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Enumerating the faces of split matroid polytopes

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Abstract. Computing f-vectors of polytopes is in general hard, and only little is known about their shape. We initiate the study of properties of f-vector of matroid base polytopes, by focusing on the class of split matroids, i.e., matroid polytopes arising from compatible splits of a hypersimplex. Unlike valuative invariants, the f-vector behaves in a much more unpredictable way, and the modular pairs of cyclic flats play a role in the face enumeration. We give a concise description of how the computation can be achieved without performing any convex hull or face lattice computation. As applications, we deduce formulas for sparse paving matroids and rank 2 matroids. These are two families that appear in other contexts within combinatorics.

Keywords: f-vectors, matroid polytopes, face numbers, split matroids, paving matroids

1 Introduction

A question that arises naturally in the study of a convex polytope $\mathcal{P} \subseteq \mathbb{R}^n$ is how many faces of each dimension $\mathcal{P}$ has. The ***f*-vector** of $\mathcal{P}$ is defined by

$$f(\mathcal{P}) := (f_0, f_1, \ldots, f_{d-1}, f_d),$$

where $f_i := \#\{i\text{-dimensional faces of } \mathcal{P}\}$ for each $i \in \{0, \ldots, d\}$ and $d := \dim \mathcal{P}$. In particular, the number of vertices of $\mathcal{P}$ is just f_0, the number of facets of $\mathcal{P}$ is f_{d-1}, and $f_d = 1$.

The difficulty of calculating the f-vector may vary drastically depending on the polytope $\mathcal{P}$, on the properties it possesses, or on how it is described. For some concrete examples of the computation of f-vectors and certain related problems, see [21]. The family of possible vectors arising as the f-vector of a polytope is notoriously hard, and their classification is open in dimensions as low as four, see [23]. Even in the case of 0/1-polytopes of fixed dimension, although the set of possible f-vectors is finite, much remains to be discovered, see [22].

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In this article we will initiate the study of the explicit face enumeration of matroid polytopes, by focusing on the well-structured subclass of **(elementary) split matroids**. There are many equivalent ways of introducing these matroids. A matroid is *elementary split* whenever it does not contain a minor isomorphic to $\mathsf{U}_{0,1} \oplus \mathsf{U}_{1,2} \oplus \mathsf{U}_{1,1}$. Similarly, one may define the class of *split matroids* via five excluded minors [14, 7]. When the matroid is connected, these two notions agree. Geometrically, a connected matroid M is split whenever every pair of facet defining hyperplanes do not intersect in the interior of the hypersimplex containing the matroid polytope $\mathcal{P}(\mathsf{M})$.

The class of split matroids was introduced by Joswig and Schröter in [14] to study tropical linear spaces. They have received considerable attention in the past few years, including a forbidden minor characterization [7], hypergraphs descriptions [5], Tutte polynomial inequalities [11], subdivisions and computation of valuations [10], and conjectures about exchange properties on the bases [6] which are related to White's conjecture.

The face structure of some special classes as positroids and lattice path matroids appeared in previous work, however without an explicit enumeration. Even though the f-vector of the matroid base polytope constitutes an invariant of the matroid M under isomorphisms, it is not valuative; see Example 2.2 below. This makes its computation considerably subtler and difficult. In particular, for the case of split matroids we require a non-trivial modification of the machinery presented in [10].

One important reason why split matroids deserve to be studied is that they encompass the classes of paving and copaving matroids. A long-standing conjecture often attributed to Crapo and Rota, appearing in print in [16], predicts that asymptotically almost all matroids are sparse paving. There is some evidence supporting this assertion [18], but another intriguing conjecture affirms that even restricting to the enumeration of *non sparse paving* matroids, the class of split matroids will continue to be predominant [10, Conjecture 4.10].

As of today, the problem of face enumeration of matroid polytopes has not been approached systematically in the literature, and to the best of our knowledge there are no prior articles addressing their computation. Some articles such as [15, 19, 3, 12, 1] may be relevant, as they discuss other aspects indirectly related to the face enumeration for (some classes of) matroid polytopes.

In particular, perhaps as a reminiscence of the situation for polytopes in general (and even for 0/1-polytopes), questions about properties of f-vectors of matroid polytopes are widely open.

Summary of results

As mentioned before, the fact that the face numbers are not valuations makes the computation of the f-vector of matroid polytopes a delicate task. In the case of split matroids, we need more data than just the number of cyclic flats of each rank and size. Some information on their pairwise intersection is necessary.

In order to express the f-vector of a polytope $\mathcal{P}$ in a more compact fashion, we will often refer to the ***f*-polynomial**, which is defined via:

$$f_{\mathcal{P}}(t) := \sum_{i=0}^{d} f_i \cdot t^i.$$

Following the notation and terminology of [10], whenever we have a matroid M of rank k and cardinality n, we will denote by $\lambda_{r,h}$ the number of stressed subsets with non-empty cusp that M has. Although one of the main results of that article establishes that the numbers $\lambda_{r,h}$ are enough to compute any valuative invariant on M, we need further data to compute the f-vector.

For a matroid M as before, we will denote by $\mu_{\alpha,\beta,a,b}$ the number of **modular pairs** of cyclic flats $\{F_1, F_2\}$ such that $a = |F_1 \smallsetminus F_2|$, $b = |F_2 \smallsetminus F_1|$, $\alpha = \operatorname{rk}(F_1) - \operatorname{rk}(F_1 \cap F_2)$, and $\beta = \operatorname{rk}(F_2) - \operatorname{rk}(F_1 \cap F_2)$; see also equation ($\star$) below.

The following constitutes the main result of this article and is stated as Theorem 2.4 further below. It tells us that the numbers $\mu_{\alpha,\beta,a,b}$ are the precise additional datum needed to perform the computation of the f-vector of a split matroid polytope. Moreover, the statement tells us concretely how to calculate the number of faces of given dimension.

Theorem *Let M be a connected split matroid of rank k on n elements. The number of faces of its base polytope $\mathcal{P}(\mathsf{M})$ is given by the polynomial*

$$f_{\mathcal{P}(\mathsf{M})}(t) = f_{\Delta_{k,n}}(t) - \sum_{r,h} \lambda_{r,h} \cdot u_{r,k,h,n}(t) - \sum_{\alpha,\beta,a,b} \mu_{\alpha,\beta,a,b} \cdot w_{\alpha,\beta,a,b}(t)$$

where the first sum ranges over all values with $0 < r < h < n$ and the second sum ranges over the values $0 < \alpha < a$, $0 < \beta < b$ for which either $a < b$ or $a = b$ and $\alpha \leq \beta$.

In the above theorem, the expressions $u_{r,k,h,n}(t)$ and $w_{\alpha,\beta,a,b}(t)$ are polynomials which depend only on their subindices. We present in Propositions 2.6 and 2.7 explicit (but complicated) formulas for them which can be used to calculate the face numbers effortlessly. A formula for the f-vector of the hypersimplex $\Delta_{k,n}$ is also given explicitly in Example 2.1. In particular, the entire calculation can be done bypassing the problem of building costly face lattices or computing convex hulls.

As two direct but interesting applications of our result, we particularize it to the classes of sparse paving and rank 2 matroids. The first is a class that made a prominent appearance in the theory of the extension complexity of independence polytopes [20]. The second bears a relevant connection with the theory of edge polytopes of graphs [17].

2 The number of faces of split matroids

2.1 The set up

Throughout this extended abstract we will assume that the reader is familiar with the usual terminology and notation in matroid theory. For the notions and machinery introduced very recently, in particular about **stressed subsets**, **relaxations**, and **cuspidal matroids** we refer the reader to our previous article [10, Sections 3–4]. Regarding **split matroids** and **elementary split matroids** the reader can consult the same article as well as [14, 5]. However, basic knowledge on polytopes should be enough to follow the arguments and methods in this manuscript.

For a d-dimensional polytope $\mathcal{P}$ we denote by $f(\mathcal{P}) := (f_0, \dots, f_d)$ its *f-vector*, and by

$$f_{\mathcal{P}}(t) := \sum_{i=0}^{d} f_i \, t^i$$

its *f-polynomial*. In both cases, f_i denotes the number of i-dimensional faces of $\mathcal{P}$. Notice that we omit the inclusion of $f_{-1} := 1$ for the empty set in both the f-vector and the f-polynomial, but we do include $f_d = 1$ for the polytope itself.

Essential notation Following our prequel [10], whenever we have a matroid M, unless specified otherwise, the rank of M is denoted by k and the size of its ground set is denoted by n. We reserve the letters r and h for the rank and the size of stressed subsets that M may possess.

Note that under the assumption of being connected the classes of split matroids and elementary split matroids coincide [5, Theorem 11]. Since the base polytope of a direct sum of matroids $\mathsf{M}_1 \oplus \mathsf{M}_2$ is the cartesian product of $\mathcal{P}(\mathsf{M}_1)$ and $\mathcal{P}(\mathsf{M}_2)$, the f-vector of any disconnected split matroid can be recovered from the f-vector of the connected components, all of which are split as well.

The most basic example of a matroid polytope is the hypersimplex $\Delta_{k,n}$, the matroid base polytope of the uniform matroid $\mathsf{U}_{k,n}$ of rank k on n elements.

Example 2.1 The face enumeration of hypersimplices is encoded in the following f-polynomial:

$$f_{\mathcal{P}(\mathsf{U}_{k,n})}(t) = f_{\Delta_{k,n}}(t) = \binom{n}{k} + \sum_{i=1}^{n-1} \binom{n}{i+1} \sum_{j=1}^{i} \binom{n-i-1}{k-j} \cdot t^i \quad .$$

For a detailed proof see for example [13, Corollary 1.4].

As we will see now, the assignment $\mathsf{M} \mapsto f_{\mathcal{P}(\mathsf{M})}(t)$ is an invariant of the matroid M that fails to be valuative. Hence its computation is a more delicate task, even for the case of paving or split matroids. In these cases, we cannot rely on the strength of [10, Theorem 6.6] — that result asserts that the evaluation of a valuative invariant on a split matroid M can be achieved by knowing relatively little about the matroid M, consisting of its rank k, its size n, and the parameters $\lambda_{r,h}$. If one is interested in knowing the f-vector of $\mathcal{P}(\mathsf{M})$, the first problem one faces is identifying what additional matroid data is required.

Example 2.2 Consider the four matroids $\mathsf{U}_{3,6}$, M, N_1 and N_2 with ground set $\{1,\ldots,6\}$ and rank three, whose families of bases are given as follows:

$$\mathcal{B}(\mathsf{U}_{3,6}) := \binom{[6]}{3}, \qquad \mathcal{B}(\mathsf{N}_1) := \binom{[6]}{3} \smallsetminus \{\{1,2,3\},\{4,5,6\}\}$$
$$\mathcal{B}(\mathsf{M}) := \binom{[6]}{3} \smallsetminus \{\{1,2,3\}\}, \qquad \mathcal{B}(\mathsf{N}_2) := \binom{[6]}{3} \smallsetminus \{\{1,2,3\},\{3,4,5\}\}.$$

The f-vectors of their base polytopes are respectively:

$$f(\mathcal{P}(\mathsf{U}_{3,6})) = (20,90,120,60,12,1), \qquad f(\mathcal{P}(\mathsf{N}_1)) = (18,72,102,60,14,1),$$
$$f(\mathcal{P}(\mathsf{M})) = (19,81,111,60,13,1), \qquad f(\mathcal{P}(\mathsf{N}_2)) = (18,72,101,59,14,1).$$

All of these matroids are sparse paving. In particular, the two matroids N_1 and N_2 have, e.g., the same Tutte polynomial and the same Ehrhart polynomial — in fact, via [10, Corollary 6.7] any valuative invariant on these two matroids yields the same result. However, observe that their f-vectors differ in the third and the fourth entries.

2.2 Cuspidal matroids

By using [10, Corollary 6.2], we see that the intersection of the hypersimplex $\Delta_{k,n}$ with the half-space of a single split hyperplane leads to the polytope:

$$\mathcal{P}(\Lambda_{k-r,k,n-h,n}) = \left\{ x \in \Delta_{k,n} \ : \ \sum_{i=1}^{h} x_i \leq r \right\} \ . \tag{2.1}$$

for appropriate values r and h. This is the base polytope of the cuspidal matroid $\Lambda_{k-r,k,n-h,n}$, a matroid having exactly three cyclic flats: the empty set, the entire ground set, and one proper cyclic flat having size h and rank r. For the purposes of this paper, the reader may regard equation (2.1) as the definition of cuspidal matroids.

Let us introduce some notation that will help us formulate later our main results in a more compact way:

$$u_{r,k,h,n}(t) := f_{\Delta_{k,n}}(t) - f_{\mathcal{P}(\Lambda_{k-r,k,n-h,n})}(t). \tag{2.2}$$

A non-obvious property is that some of these coefficients may be negative while other are positive — moreover, the actual sign of each individual coefficient a priori depends on the four parameters r, k, h, n.

Before we go on, let us introduce a second polynomial, which will play an important role in the sequel. For fixed numbers $0 < \alpha < a$ and $0 < \beta < b$ let us define,

$$\begin{aligned} w_{\alpha,\beta,a,b}(t) &:= f_{\Delta_{\alpha+\beta,a+b}}(t) - f_{\Delta_{\alpha,a}}(t) \cdot f_{\Delta_{\beta,b}}(t) - u_{\alpha,\alpha+\beta,a,a+b}(t) - u_{\beta,\alpha+\beta,b,a+b}(t) \\ &= f_{\mathcal{P}(\Lambda_{\beta,\alpha+\beta,b,a+b})}(t) + f_{\mathcal{P}(\Lambda_{\alpha,\alpha+\beta,a,a+b})}(t) - f_{\Delta_{\alpha+\beta,a+b}}(t) - f_{\Delta_{\alpha,a}}(t) \cdot f_{\Delta_{\beta,b}}(t). \end{aligned}$$

Later, in Proposition 2.6, we provide a compact formula for the polynomials $w_{\alpha,\beta,a,b}(t)$ and a formula for the polynomials $u_{r,k,h,n}(t)$ in Proposition 2.7 both of which can be used to calculate these polynomials, bypassing the computation of f-vectors of cuspidal matroids using the polytopes themselves.

Remark 2.3 The intuition of why it is reasonable to consider and define the complicated expression above stems from [10, Example 6.5]. As follows from the explanation there, if the assignment $\mathsf{M} \mapsto f_{\mathcal{P}(\mathsf{M})}(t)$ were valuative, then the defining formula for $w_{\alpha,\beta,a,b}(t)$ would actually be identically zero. The polynomial $w_{\alpha,\beta,a,b}(t)$ quantifies (in a certain way) how far the map $\mathsf{M} \mapsto f_{\mathcal{P}(\mathsf{M})}(t)$ is from being valuative.

2.3 Face counting of split matroids

For a connected split matroid M, let us define the following numbers that we have already mentioned in the introduction. The number of stressed subsets with non-empty cusp having rank r and size h, denoted $\lambda_{r,h}$ — recall that by [10, Proposition 3.9], in a connected split matroid this is the same as the number of proper non-empty cyclic flats of rank r and size h. We also need the numbers $\mu_{\alpha,\beta,a,b}$ of (unordered) modular pairs $\{F_1, F_2\}$ of proper non-empty cyclic flats, i.e., F_1 and F_2 fulfilling the *modularity property*,

$$\operatorname{rk}(F_1) + \operatorname{rk}(F_2) = \operatorname{rk}(F_1 \cap F_2) + \operatorname{rk}(F_1 \cup F_2), \tag{$\star$}$$

where the indices denote the following quantities:

$$\begin{aligned} a &= |F_1 \smallsetminus F_2|, & \alpha &= \operatorname{rk} F_1 - \operatorname{rk}(F_1 \cap F_2) \\ b &= |F_2 \smallsetminus F_1|, & \beta &= \operatorname{rk} F_2 - \operatorname{rk}(F_1 \cap F_2)\ . \end{aligned}$$

Note that the set $F_1 \cap F_2 \subsetneq F_1 \subsetneq [n]$ can not contain a circuit if M is a connected split matroid, thus it is an independent set, i.e., $\operatorname{rk}(F_1 \cap F_2) = |F_1 \cap F_2|$.

Theorem 2.4 *Let M be a connected split matroid of rank k on n elements. The number of faces of its base polytope $\mathcal{P}(\mathsf{M})$ is given by the polynomials*

$$f_{\mathcal{P}(\mathsf{M})}(t) = f_{\Delta_{k,n}}(t) - \sum_{r,h} \lambda_{r,h} \cdot u_{r,k,h,n}(t) - \sum_{\alpha,\beta,a,b} \mu_{\alpha,\beta,a,b} \cdot w_{\alpha,\beta,a,b}(t) \tag{2.3}$$

where the first sum ranges over all values with $0 < r < h < n$ *and the second sum ranges over the values* $0 < \alpha < a$, $0 < \beta < b$ *for which either* $a < b$ *or* $a = b$ *and* $\alpha \leq \beta$.

On one hand, note that the polynomials $f_{\Delta_{k,n}}(t)$, $u_{r,k,h,n}(t)$ and $w_{\alpha,\beta,a,b}(t)$ can be precomputed for all the occurring instances of the variables which appear as subindices. The first non-trivial fact that is deduced by our statement is that in addition to the parameters $\lambda_{r,h}$, which always appear in the computation of a valuative invariant, the precise additional matroidal datum needed to compute the f-vector consists of the numbers $\mu_{\alpha,\beta,a,b}$. Strikingly, the last sum in equation (2.3) does not take into consideration the rank nor the size of the matroid M itself, only the intersection data for the modular pairs of flats. The second non-trivial fact is that it explains how to put together this information in order to effectively computing the f-vector of $\mathcal{P}(\mathsf{M})$ for a split matroid, circumventing the necessity of constructing the polytope.

Example 2.5 Let us take a look again at Example 2.2. The matroids N_1 and N_2 are sparse paving, have rank $k = 3$ and size $n = 6$. In each case the proper non-empty cyclic flats are exactly the non-bases, yielding for both matroids $\lambda_{2,3,3,6} = 2$. One can compute the corresponding polynomial, $u_{2,3,3,6}(t) = 1 + 9t + 9t^2 - t^4$. In N_1, the intersection of the only pair of proper non-empty cyclic flats, $F_1 = \{1,2,3\}$ and $F_2 = \{4,5,6\}$, does not satisfy the property ($\star$), because $\operatorname{rk}(F_1 \cap F_2) + \operatorname{rk}(F_1 \cup F_2) = 0 + 3$, whereas $\operatorname{rk}(F_1) + \operatorname{rk}(F_2) = 2 + 2 = 4$.

For N_2, the situation is different, as $F_1 = \{1,2,3\}$ and $F_2 = \{3,4,5\}$ indeed satisfy ($\star$), and we have $a = |F_1 \smallsetminus F_2| = 2$, $b = |F_2 \smallsetminus F_1| = 2$, $\alpha = \operatorname{rk}(F_1) - |F_1 \cap F_2| = 2 - 1 = 1$, and $\beta = \operatorname{rk}(F_2) - |F_1 \cap F_2| = 2 - 1 = 1$, so that $\mu_{1,1,2,2} = 1$ and we need to subtract $w_{1,1,2,2}(t) = t^2 + t^3$ to obtain the correct f-polynomial, as we expected.

2.4 Explicit formulas

The polynomials $u_{r,k,h,n}(t)$ and $w_{\alpha,\beta,a,b}(t)$ in Theorem 2.4 are defined in terms of f-vectors of specific matroid polytopes. In this subsection we will present explicit descriptions for these polynomials, enabling us to do the face enumeration of a split matroid polytope, without any convex hull or face lattice computation. To express the formulas in a compact form, we will make use of multinomial coefficients. Let i, j, ℓ be non negative integers, then

$$\binom{i+j+\ell}{i,j} := \binom{i+j+\ell}{i,j,\ell} = \frac{(i+j+\ell)!}{i!j!\ell!} \ .$$

We begin with an explicit formula for the polynomials $w_{\alpha,\beta,a,b}(t)$.

Proposition 2.6 *For any $0 < \alpha < a$ and $0 < \beta < b$, the following formula holds:*

$$w_{\alpha,\beta,a,b}(t) = \sum_{i=0}^{a-\alpha-1} \sum_{j=0}^{\alpha-1} \sum_{i'=0}^{b-\beta-1} \sum_{j'=0}^{\beta-1} \binom{a}{i,j} \binom{b}{i',j'} \cdot (1+t) \cdot t^{a+b-i-j-i'-j'-2} \quad .$$

For the polynomials $u_{r,k,h,n}(t)$ we provide the following formula.

Proposition 2.7 *For any $0 < r < k < n$ and $r < h < n$ the following formula holds*

$$u_{r,k,h,n}(t) = p_{r,k,h,n}(t) - p'_{r,h}(t) \cdot p'_{k-r,n-h}(t) \cdot (1+t) + \sum_{i=r+1}^{k} \binom{h}{i} \binom{n-h}{k-i}$$

where $p'_{r,h}(t) = f_{\Delta_{r,h}}(t) - \binom{h}{r}$ and

$$p_{r,k,h,n}(t) = \sum_{j=0}^{h-r-1} \sum_{i=0}^{\min\{h-j,k-1\}} \sum_{\ell=0}^{\min\{k-i,k-r\}-1} \sum_{m=0}^{\min\{n-h-\ell,n-k-j-1\}} \binom{h}{i,j} \binom{n-h}{\ell,m} t^{n-1-s}.$$

where s denotes $i+j+\ell+m$ in the above sum.

Example 2.8 Let M be the projective geometry $\mathrm{PG}(2,3)$. This is a matroid on $n = 13$ elements of rank $k = 3$. It is split as it is in fact paving. This matroid has 13 stressed hyperplanes, i.e., rank $k-1 = 2$ flats, all of which have cardinality $h = 4$. In other words, we have $\lambda_{2,4} = 13$. In particular, to use the formula of Theorem 2.4, the polynomial

$$\begin{aligned} u_{2,3,4,13}(t) = &- t^{11} - 11\,t^{10} - 54\,t^{9} - 156\,t^{8} - 294\,t^{7} - 378\,t^{6} \\ &- 336\,t^{5} - 195\,t^{4} + t^{3} + 166\,t^{2} + 114\,t + 4 \end{aligned}$$

is required. Since projective geometries are modular matroids, any pair of distinct proper non-empty cyclic flats fulfills the property ($\star$). Also, every pair of them intersect in a single element. Moreover, for every pair of these cyclic flats we have $a = |F_1 \smallsetminus F_2| = 3$, and by symmetry $b = |F_1 \smallsetminus F_2| = 3$. Additionally, $\alpha = \mathrm{rk}(F_1) - |F_1 \cap F_2| = 2 - 1 = 1$ and again by symmetry $\beta = \mathrm{rk}(F_2) - |F_1 \cap F_2| = 1$. Therefore there is a single non-vanishing coefficient $\mu_{\alpha,\beta,a,b}$ which is

$$\mu_{1,1,3,3} = \binom{13}{2} = 78 \quad .$$

It remains to compute:

$$w_{1,1,3,3}(t) = t^5 + 7t^4 + 15t^3 + 9t^2 \quad .$$

Now applying Theorem 2.4, we obtain:

$$\begin{aligned} f_{\mathcal{P}(\mathrm{PG}(2,3))}(t) &= f_{\Delta_{3,13}}(t) - 13\,u_{2,3,4,13}(t) - 78\,w_{1,1,3,3}(t) \\ &= t^{12} + 39\,t^{11} + 455\,t^{10} + 2704\,t^{9} + 9893\,t^{8} + 24414\,t^{7} + 42666\,t^{6} + \\ &\quad 54054\,t^{5} + 49608\,t^{4} + 31707\,t^{3} + 12870\,t^{2} + 2808\,t + 234 \quad . \end{aligned}$$

2.5 Face numbers of sparse paving matroids

As mentioned in the introduction, it is conjectured that almost all matroids are sparse paving; see [16] for the details. Furthermore, many famous examples of matroids fall into this class; notable examples are the Fano matroid, the Vámos matroid, the complete graph on four vertices, and the duals of each of them. Sparse paving and paving matroids are split, so we can make use of our main result.

Corollary 2.9 *Let* M *be a connected sparse paving matroid of rank* k *on* n *elements having exactly* λ *circuit-hyperplanes, and let* μ *count the pair of circuit-hyperplanes which have* $k-2$ *elements in common. Then*

$$f_{\mathcal{P}(\mathsf{M})}(t) = f_{\Delta_{k,n}}(t) - \lambda \cdot u(t) - \mu \cdot (t^2 + t^3)$$

where $u(t)$ *is given by*

$$1 - k \cdot (n-k) \cdot (t+1) + \Big((n-k) \cdot (t+1)^{k+1} + k \cdot (t+1)^{n-k+1} - n \cdot (t+1) \Big) \cdot t^{-1} + \Big((t+1)^k + (t+1)^{n-k} - (t+1)^n - 1 \Big) \cdot t^{-2} \ .$$

Remark 2.10 This formula can be used to prove that the number of facets of the base polytope of a matroid on n elements may be as large as $c2^n/n^{3/2}$ for an absolute constant c. However, for arbitrary $0/1$-polytopes in $\mathbb{R}^n$ it is known that the number of facets can be larger than $\left(\frac{cn}{\log n}\right)^{n/4}$, via a random construction [4].

Given a lattice polytope $\mathcal{P} \subseteq \mathbb{R}^n$, an *extended formulation* of $\mathcal{P}$ is another lattice polytope $\mathcal{Q} \subseteq \mathbb{R}^m$ together with a projection map $\pi : \mathbb{R}^m \to \mathbb{R}^n$ which projects $\mathcal{Q}$ onto $\mathcal{P}$. The *complexity* of an extended formulation is the number of facets of the polytope $\mathcal{Q}$. The *extension complexity* of $\mathcal{P}$, denoted $\mathrm{xc}(\mathcal{P})$, is the minimum complexity of an extended formulation of $\mathcal{P}$.

In a landmark paper [20, Corollary 6], Rothvoss proved [1] that for all n there exists a matroid M on n elements whose base polytope has extension complexity $\mathrm{xc}(\mathcal{P}(\mathsf{M})) \in \Omega\left(\frac{2^{n/2}}{n^{5/4}\sqrt{\log(2n)}}\right)$. Moreover, Rothvoss' proof is non-constructive and relies only on an enumerative result of matroids, that therefore guarantees that whatever these examples are, they must belong to the class of sparse paving matroids, and are therefore split matroids. It remains a notorious open problem to find an explicit family of matroids having exponential extension complexity. In fact, having one would yield an explicit infinite family of Boolean functions requiring superlogarithmic depth circuits, according

[1] To be precise, Rothvoss proved that the extension complexity of the *independence polytope* of some matroid is exponential, but an elementary reasoning shows that this is equivalent to an analogous statement for the base polytope. See for example the short explanation in [2, p. 1].

to an observation attributed to Göös in [2, Section 8]. We conjecture, however, that a certain class of "extremal" sparse paving matroids must already constitute such an example; for the details of the conjecture we refer to the extended version of the present paper [9].

2.6 Face numbers of rank two matroids

A loopless matroid of rank two is trivially paving, and hence a split matroid. This allows us to use the strength of Theorem 2.4 to compute their f-vectors. The hyperplanes, i.e., the flats of rank one, of a loopless matroid of rank two form a partition of the ground, and conversely, any partition of the ground set defines precisely a single rank two matroid having each part as a flat. The bases of the matroid are obtained by taking two elements of the ground set, not belonging to the same part.

Base polytopes of matroids of rank two have made prominent appearances throughout algebraic combinatorics, under various guises. Notably, as is pointed out in [8, Section 6.1], they coincide with edge polytopes of complete multipartite graphs — we refer to that paper for the precise definition of edge polytopes and a short overview of them. In this vein, the work of Ohsugi and Hibi [17] addresses the edge polytopes of complete multipartite graphs, motivated both from toric geometry and graph theory. In particular, the content of [17, Theorem 2.5] provides a formula for the f-vector of the edge polytope of an arbitrary complete multipartite graph, and thus for general rank two matroid polytopes. Let us point out that there appears to be an error in the formula as they stated it — in particular within the quantity they denote by α_i. As an application of Theorem 2.4 we can give another formula for the f-vector of these polytopes.

Corollary 2.11 *Let M be a loopless matroid of rank two having s hyperplanes with cardinalities $h_1, \dots, h_s$. Then, the number of i-dimensional faces of $\mathcal{P}(\mathsf{M})$ or, equivalently, the edge polytope of a complete multipartite graph with parts of sizes $h_1, \dots, h_s$ is given by:*

$$f_i(\mathcal{P}(\mathsf{M})) = \binom{n+1}{i+2} + (s-1)\binom{n}{i+2} - \sum_{j<\ell} \binom{h_j + h_\ell + 1}{i+2} + (s-2)\sum_{j=1}^{s}\binom{h_j+1}{i+2} - \sum_{j=1}^{s}\binom{n-h_j}{i+2}.$$

2.7 Questions on the shape of f-vectors of matroids

A recent trend in matroid theory is that of proving unimodal and log-concave inequalities for various vectors of numbers associated to matroids. A finite sequence of numbers $(a_0, \dots, a_n)$ is said to be *unimodal* if there exists some index $0 \leq j \leq n$ with the property that

$$a_0 \leq \cdots \leq a_{j-1} \leq a_j \geq a_{j+1} \geq \cdots \geq a_n.$$

If all the a_i's are positive, a stronger condition is that of *log-concavity*, which asserts that for each index $1 \leq j \leq n-1$ the inequalities $a_j^2 \geq a_{j-1}a_{j+1}$ hold.

It is quite inviting to ask the following question.

Question 2.12 Are the f-vectors of matroid base polytopes unimodal, or even log-concave?

It is known that there are simplicial polytopes having a non-unimodal f-vector; see [21, Chapter 8.6]. Within the existing literature we were not able to find any examples of non-unimodal f-vectors for the general class of 0/1-polytopes. We have been able to verify the log-concavity of the f-vectors of the following classes of matroids, in some cases relying critically on the results of this paper:

- All matroids on a ground set of size at most 9.
- Split matroids on a ground set of size at most 15.
- Sparse paving matroids on a ground set of size at most 40.
- Lattice path matroids on a ground set of size at most 13.
- Rank two matroids on a ground set of size at most 60.

Note: An extended version of this manuscript including all proofs can be found on the arXiv, see [9].

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