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Shi arrangements and low elements in Coxeter groups

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Abstract. Given an arbitrary Coxeter system (W,S) and a nonnegative integer m, the m-Shi arrangement of (W,S) is a subarrangement of the Coxeter hyperplane arrangement of (W,S). The classical Shi arrangement ($m=0$) was introduced in the case of affine Weyl groups by Shi to study Kazhdan-Lusztig cells for W. As two key results, Shi showed that each region of the Shi arrangement contains exactly one element of minimal length in W and that the union of their inverses form a convex subset of the Coxeter complex. The set of m-low elements in W were introduced to study the word problem of the corresponding Artin-Tits (braid) group and they turn out to produce automata to study the combinatorics of reduced words in W.

We generalize and extend Shi's results to any Coxeter system. First, for $m \in \mathbb{N}$ the set of minimal length elements of the regions in a m-Shi arrangement is precisely the set of m-low elements, settling a conjecture of the first and third authors in this case. Second, for $m = 0$ the union of the inverses of the (0-)low elements form a convex subset in the Coxeter complex, settling a conjecture by the third author, Nadeau and Williams.

Keywords: Coxeter groups, low elements, Shi arrangements, Garside shadows

1 Introduction

Let (W,S) be a Coxeter system with length function $\ell : W \to \mathbb{N}$ and set of reflections $T = \cup_{w\in W} wSw^{-1} = \{s_\alpha \mid \alpha \in \Phi^+\}$, where Φ^+ is a set of positive roots in a root system Φ for (W,S). As a reflection group, W acts on the *Coxeter complex* $\mathcal{U}(W,S)$ that arises naturally from the Coxeter (hyperplane) arrangement $\mathcal{A}(W,S) = \{H_\alpha \mid \alpha \in \Phi^+\}$. The maximal simplices of $\mathcal{C}(W,S)$ are called *chambers* and they correspond to the connected

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components of the complement of $\mathcal{A}(W,S)$. The map $w \mapsto C_w$ is a bijection between W and the set of chambers; see for instance Figure 1 and Figure 2 below.

Let $m \in \mathbb{N}$. A positive root $\beta \in \Phi^+$ is *m-small* if there are at most m parallel, or ultraparallel, hyperplanes separating H_β from the fundamental chamber C_e (not counting H_β). Denote by Σ_m the set of m-small roots. Small roots were introduced by Brink and Howlett to prove that any finitely generated Coxeter system is automatic [2]; a key and remarkable result in their article was to prove that Σ_0 is a finite set. Later, Fu [6] proved that Σ_m is finite for all $m \in \mathbb{N}$. The sets of m-small roots are the building blocks of a family of regular automata that recognize the language of reduced words in (W,S).

The *m-Shi arrangement* $\mathrm{Shi}_m(W,S)$ *of* (W,S) is the hyperplane subarrangement of $\mathcal{A}(W,S)$:

$$\mathrm{Shi}_m(W,S) = \{H_\alpha \mid \alpha \in \Sigma_m\}.$$

The regions of $\mathrm{Shi}_m(W,S)$ are union of chambers and define therefore an equivalence relation $\sim_m$ on W. It was conjectured in [5, Conjecture 2] that each equivalence class under $\sim_m$ contains a unique minimal length element and that the set of these minimal length elements is the set of m-low elements. An element $w \in W$ is *m-low* if the inversion set $\Phi(w)$ of w is spanned by the m-small roots it contains. The set L_m of m-low elements turns out to be a finite Garside shadow [5, 3], that is, it shadows a finite Garside family in a corresponding Artin-Tits group.

The following two theorems are the main results of this abstract: the first theorem settles [5, Conjecture 2] and the second settles [7, Conjecture 3].

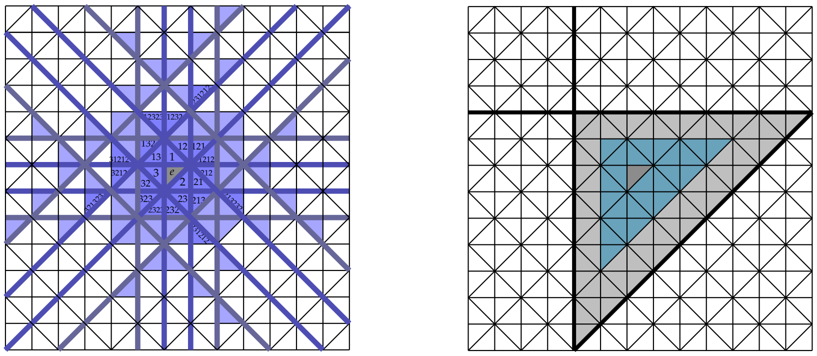
Theorem 1.1. *Let (W,S) be a Coxeter system and $m \in \mathbb{N}$.*

1. *Each region of $\mathrm{Shi}_m(W,S)$ contains a unique element of minimal length.*
2. *The set of the minimal length elements of $\mathrm{Shi}_m(W,S)$ is equal to the set L_m of m-low elements.*

A noteworthy consequence of the Theorem 1.1 and of the fact that L_m is a Garside shadow is that if the join z (under the right weak order) of two minimal elements of $\mathrm{Shi}_m(W,S)$ exists, then z is also the minimal element of a region of $\mathrm{Shi}_m(W,S)$.

Theorem 1.2. *Let (W,S) be a Coxeter system. The union of the chambers C_w for $w^{-1} \in L_0$ is a convex set.*

These theorems are illustrated in Figures 1 and 2. The proofs of these theorems depend on the *sandwich property* of *short inversion posets*, discussed in §3. The first author showed in 2019 that the *inversion set* $\Phi(w)$ *of* $w \in W$ is spanned by its *set of short inversions* $\Phi^1(w)$. We endow $\Phi^1(w)$ with a poset structure arising from the configuration of maximal dihedral reflection subgroups: $\alpha \dot{\prec}_w \beta$ if β is not in the simple system of the maximal dihedral reflection subgroup containing $\alpha, \beta \in \Phi^1(w)$, see §3.2. Then we prove that any short inversion $\beta \in \Phi^1(w)$ is *sandwiched* between a left-descent root and a



(a) The 1-Shi arrangement (with the 0-Shi arrangement in darker blue)

(b) The 1-Shi polyhedron (with the 0-Shi polyhedron inside)

Figure 1: The 1-Shi arrangement and the 1-Shi polyhedron for $\tilde{B}_2$.

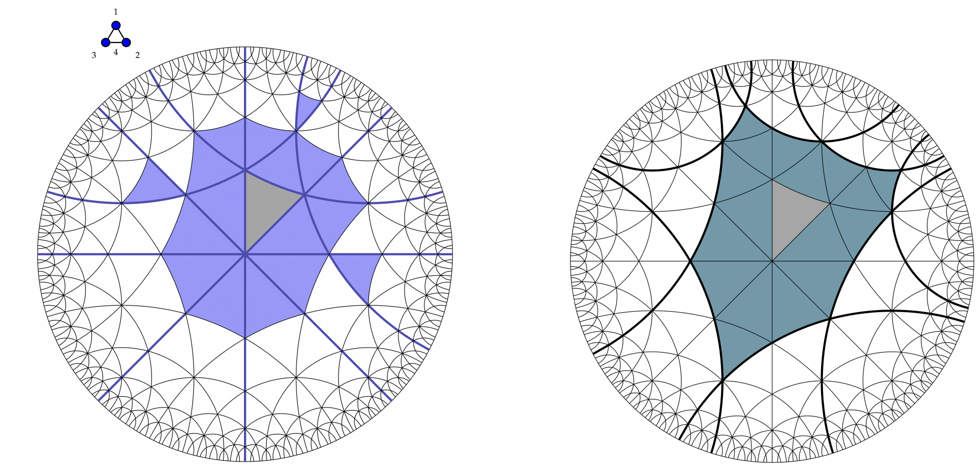
right-descent root, roots naturally defined from the left and right descent sets of w; this is Theorem 3.6, which is the core result of this abstract. We emphasize that these posets are new and have been very useful in analyzing elements of W.

In order to properly introduce m-Shi arrangements in as many realizations of the Coxeter arrangement as possible (e.g. Tits cones, Davis complexes, Euclidean and Hyperbolic spaces, etc.), the full paper uses the notion of *chambered sets*. Our discussion of chambered sets is omitted in the extended abstract.

Finally, in §5, we introduce extended Shi arrangements and we focus on Theorem 1.1 and Theorem 1.2. Combinatorics of roots and reduced words are surveyed in §2 while m-small roots and m-low elements are discussed in §4.

Let us give a bit of history about the m-Shi arrangement. For more details and references, see [4]. In 1986, Shi introduced the Shi arrangement $\mathrm{Shi}(W,S) = \mathrm{Shi}_0(W,S)$ in the case of irreducible affine Weyl groups to study Kazhdan-Lusztig cells for W. Surprising connections to Shi arrangements have been studied: to *ad*-nilpotent ideals of Borel subalgebras, and to Catalan arrangements, for example. In 1988, Shi proved a conjecture by Carter on the number of sign-types of an affine Weyl group. In order to prove that conjecture, Shi enumerated the number of regions in $\mathrm{Shi}_0(W,S)$. In particular, Shi proves that each region of the Shi arrangement contains a unique minimal element and that the union of the chambers corresponding to the inverses of those minimal elements is a convex subset of the Euclidean space. Theorems 1.1 and 1.2 are a generalization of both results to arbitrary Coxeter systems. Notice that in the case of affine Coxeter systems and for $m = 0$, Theorem 1.1 was proven by Chapelier-Laget and the second author, while for rank 3 and $m = 0$ it was proven by Charles. Osajda and Przytycki independently, in 2022, have a proof of Theorem 1.1(1) in the case $m = 0$,

As far as we know, the *m-(extended) Shi arrangements* were defined for affine Coxeter systems in Armstrong's thesis, but were implicit in Athanasiadis's work on *generalized*



(a) The 0-Shi arrangement. The low elements are shaded.

(b) The 0-Shi polyhedron

Figure 2: The 0-Shi arrangement and polyhedron of the Coxeter system with Coxeter graph given in the upper left.

Catalan numbers. In the extended case, the regions in $\mathrm{Shi}_m(W,S)$ were first enumerated by Yoshinaga using techniques from representation theory. In his thesis, Thiel gives a direct proof by extending Shi's result to any m in the case of affine Coxeter systems.

Theorem 1.1 shows that Thiel's minimal elements for $\mathrm{Shi}_m(W,S)$ are precisely the m-low elements. We recover Thiel's results as a direct consequence of the proof of Theorem 1.2.

Theorem 1.3. *If (W,S) is of affine type, then the union of the chambers C_w for $w^{-1} \in L_m$ is a convex set.*

Theorem 1.3 is not true for an indefinite Coxeter system, i.e., neither affine nor finite; for a counterexample see Figure 4. There are many new questions about the Shi arrangement in indefinite type; see [4] for a few of them.

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2 Preliminaries

Fix a Coxeter system (W,S) with length function $\ell : W \to \mathbb{N}$; the *rank of* (W,S) is the cardinality of S. We assume the reader familiar with the basics of the theory of Coxeter groups; see for instance [8, 1].

Combinatorics of reduced words We say that a word $s_1 \dots s_k$ ($s_i \in S$) is a *reduced word for* $w \in W$ if $w = s_1 \dots s_k$ and $k = \ell(w)$. For $u, v, w \in W$, we say u *is a prefix of* w if a reduced word for u can be obtained as a prefix of a reduced word for w; v *is a suffix of* w if a reduced word for u can be obtained as a suffix of a reduced word for w; and $w = uv$ is a *reduced product* if $\ell(w) = \ell(u) + \ell(v)$. More generally, we say that $w = u_1 \dots u_k$ is a *reduced product* if $\ell(w) = \ell(u_1) + \cdots + \ell(u_k)$, $u_i, w \in W$.

Weak and Bruhat orders This suffix/prefix terminology is best embodied by the *weak order*. The *right weak order* is the poset $(W, \leq_R)$ defined by $u \leq_R w$ if u is a prefix of w. The right weak order gives a natural orientation of the right Cayley graph of (W, S): for $w \in W$ and $s \in S$, we orient the edge $w \to ws$ if $w \leq_R ws$.

Recall that the *Bruhat order* is the poset $(W, \leq)$ defined as follows: $u \leq w$ if and only if a word for u can be obtained as a subword of a reduced word for w. We denote covering in the Bruhat order by $x \lhd y$.

Root system Please see [8] for information on geometric representations of (W, S), the symmetric bilinear form B, and root systems. We note that (1) if B is positive definite, then W is finite; if it is positive semi-definite but not positive definite, then W is affine; and otherwise W is indefinite; and (2) there is a bijection between the positive roots Φ^+ and the reflections T.

Depth of positive roots The *depth on* Φ^+ [2] is the function $\mathrm{dp} : \Phi^+ \to \mathbb{N}$ defined by:

$$\mathrm{dp}(\beta) = \min\{\ell(g) \mid g(\beta) \in \Delta\}.$$

There is a recursion for depth [1, Lemma 4.6.2] and $\mathrm{dp}(\alpha_s) = 0$ for all $s \in S$. The depth may be seen as measuring how far a positive root is from Δ in the orbit $\Phi = W(\Delta)$. There are many different depths and they are not equivalent. In this article we also consider the ∞-depth. For more on depths, lengths and weak orders on root systems, see [5, §5.1].

Inversion sets The *inversion set* $\Phi(w)$ *of* $w \in W$ is defined by:

$$\Phi(w) = \Phi^+ \cap w(\Phi^-) = \{\beta \in \Phi^+ \mid \ell(s_\beta w) < \ell(w)\}.$$

Its cardinality is $\ell(w)$ and is sometimes denoted in the literature by $N(w)$ or $\mathrm{inv}(w)$.

Reflection subgroups We end this section by recalling some useful facts about reflection subgroups and, in particular, about maximal dihedral reflection subgroups [5, §2.8]. A *reflection subgroup* W' of W is a subgroup $W' = \langle s_\beta \mid \beta \in A \rangle$ generated by the reflections associated to the roots in some $A \subseteq \Phi^+$. We set $\Phi_{W'} := \{\beta \in \Phi \mid s_\beta \in W'\}$ and $\Delta_{W'} := \{\alpha \in \Phi^+ \mid \Phi(s_\alpha) \cap \Phi_{W'} = \{\alpha\}\}$. The first author showed in 1990 that $\Phi_{W'}$ is a root system in (V, B) with simple root system $\Delta_{W'}$ and simple reflections $\chi(W') := \{s_\alpha \mid \alpha \in \Delta_{W'}\}$. There are corresponding positive roots: $\Phi^+_{W'} = \Phi_{W'} \cap \Phi^+$; both notions depend on (W, S) and not just W.

Maximal dihedral reflection subgroups A reflection subgroup W' of rank 2 is well-known to be isomorphic to a dihedral group and is so called a *dihedral reflection subgroup*. This following result gives a criterion for comparing depths of roots.

Proposition 2.1. *Let $\alpha, \beta \in \Phi^+$. Assume there is a dihedral reflection subgroup W' such that such that $\alpha \in \Delta_{W'}$ and $\beta \in \Phi^+_{W'} \setminus \Delta_{W'}$, then* $\mathrm{dp}(\alpha) < \mathrm{dp}(\beta)$.

A dihedral reflection subgroup W' is a *maximal dihedral reflection subgroup* if it is not contained in any other dihedral reflection subgroup but itself. Our partial order on the short inversions is based on maximal dihedral reflection subgroups. The following result is useful: it gives the form of inversion sets in maximal dihedral reflection subgroup.

Proposition 2.2. *Let W' be a maximal dihedral reflection subgroup. The inversion set of $u \in W'$, $u \neq e$, is of the form $\Phi_{W'}(u) = \mathrm{cone}_\Phi(\alpha, \beta)$ with $\alpha \in \Delta_{W'}$ and $\beta \in \Phi^+_{W'}$.*

Any dihedral reflection subgroup is contained in a unique maximal dihedral reflection subgroup. In particular, for $\alpha, \beta \in \Phi$ such that $\mathbb{R}\alpha \neq \mathbb{R}\beta$, the dihedral reflection subgroup $\langle s_\alpha, s_\beta \rangle$ is contained in the unique maximal dihedral reflection subgroup $\mathcal{M}_{\alpha,\beta}$, with root subsystem $\Phi_{\alpha,\beta} = (\mathbb{R}\alpha \oplus \mathbb{R}\beta) \cap \Phi$, and simple system $\Delta_{\mathcal{M}_{\alpha,\beta}}$. For simplicity, if $s = s_\alpha \in T$ and $t = s_\beta \in T$, we write $\mathcal{M}_{s,t} = \mathcal{M}_{\alpha,\beta}$.

Remark 2.3. The finite maximal dihedral reflection subgroups of (W, S) are precisely the finite parabolic subgroups of rank 2, that is, the conjugates of the standard parabolic subgroups $W_{s,t} = \langle s, t \rangle$ for $s, t \in S$ distinct such that the order $m_{s,t}$ of st is finite. Conversely, any conjugate of a rank 2 finite parabolic subgroup is maximal [3, Theorem 3.11(b)].

3 Short inversion posets

Among all inversions of an element of W, the short inversions span all the others. The key to proving Theorem 4.3 is to exhibit an order on the short inversions and to show that any short inversion is *sandwiched* between a *left descent-root* and a *right descent-root*.

3.1 Short inversions and descent roots

We think of $\Phi(w)$ as a polyhedral cone in $\Phi \subseteq V$ since $\Phi(w) = \mathrm{cone}_\Phi(\Phi(w))$. The *set of short inversions of* $\Phi(w)$ is the set

$$\Phi^1(w) = \{\beta \in \Phi^+ \mid \ell(s_\beta w) = \ell(w) - 1\} = \{\beta \in \Phi^+ \mid s_\beta w \lhd w\} \subseteq \Phi(w).$$

The first author showed in 1994 that $\Phi^1(w)$ is a basis of cone$(\Phi(w))$: the set of extreme rays of cone$_\Phi(\Phi(w))$ is indeed $\{\mathbb{R}_{\geq 0}\beta \mid \beta \in \Phi^1(w)\}$.

Proposition 3.1. *Let $w \in W$ and $\alpha, \beta \in \Phi^1(w)$ with $\alpha \neq \beta$. Then $\alpha \in \Delta_{\mathcal{M}_{\alpha,\beta}}$ or $\beta \in \Delta_{\mathcal{M}_{\alpha,\beta}}$. In particular: (1) if $\Delta_{\mathcal{M}_{\alpha,\beta}} = \{\alpha, \alpha'\}$ and $\beta \neq \alpha'$, then $\alpha' \notin \Phi(w)$; or (2) if $\Delta_{\mathcal{M}_{\alpha,\beta}} = \{\alpha, \beta\}$, then $\Phi^+_{\mathcal{M}_{\alpha,\beta}} \subseteq \Phi(w)$ and $\mathcal{M}_{\alpha,\beta}$ is finite.*

The well-known left and right descent sets of $w \in W$ have their natural counterparts in $\Phi^1(w)$. The *left descent set* $D_L(w) = \{s \in S \mid sw \lhd w\}$ is in bijection with the set of *left descent-roots:* $\Phi^L(w) = \Phi(w) \cap \Delta$. The *right descent set* $D_R(w) = \{s \in S \mid ws \lhd w\}$ is in bijection with the set of *right descent-roots:* $\Phi^R(w) = \{-w(\alpha_s) \mid s \in D_R(w)\}$.

3.2 Short inversion posets

Let $w \in W$. For $\alpha, \beta \in \Phi^1(w)$, we write $\alpha \dot{\prec}_w \beta$ if $\beta \notin \Delta_{\mathcal{M}_{\alpha,\beta}}$. By Proposition 3.1, this is equivalent to $\alpha \in \Delta_{\mathcal{M}_{\alpha,\beta}}$ and $\beta \notin \Delta_{\mathcal{M}_{\alpha,\beta}}$. Proposition 3.2 is a direct consequence of Proposition 2.1.

Proposition 3.2. *Let $w \in W$ and $\alpha, \beta \in \Phi^1(w)$. If $\alpha \dot{\prec} \beta$, then* $\mathrm{dp}(\alpha) < \mathrm{dp}(\beta)$.

For $w \in W$, we define the relation $\preceq_w$ to be the transitive and reflexive closure of $\dot{\prec}_w$, which turns out to be a partial order on $\Phi^1(w)$.

Proposition 3.3. *The relation $\preceq_w$ is a partial order on $\Phi^1(w)$. Moreover, for any reduced word $w = s_1 \dots s_k$ consider the following total order $\leq$ on $\Phi(w)$: $\alpha_{s_1} < s_1(\alpha_{s_2}) < \cdots < s_1 \dots s_{k-1}(\alpha_{s_k})$. Then $\alpha \preceq_w \beta$ implies $\alpha \leq \beta$ and* $\mathrm{dp}(\alpha) \leq \mathrm{dp}(\beta)$ *for any $\alpha, \beta \in \Phi^1(w)$.*

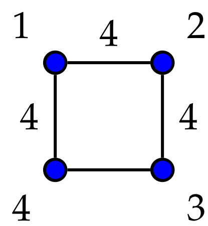
Remark 3.4. (1) The relation $\dot{\prec}_w$ is not the cover relation for $\preceq_w$. (2) The total order on $\Phi(w)$ in the statement of Proposition 3.3 is in fact the restriction of an *admissible order* on Φ^+ to $\Phi(w)$. Admissible orders on Φ^+ are in bijection with *reflection orders*, which plays a role in Kazhdan-Lusztig theory

Example 3.5. Consider (W, S) with $S = \{1, 2, 3, 4\}$ and the Coxeter graph in Figure 3. This is an indefinite Coxeter system. Let $w = 1234232314$, so that $\Phi^L(w) = \{\alpha_1\}$, $\Phi^R(w) = \{123432321(\alpha_4)\}$ and $\Phi^1(w) = \{\alpha_1, 1(\alpha_2), 31(\alpha_2), \gamma = 1234232(\alpha_1), 1(\alpha_4), \beta = 123432321(\alpha_4)\}$. The Hasse diagram of the short inversion poset of w is in Figure 3.

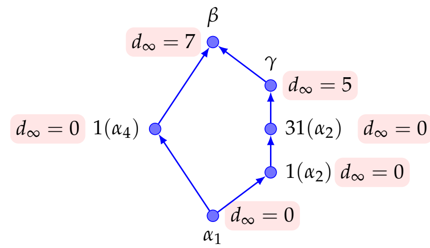
We state now the main result of this section, the sandwich theorem.

Theorem 3.6. *Let $w \in W$. For the poset $(\Phi^1(w), \preceq_w)$, the minimal elements are the left-descent roots in $\Phi^L(w)$ and the maximal elements are the right-descent roots in $\Phi^R(w)$. More precisely, for any $\beta \in \Phi^1(w)$ there is $\alpha \in \Phi^L(w)$ and $\gamma \in \Phi^R(w)$ such that $\alpha \preceq_w \beta \preceq_w \gamma$.*

The key to proving Theorem 3.6 is to explicitly construct, for $w \in W$ and for each $\beta \in \Phi^1(w) \setminus \Phi^L(w)$, a short inversion $\alpha \in \Phi^1(w)$ such that $\alpha \dot{\prec}_w \beta$. For such a $\beta \in \Phi^1(w) \setminus \Phi^L(w)$, we consider $g \in W$ such that $g(\beta) \in \Delta$ and $\ell(g) = \mathrm{dp}(\beta)$, which exists by definition of the depth.



(a) The Coxeter graph



(b) The short root poset for w.

Figure 3: Observe that any short inversion is *sandwiched* between a left descent-root and a right descent-root in the short root poset. To the side of each root is its ∞-depth. See Example 3.5 and Section 4.1.

4 m-Small roots and m-low elements

Let (W, S) be a Coxeter system and $m \in \mathbb{N}$. In this section, we provide, as a consequence of Theorem 3.6, a key characterization of m-low elements: an element $w \in W$ is m-low if and only if $\Phi^R(w)$ consists of m-small roots, see Theorem 4.3 below.

4.1 Dominance order, dominance-depth, and m-small roots

Defined by Brink and Howlett [2], the *dominance order* is the partial order $\preceq_{\text{dom}}$ on Φ^+:

$$\alpha \preceq_{\text{dom}} \beta \iff (\forall w \in W,\ \beta \in \Phi(w) \implies \alpha \in \Phi(w)).$$

In the same paper, they introduced, in relation to the dominance order, another depth-statistic: the *dominance-depth* or ∞-*depth* $\text{dp}_\infty : \Phi^+ \to \mathbb{N}$ is defined by

$$\text{dp}_\infty(\beta) = |\{\alpha \in \Phi^+ \setminus \{\beta\} \mid \alpha \prec_{\text{dom}} \beta\}|.$$

In particular, $\text{dp}_\infty(\alpha_s) = 0$ for all $s \in S$ and there is a recurrence analogous to the recursion for depth. For $m \in \mathbb{N}$, the set Σ_m of *m-small roots* is the set of positive roots that dominate at most m distinct proper positive roots; that is, $\Sigma_m = \{\beta \in \Phi^+ \mid \text{dp}_\infty(\beta) \leq m\}$. The set Φ^+ is then $\bigcup_{m \in \mathbb{N}} \Sigma_m$. The m-small roots are defined in the introduction in relation with parallelism. Brink and Howlett [2] (for $m = 0$) and Fu [6] (for all m) proved that the set Σ_m is finite for all $m \in \mathbb{N}$ and finite S, which implies that the sets of m-small roots provides a decomposition of the positive roots into finite sets whenever S is finite.

4.2 m-small inversion sets and m-low elements

The *m-small inversion set of* $w \in W$ is the set:

$$\Sigma_m(w) = \Phi(w) \cap \Sigma_m.$$

The *set L_m of m-low elements* is, see [5] for more details:

$$L_m = \{w \in W \mid \Phi(w) = \mathrm{cone}_\Phi(\Sigma(w))\} = \{w \in W \mid \Phi^1(w) \subseteq \Sigma_m\}.$$

Example 4.1. (1) If W is finite, then $\Sigma_m = \Sigma_0 = \Phi^+$ for all $m \in \mathbb{N}$. Hence $L_m = L_0 = W$. (2) The elements of the set L_0 in affine type $\tilde{B}_2$ are the darker blue regions in Figure 1 (a), and the elements of L_1 are shaded a lighter blue. (3) The set L_0 of a non-affine Coxeter arrangement consists of the elements in the blue regions in Figure 2.

If S is finite, the set Σ_m is finite and therefore the set L_m is also finite. Actually, if S is finite, the set of m-low elements is a finite Garside shadow, that is, L_m contains S and is closed under taking suffixes and under taking join in the right weak order.

The key notion to prove that L_m is a Garside shadow is *bipodality*: a set $A \subseteq \Phi^+$ is *bipodal* if for any $\beta \in A$ and maximal dihedral reflection subgroup W' such that $\beta \in \Phi_{W'} \setminus \Delta_{W'}$ we have $\Delta_{W'} \subseteq A$; see [5, 3] for more information. Because L_m is bipodal and a Garside shadow, we have the following useful corollary.

Corollary 4.2. *Let $w \in W$, $\alpha, \beta \in \Phi^1(w)$ with $\alpha \preceq_w \beta$, then $\mathrm{dp}_\infty(\alpha) \leq \mathrm{dp}_\infty(\beta)$.*

As a direct consequence of Theorem 3.6 (the sandwich theorem) and Corollary 4.2, we obtain the following theorem. Together with Corollary 4.2, it establishes the relationship between our partial order $\preceq_w$ on Φ^1 and the ∞-depth.

Theorem 4.3. *Let $w \in W$ and set $d_w = \max\{\mathrm{dp}_\infty(\gamma) \mid \gamma \in \Phi^R(w)\}$. (1) The ∞-depth on $\Phi^1(w)$ is maximum on $\Phi^R(w)$: $\mathrm{dp}_\infty(\beta) \leq d_w$, for all $\beta \in \Phi^1(w)$. (2) The element w is a d_w-low element; (3) For $m \in \mathbb{N}$, $w \in L_m$ if and only if $m \geq d_w$.*

The following corollary proves [5, Conjecture 2], which is key to proving Theorem 1.1.

Corollary 4.4. *Let $m \in \mathbb{N}$. The map $\lambda_m : L_m \to \Lambda_m = \{\Sigma_m(w) \mid w \in W\}$, defined by $w \mapsto \Sigma_m(w)$, is a bijection.*

The next proposition is crucial to proving Theorem 1.2 and Theorem 1.3. For their proofs, we need the existence of a supporting hyperplane of C_{sw} which is not m-low and which separates C_{sw} from C_e.

Proposition 4.5. *Let $m \in \mathbb{N}$, $w \in L_m$ and $s \in S$. Then $sw \in L_{m+1}$. Moreover: (1) $sw \in L_{m+1} \setminus L_m$ if and only if $w < sw$ and there is $r \in D_R(w)$ such that $\mathrm{dp}_\infty(-sw(\alpha_r)) = m+1$. (2) Under the conditions above, $\alpha_s \prec_{\mathrm{dom}} -sw(\alpha_r)$ for any $r \in D_R(w)$ with $\mathrm{dp}_\infty(-sw(\alpha_r)) = m+1$.*

5 Extended Shi arrangements and low elements

Let (W, S) be a Coxeter system and $m \in \mathbb{N}$. In this section we first introduce extended Shi arrangements and discuss Theorem 1.1 and Theorem 1.2. We also discuss how we obtained, as a byproduct, a direct proof of Thiel's Theorem 1.3. Then we provide in a counterexample to the convexity of the inverses of L_m if $m > 0$ and (W, S) is indefinite.

5.1 Extended Shi arrangements and proof of Theorem 1.1

Let $m \in \mathbb{N}$. The *(extended) m-Shi arrangement* $\mathrm{Shi}_m(W,S)$ is the set of *m-small hyperplanes*:

$$\mathrm{Shi}_m(W,S) = \{H_\beta \mid \beta \in \Sigma_m\},$$

which consists of the hyperplanes in $\mathcal{A}$ that are separated from the fundamental chamber C by at most m parallel hyperplanes.

The closed regions for $\mathrm{Shi}_m(W,S)$ are called the *m-Shi regions*. The corresponding equivalence relation $\sim_{\Sigma_m}$ on W is abbreviated $\sim_m$ in this case. We have $u \sim_m v$ if and only if C_u and C_v are contained in the same m-Shi region.

Example 5.1. See Figures 1(a) and 2(a) where the blue chambers correspond to the m-low elements and are the unique minimal chamber of their corresponding m-Shi region. For $m = 0$, observe that the small hyperplanes (thick blue lines) do not have any other hyperplanes between them and C. In Figure 1, the 1-small hyperplanes consist of the small hyperplanes plus hyperplanes that have exactly one hyperplane between them and C.

Proposition 5.2. *For $m \in \mathbb{N}$ and $u, v \in W$, we have $u \sim_m v \iff \Sigma_m(u) = \Sigma_m(v)$. In other words, two chambers C_u and C_u are in the same m-Shi region if and only if u and v have the same m-small inversion set.*

In affine Weyl group and in the case $m = 0$, the map $w \mapsto \Sigma_0(w)$ from W to Λ_0 is the generalization of Shi's admissible sign type map. The following theorem proves in particular Theorem 1.1.

Theorem 5.3. *Let $m \in \mathbb{N}$. For any $w \in W$, there is a unique m-low element $u \in L_m$ such that $u \sim_m w$. Moreover $u \leq_R w$. In particular, each region of $\mathrm{Shi}_m(W,S)$ contains a unique element of minimal length, which is a low element.*

Remark 5.4. (1) The proof of Theorem 5.3 depends on the bijection between m-low elements and m-small short inversions given in Corollary 4.4. (2) In the terminology of Parkinson and Yau, Theorem 5.3 means that any m-Shi arrangement is *gated* and that L_m is the set of *gates* of $\mathrm{Shi}_m(W,S)$.

5.2 The m-Shi polyhedron and convexity

We now discuss the proofs of Theorem 1.2 and Theorem 1.3. Let $m \in \mathbb{N}$. Consider the set

$$\mathcal{B}_m = \{x^{-1}(\alpha_s) \mid x \in L_m,\ s \in S,\ sx \notin L_m\}.$$

Since the set L_m is a Garside shadow, it is stable under taking suffixes, so $s \in S \setminus D_L(x)$ in the definition above. The set $\mathcal{B}_m \subseteq \Phi^+$ and is finite if S is.

Definition 5.5. We define the *m-Shi polyhedron* to be the convex set:

$$\mathscr{S}_m = \bigcap_{\beta \in \mathcal{B}_m} H_\beta^+.$$

In the case of irreducible affine Weyl groups, Shi proved in in 1987 that $\mathscr{S}_0$ is a simplex with $|S|$ half-spaces in the above definition. See Figures 1 (b) and 2 (b) where the shaded regions correspond to the corresponding m-Shi polyhedron.

The following two theorems are Theorem 1.2 and Theorem 1.3.

Theorem 5.6. *The* 0*-Shi polyhedron is:*

$$\mathscr{S}_0 = \bigcup_{w \in L_0} C_{w^{-1}}.$$

Theorem 5.7. *Let (W, S) be an affine Coxeter system and let $m \in \mathbb{N}$. The m-Shi polyhedron is the union of $C_{w^{-1}}$ for $w \in L_m$.*

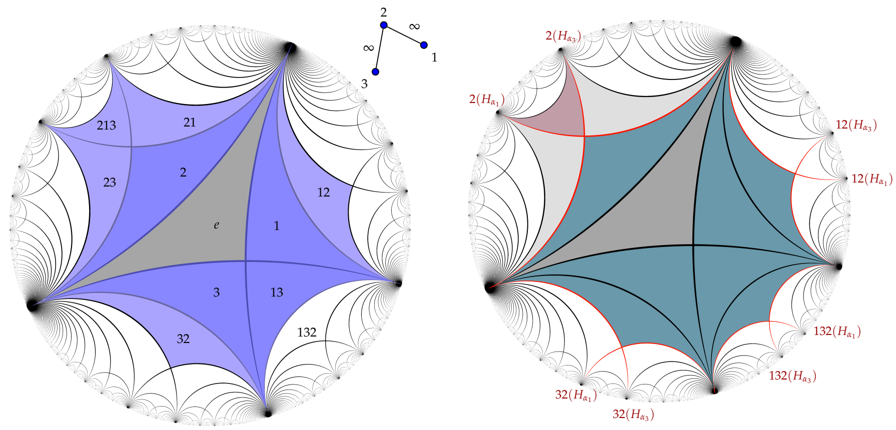
The proof that the m-Shi polyhedron is contained in the union of $C_{w^{-1}}$ for $w \in L_m$ is relatively straightforward and is a consequence of Lemma 5.8.

Lemma 5.8. *Let $m \in \mathbb{N}$ and $w \in W$ such that $\Phi(w^{-1}) \cap \mathscr{B}_m = \emptyset$. Then $w \in L_m$. In other words: $L_m \supseteq \{w \in W \mid \Phi(w^{-1}) \cap \mathscr{B}_m = \emptyset\}$.*

Proving that the union of $C_{w^{-1}}$ for $w \in L_m$ is contained in the Shi polyhedron is trickier and is not true in general for $m > 0$ in indefinite types–see Remark 5.9. It amounts to showing $L_m \subseteq \{w \in W \mid \Phi(w^{-1}) \cap \mathscr{B}_m = \emptyset\}$. The proof of this boils down to showing that if we have a $w \in W$ such that $\Phi(w^{-1}) \cap \mathscr{B}_m \neq \emptyset$, then w is not low. Here we need the existence of a supporting hyperplane which is not m-low (and some other conditions) and use Proposition 4.5 to obtain it.

Remark 5.9. In the proof of Theorem 5.7, we needed and proved the following property : if $\alpha, \beta, \gamma \in \Phi^+$ are such that $\alpha \preceq_{\text{dom}} \gamma$ and $\beta \preceq_{\text{dom}} \gamma$, then either $\alpha \preceq_{\text{dom}} \beta \preceq_{\text{dom}} \gamma$ or $\beta \preceq_{\text{dom}} \alpha \preceq_{\text{dom}} \gamma$. This property arises from the transitivity of the parallelism relation in Euclidean geometry. Unfortunately, it is not true in non-Euclidean space.

Convexity and extended Shi arrangements in indefinite Coxeter systems There can be no result analogous to Theorem 1.2 for all indefinite systems and $m > 0$. For instance, consider (W, S) be the indefinite system whose Coxeter graph is given in Figure 4. The red hyperplanes on the right do form a polyhedron, but C_{21} and C_{23} are not enclosed in it (light gray Figure 4(b)), although 12 and 32 are 1-low. The union of $C_{w^{-1}}$ for $w \in L_1$ is not even convex, since C_{213} (red) is not in the union as $312 = 132 \notin L_1$.



(a) The 0 and 1-Shi arrangements (b) The 0- and 1-polyhedron.

Figure 4: The 0 and 1-Shi arrangements and a counterexample of convexity for the indefinite system whose Coxeter graph is in the top middle of the picture. See Section 5.2.

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