = 1, \ldots, n-1$, and the diagram of Type D_n has an automorphism ε_D interchanging $\delta_0 \leftrightarrow \delta_1$. If $\varepsilon \in \{\varepsilon_A, \varepsilon_D\}$, then $h(w) = h(\varepsilon_D(w))$.*

2.5.1 Reflection subgroups

In light of Theorem 14, we will be concerned with reflection subgroups isomorphic to $W(A_3)$ and $W(D_4)$ inside $W(A_{n-1})$ and $W(D_n)$.

Proposition 16. *Reflection subgroups isomorphic to $W(A_3)$ and $W(D_4)$ inside $W(A_{n-1})$ and $W(D_n)$ are characterized as follows:*

(a) No reflection subgroup $W' \subset W(A_{n-1})$ is isomorphic to $W(D_4)$.

(b) Reflection subgroups $W' \cong W(A_3)$ inside $W(A_{n-1})$ are conjugate to the parabolic subgroup $W(A_{n-1})_{\{1,2,3\}}$.

(c) Reflection subgroups $W' \cong W(D_4)$ inside $W(D_n)$ are conjugate to the parabolic subgroup $W(D_n)_{\{0,1,2,3\}}$.

(d) Reflection subgroups $W' \cong W(A_3)$ inside $W(D_n)$ come in two classes: those related to $W(D_n)_{\{1,2,3\}}$ by conjugacy and ε_D (Class I), and those conjugate to $W(D_n)_{\{0,1,2\}}$ (Class II).

2.5.2 One line notation and patterns

We will be interested in occurrences of the patterns from Theorem 14 in elements $w \in W(A_{n-1})$ or $W(D_n)$. For $w \in W(D_n)$, it will sometimes be useful for us to distinguish between Class I and II patterns (see Proposition 16(d)). Realizing these Weyl groups as $\mathfrak{S}_n$ and $\mathfrak{D}_n$, respectively, allows for one-line interpretations of pattern containment (summarized in Figure 2). This approach to pattern containment is in some sense a hybrid between the approaches of Billey [3] using signed patterns and of Billey, Braden, and Postnikov [2, 4] using patterns in the sense of Definition 12. Our distinction between Class I and II patterns is seemingly novel and reflects the disparate effects that occurrences of these patterns can have on $h(w)$.

Type	Class	Pattern	One-line
A_3	I	$s_2s_1s_3s_2$	3412
A_3	II	$s_2s_1s_3s_2$	$\pm 12\bar{3}$
A_3	I	$s_1s_2s_3s_2s_1$	4231
A_3	II	$s_1s_2s_3s_2s_1$	$\pm 1\bar{3}\bar{2}$
D_4		$s_2s_0s_1s_3s_2$	$\pm 14\bar{3}2$

Figure 2: The patterns from Theorem 14 with their one-line notations, divided according to type and class.

Definition 17.

(i) For p a signed permutation of $[k]$, we say $w \in \mathfrak{D}_n$ contains p at positions $1 \leq i_1 < \cdots < i_k \leq n$ if $\mathrm{sign}(w(i_j)) = \mathrm{sign}(p(j))$ for $j = 1, \ldots, k$ and $|w(i_1)|, \ldots, |w(i_k)|$ are in the same relative order as $|p(1)|, \ldots, |p(k)|$.

(ii) For $p \in \mathfrak{S}_k$, we say $w \in \mathfrak{S}_n$ contains p at positions $1 \leq i_1 < \cdots < i_k \leq n$ if $w(i_1), \ldots, w(i_k)$ have the same relative order as $p(1), \ldots, p(k)$. We say $u \in \mathfrak{D}_n$ contains p at positions $i_1 < \cdots < i_k$, where each $i_j \in \pm[n]$ if $u(i_1), \ldots, u(i_k)$ have the same relative order as $p(1), \ldots, p(k)$ and $|i_1|, \ldots, |i_k|$ are distinct.

In each case, we say that the *values* of the occurrence are $w(i_1), \ldots, w(i_k)$.

The following is a translation of Theorem 14 in light of our conventions for patterns.

Proposition 18. *Let G be simply-laced; then $X_w \subset G/B$ is smooth if and only if w avoids the patterns* $3412, \pm 12\bar{3}, 4231, \pm 1\bar{3}\bar{2}$, *and* $\pm 14\bar{3}2$ *(see Figure 2).*

The following statistic on occurrences of the pattern 3412 will be of special importance for us (see Theorem 6).

Definition 19 (See [10, 21])**.** We say an occurrence of 3412 in $w \in \mathfrak{S}_n$ or $\mathfrak{D}_n$ at positions $a < b < c < d$ has *height* equal to $w(a) - w(d)$. We let $\mathrm{mHeight}(w)$ denote the minimum height over all occurrences of 3412 in w.

3 Upper bounds on $h(w)$

3.1 Proof strategy

We will identify certain patterns p (among those from Proposition 18) such that if w contains p, then $h(w)$ can be computed using Theorem 4 and an analysis of the Bruhat covers of w. Then, for w avoiding these patterns and containing others, we will—by a combination of parabolic reduction (Theorem 13), inversion (Proposition 8), and diagram automorphisms (Proposition 15)—obtain a bound $h(w) \leq h(u)$ for u in some special family $\mathcal{S}$. Finally, we will show that elements $u \in \mathcal{S}$ have distinguished BP-decompositions such that the base and fiber in the bundle (Theorem 10) with total space X_u can be understood, allowing for the computation of $h(u)$. In the remainder, we refer primarily to the elements $w \in W$ rather than the Schubert varieties X_w that they index, although each of these steps has a geometric basis. We say w is *smooth* (resp. *singular*) if X_w is smooth (resp. singular).

We only have space to give a few representative proofs and proof ideas in this extended abstract.

Proposition 20. *Let $w \in \mathfrak{S}_n$ or $\mathfrak{D}_n$; we have $h(w) = 1$ if w contains:*

(i) 4231 *and $w \in \mathfrak{S}_n$,*

(ii) $\pm 12\bar{3}$,

(iii) $\pm 14\bar{3}2$, *or*

(iv) 3412 *of height one.*

Proof idea. The strategies for all cases are similar: containment of any of these patterns implies a relation $\tau_1 + \tau_2 = \tau_3 + \tau_4$ for $\tau_1, \tau_2, \tau_3, \tau_4 \in \mathrm{Inv}(w)$. We show that this implies a relation between roots indexing lower Bruhat covers of w. By results of Dyer [12], this implies that $[q^{\ell(w)-1}]L(w) > [q]L(w)$, so that $h(w) = 1$ by Theorem 4. □

3.2 Proof of Theorem 5 in Type A

In this section we obtain an upper bound on $h(w)$ for $w \in \mathfrak{S}_n$ in terms of $\mathrm{mHeight}(w)$; this establishes Theorem 5 for $W = \mathfrak{S}_n$ as well as one direction of Theorem 6.

Lemma 21. *For $n \geq 4$, consider $w \in \mathfrak{S}_n$ where $w(1) = n-1$, $w(2) = n$, $w(n-1) = 1$, $w(n) = 2$ and $w(i) = n-i+1$ for $3 \leq i \leq n-2$. Then $h(w) = n-3$.*

Proof. Let $J = \{2, 3, \ldots, n-2\}$ so that $w_J = w_0(J)$. The parabolic decomposition $w = w^J w_J$ is a Billey–Postnikov decomposition. Moreover, $L(w_J) = L(w_0(J))$ is palindromic, since $X_{w_0(J)}$ is a product of flag varieties and therefore smooth. Every $u \in W^J$ satisfies $u(2) < u(3) < \cdots < u(n-1)$ so by counting inversions with $u(1)$ and $u(n)$, we see $\ell(u) = (u(1)-1) + (n-u(n)) - \mathbf{1}_{u(1)>u(n)}$. Elements $u \in [e, w^J]^J$ are characterized by $u(1) \leq n-1$ and $u(n) \geq 2$ with $u(2) < \cdots < u(n-1)$. We are now able to count the rank sizes of $[e, w^J]^J$ to be $1, 2, 3, \ldots, n-4, n-3, n-2, n-1, n-3, n-4, \ldots, 2, 1$. Thus, $h(L^J(w^J)) = n-3$ and $h(w) = \min(h(L^J(w^J)), h(L(w_J))) = \min(n-3, \infty) = n-3$. □

For an occurrence of a 3412 in w at indices $a < b < c < d$ with $w(c) < w(d) < w(a) < w(b)$ its *content* is $1 + |\{i \mid b < i < c, w(d) < w(i) < w(a)\}|$. Let $\mathrm{mCont}(w)$ be the minimum content of a 3412 pattern in w.

Lemma 22. *For $w \in \mathfrak{S}_n$ that contains* 3412*,* $\mathrm{mHeight}(w) = \mathrm{mCont}(w)$*.*

One advantage of working with content instead of height is that we evidently have $\mathrm{mCont}(w) = \mathrm{mCont}(w^{-1})$.

Lemma 23. *Suppose that $w \in \mathfrak{S}_n$ avoids* 4231 *and contains* 3412*. Then $h(w) \leq \mathrm{mHeight}(w)$.*

Proof. We use induction on n. The statement is true when $n = 4$, where $h(3412) = \mathrm{mHeight}(3412) = 1$. For a general n and $w \in \mathfrak{S}_n$, let $k = \mathrm{mHeight}(w) = \mathrm{mCont}(w)$. For $J = \{2, 3, \ldots, n-1\}$, if w_J has $\mathrm{mCont}(w_J) = k$, then we are done by the induction hypothesis and Theorem 13 which says $h(w) \leq h(w_J) \leq \mathrm{mCont}(w_J) = k$. We can thus assume without loss of generality that the index 1 appears in all 3412's of w with content k. Similarly, by considering $J = \{1, 2, \ldots, n-2\}$, we can also assume that the index n appears in all 3412's of w with content k. As $h(w) = h(w^{-1})$, with the same argument on w^{-1}, we can reduce to the case that w contains a unique 3412 of content k on indices $1 < w^{-1}(n) < w^{-1}(1) < n$ (see Figure 3). As we assume that w_J does not contain a

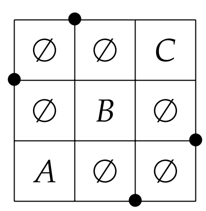

Figure 3: The permutation diagram for w with an occurrence of 3412 on the boundary. We draw permutation diagrams by putting $\bullet$'s at Cartesian coordinates $(i, w(i))$.

3412 of content k, there does not exist i such that $1 < i < w^{-1}(n)$ with $w(i) > w(n)$. By symmetry, we know six of the regions in Figure 3 are empty as shown, and label the other three regions as A, B, C. By definition, $|B| = k - 1$. If $k = 1$, then $h(w) = 1$ by Proposition 20. If $k > 1$, B is not empty; since w avoids 4231, A and C must be empty. Thus w is exactly the permutation in Lemma 21, which gives $h(w) = n - 3 = k$. □

3.3 Extension to Type D

Proposition 24. *If $w \in \mathfrak{D}_n$ contains* 4231*, then $h(w) \leq 2$.*

Proof idea. We adapt the strategy for Proposition 20 to show that for most occurrences of 4231, we in fact have $h(w) = 1$. The few remaining cases are analyzed separately. □

Definition 25. Define the *magnitude* $\mathrm{mag}(w)$ as the smallest $b > 0$ such that w has an occurrence of $\pm 1\bar{3}\bar{2}$ with values $a\bar{c}\bar{b}$.

Proposition 26. *Suppose $w \in \mathfrak{D}_n$ contains $\pm 1\bar{3}\bar{2}$ and avoids 4231, then $h(w) \leq \mathrm{mag}(w) - 1$.*

Proposition 27. *Let $W = \mathfrak{D}_n$ for $n \geq 5$, let $J = S \setminus \{1\}, J' = S \setminus \{0\}, K = S \setminus \{n-1\}$, and suppose $w \in \mathfrak{D}_n$ is singular, but satisfies:*

(i) w avoids $4231, \pm 1\bar{3}\bar{2}, \pm 12\bar{3}, \pm 14\bar{3}2$, and neither w nor $\varepsilon_D(w)$ contains any occurrences of 3412 of height one,

(ii) $w_J, w_{J'}, w_K, {}_Jw, {}_{J'}w, {}_Kw$ are smooth.

Then $w = u := [n, 2, \bar{3}, \bar{4}, \dots, \overline{n-1}, \pm 1]$ or $w = \varepsilon_D(u)$.

We are now ready to complete the proof of Theorem 5, resolving Conjecture 3.

Proof of Theorem 5. First suppose $G = \mathrm{SL}_{r+1}$, and let $w \in W(A_r) = \mathfrak{S}_{r+1}$ such that X_w is singular. By Theorem 14, w contains 4231 or 3412. If w contains 4231, then $h(w) = 1$ by Proposition 20. Otherwise w avoids 4231 and contains 3412, so $h(w) \leq \mathrm{mHeight}(w)$ by Lemma 23. It is clear by definition that $\mathrm{mHeight}(w) \leq r - 2$ for any w, so we are done.

Now suppose $G = \mathrm{SO}_{2r}$ for $r \geq 5$, and let $w \in W(D_r) = \mathfrak{D}_r$. Suppose by induction that the claim is true for $G = \mathrm{SO}_{2r'}$ for $r' < r$ (the base case $r' = 4$ is covered by the computations in [2]). If w contains 4231, then $h(w) \leq 2 \leq r - 2$ by Proposition 24, so we may assume that w avoids 4231. Then by Proposition 26, if w contains $\pm 1\bar{3}\bar{2}$ we have $h(w) \leq \mathrm{mag}(w) \leq r - 2$. If w contains any of the patterns from Proposition 20, then $h(w) = 1 \leq r - 2$. Let $J = S \setminus \{2\}, J' = S \setminus \{0\}, K = S \setminus \{r-1\}$; if any of $w_J, w_{J'}, w_K, {}_Jw, {}_{J'}w, {}_Kw$ is singular, then by the type A result, or by the induction hypothesis, we have $h(w) \leq r - 3$. Finally, if w does not fall into any of the above cases, then w satisfies the hypotheses (i) and (ii) of Proposition 27, so $w = u := [r, 2, \bar{3}, \bar{4}, \dots, \overline{r-1}, \pm 1]$ or $w = \varepsilon_D(u)$.

We will now compute $h(u) = h(\varepsilon_D(u))$; suppose for convenience that r is even, the other case being exactly analogous. Let $I = \{1, 2 \dots, r-2\}$, then we have $u_I = w_0(I)$ is the longest element of $\mathfrak{S}_{r-1}$, so $h(u_I) = \infty$. Thus we need to compute $h(L^I(u^I))$ with $u^I = [\overline{r-1}, \dots, \bar{4}, \bar{3}, 2, r, \bar{1}]$. Notice $\ell(u^I) = N := \frac{1}{2}(r^2 - 3r + 4)$ with reduced word:

$$s_0(s_2s_0)(s_3s_2s_1)\cdots(s_{r-4}s_{r-5}\cdots s_3s_2s_0)(s_{r-3}\cdots s_3s_2s_1)(s_{r-2}\cdots s_3s_2s_0)s_{r-1}.$$

We claim that $L^I(u^I) = 1 + 2q + 3q^2 + \cdots + aq^{N-2} + 2q^{N-1} + q^N$, with $a \geq 4$, so that $h(u) = h(L^I(u^I)) = 2 < r - 2$. Indeed, the elements of length one in $[e, u^I]^I$ are $\{s_0, s_{r-1}\}$, the elements of length two are $\{s_0s_{r-1}, s_2s_0, s_{r-2}s_{r-1}\}$, and the elements of length $N - 1$ are $\{s_0u^I, s_2u^I\}$. Consider the four elements $z_1 = s_0s_2u^I, z_2 = s_2s_0u^I, z_3 = s_0u^Is_{r-1}, z_4 = s_3s_2u^I$. It is easy to check for $i = 1, 2, 3, 4$ that $\ell(z_i) = N - 2$, that $z_i \leq u^I$, and that $z_i \in W^I$; thus $a \geq 4$ as desired. □

4 Exact formula when $G = \mathrm{SL}_n$

For $w \in \mathfrak{S}_n$, we have proved the upper bound in Theorem 6 in Section 3.2. The lower bound follows from Lemma 28 below.

Lemma 28. *Suppose that* $w \in \mathfrak{S}_n$ *avoids* 4231 *and contains* 3412. *Then* $h(w) \geq \mathrm{mHeight}(w)$.

Proof idea. This is an inductive argument using a diagram analysis, analogous to but more involved than the proof of Lemma 23. The relevant diagram is shown in Figure 4.

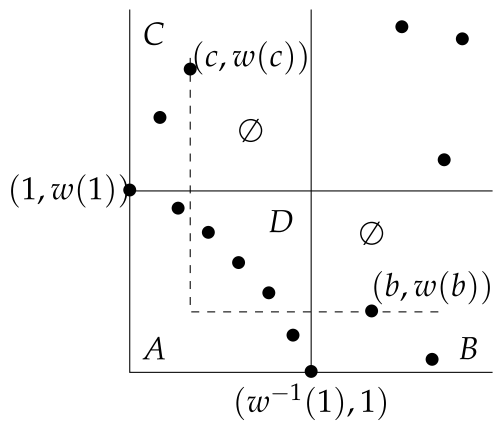

Figure 4: The permutation diagram of w used in the proof of Lemma 28.

□

Acknowledgements

We are grateful for our very fruitful conversations with Alexander Woo and Sara Billey.

References

[1] H. Abe and S. Billey. "Consequences of the Lakshmibai-Sandhya theorem: the ubiquity of permutation patterns in Schubert calculus and related geometry". *Schubert calculus—Osaka 2012*. Vol. 71. Adv. Stud. Pure Math. Math. Soc. Japan, [Tokyo], 2016, pp. 1–52. DOI.

[2] S. Billey and A. Postnikov. "Smoothness of Schubert varieties via patterns in root subsystems". *Adv. in Appl. Math.* **34**.3 (2005), pp. 447–466. DOI.

[3] S. C. Billey. "Pattern avoidance and rational smoothness of Schubert varieties". *Adv. Math.* **139**.1 (1998), pp. 141–156. DOI.

[4] S. C. Billey and T. Braden. "Lower bounds for Kazhdan-Lusztig polynomials from patterns". *Transform. Groups* **8**.4 (2003), pp. 321–332. DOI.

[5] S. C. Billey and G. S. Warrington. "Maximal singular loci of Schubert varieties in $\mathrm{SL}(n)/B$". *Trans. Amer. Math. Soc.* **355**.10 (2003), pp. 3915–3945. DOI.

[6] A. Björner and T. Ekedahl. "On the shape of Bruhat intervals". *Ann. of Math. (2)* **170**.2 (2009), pp. 799–817. DOI.

[7] T. Braden and R. MacPherson. "From moment graphs to intersection cohomology". *Math. Ann.* **321**.3 (2001), pp. 533–551. DOI.

[8] F. Brenti. "A combinatorial formula for Kazhdan-Lusztig polynomials". *Invent. Math.* **118**.2 (1994), pp. 371–394. DOI.

[9] J. B. Carrell and J. Kuttler. "Smooth points of T-stable varieties in G/B and the Peterson map". *Invent. Math.* **151**.2 (2003), pp. 353–379. DOI.

[10] A. Cortez. "Singularités génériques et quasi-résolutions des variétés de Schubert pour le groupe linéaire". *Adv. Math.* **178**.2 (2003), pp. 396–445. DOI.

[11] V. V. Deodhar. "Local Poincaré duality and nonsingularity of Schubert varieties". *Comm. Algebra* **13**.6 (1985), pp. 1379–1388. DOI.

[12] M. Dyer. "On the "Bruhat graph" of a Coxeter system". *Compositio Math.* **78**.2 (1991), pp. 185–191. Link.

[13] C. Gaetz and Y. Gao. "On the minimal power of q in a Kazhdan-Lusztig polynomial". 2023. arXiv:2303.13695.

[14] D. Kazhdan and G. Lusztig. "Representations of Coxeter groups and Hecke algebras". *Invent. Math.* **53**.2 (1979), pp. 165–184. DOI.

[15] D. Kazhdan and G. Lusztig. "Schubert varieties and Poincaré duality". *Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979)*. Proc. Sympos. Pure Math., XXXVI. Amer. Math. Soc., Providence, R.I., 1980, pp. 185–203.

[16] V. Lakshmibai and B. Sandhya. "Criterion for smoothness of Schubert varieties in $\mathrm{Sl}(n)/B$". *Proc. Indian Acad. Sci. Math. Sci.* **100**.1 (1990), pp. 45–52. DOI.

[17] L. Manivel. "Le lieu singulier des variétés de Schubert". *Internat. Math. Res. Notices* 16 (2001), pp. 849–871. DOI.

[18] P. Polo. "Construction of arbitrary Kazhdan-Lusztig polynomials in symmetric groups". *Represent. Theory* **3** (1999), pp. 90–104. DOI.

[19] E. Richmond and W. Slofstra. "Rationally smooth elements of Coxeter groups and triangle group avoidance". *J. Algebraic Combin.* **39**.3 (2014), pp. 659–681. DOI.

[20] E. Richmond and W. Slofstra. "Billey-Postnikov decompositions and the fibre bundle structure of Schubert varieties". *Math. Ann.* **366**.1-2 (2016), pp. 31–55. DOI.

[21] A. Woo. "Permutations with Kazhdan-Lusztig polynomial $P_{id,w}(q) = 1 + q^h$". *Electron. J. Combin.* **16**.2, Special volume in honor of Anders Björner (2009). Link.

[22] A. Woo and A. Yong. "Governing singularities of Schubert varieties". *J. Algebra* **320**.2 (2008), pp. 495–520. DOI.