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The Poincaré-extended **ab**-index

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Abstract. Motivated by a conjecture concerning Igusa local zeta functions for intersection posets of hyperplane arrangements, we introduce and study the *Poincaré-extended* **ab***-index*, which generalizes both the **ab**-index and the Poincaré polynomial. For posets admitting R-labelings, we give a combinatorial description of the coefficients of the extended **ab**-index, proving their nonnegativity. In the case of intersection posets of hyperplane arrangements, we prove the above conjecture of the second author and Voll.

Keywords: poset, matroid, oriented matroid, ab-index, hyperplane arrangement, R-labeling, quasisymmetric function

Grunewald, Segal, and Smith introduced the subgroup zeta function of finitely-generated groups [14], and Du Sautoy and Grunewald gave a general method to compute such zeta functions using p-adic integration and resolution of singularities [25]. This motivated Voll and the second author to examine the setting where the multivariate polynomials factor linearly. They found that the p-adic integrals are specializations of multivariate rational functions depending only on the combinatorics of the corresponding hyperplane arrangement [19]. After a natural specialization, its denominator greatly simplifies, and they conjecture that the numerator polynomial has nonnegative coefficients.

In this work, we prove their conjecture, which is related to the poles of these zeta functions; see Remark 1.19. Specifically, we reinterpret these numerator polynomials by introducing and studying the *(Poincaré-)extended* **ab***-index*, a polynomial generalizing both the *Poincaré polynomial* and **ab***-index* of the *intersection poset* of the arrangement. These polynomials have been studied extensively in combinatorics, although from different perspectives. The coefficients of the Poincaré polynomial have interpretations in terms of the combinatorics and the topology of the arrangement [8, Section 2.5]. The **ab**-index, on the other hand, carries information about the order complex of the poset and is particularly well-understood in the case of face posets of oriented matroids—or, more generally, Eulerian posets. In those settings, the **ab**-index encodes topological data via the *flag f-vector* [2].

We study the extended **ab**-index in the generality of graded posets admitting R-labelings. This class of posets includes intersection posets of hyperplane arrangements

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and, more generally, geometric lattices and geometric semilattices. We show that the extended **ab**-index has nonnegative coefficients by interpreting them in terms of a combinatorial statistic. This generalizes statistics given for the **ab**-index by Billera, Ehrenborg, and Readdy [6] and for the pullback **ab**-index (defined below) by Bergeron, Mykytiuk, Sottile and van Willigenburg [5]. This interpretation proves the aforementioned conjecture [19], as well as a related conjecture from Kühne and the second author [18].

Motivated by the proofs of these conjectures, we describe a close relationship between the Poincaré polynomial and the **ab**-index by showing that the extended **ab**-index can be obtained from the **ab**-index by a suitable substitution. This recovers, generalizes and unifies several results in the literature. Concretely, special cases of this substitution were observed by Billera, Ehrenborg and Readdy for lattices of flats of *oriented matroids* [6], by Saliola and Thomas for lattices of flats of *oriented interval greedoids* [24], and by Ehrenborg for *distributive lattices* [11].

1 The Poincaré-Extended ab-index

1.1 Main definitions

Unless otherwise specified, P is a finite ***graded poset*** of rank n, that is, P is a finite poset with unique minimum element $\hat{0}$ of rank 0 and unique maximum element $\hat{1}$ of rank n such that $\mathsf{rank}(X)$ is equal to the length of any maximal chain from $\hat{0}$ to X. The ***Möbius function*** μ of P is given by $\mu(X,X) = 1$ for all $X \in P$ and $\mu(X,Y) = -\sum_{X \leq Z < Y} \mu(X,Z)$ for all $X < Y$ in P. The ***Poincaré polynomial*** of P is

$$\mathsf{Poin}(P;y) = \sum_{X \in P} |\mu(\hat{0},X)| \cdot y^{\mathsf{rank}(X)} \ \in \mathbb{Z}[y].$$

The ***chain Poincaré polynomial*** of a chain $\mathcal{C} = \{\mathcal{C}_1 < \cdots < \mathcal{C}_k\}$ in $P \setminus \{\hat{1}\}$ is

$$\mathsf{Poin}_{\mathcal{C}}(P;y) = \prod_{i=1}^{k} \mathsf{Poin}([\mathcal{C}_i, \mathcal{C}_{i+1}];y) \ \in \mathbb{Z}[y],$$

where we set $\mathcal{C}_{k+1} = \hat{1}$. By taking the singleton chain $\{\hat{0}\}$, we recover the usual Poincaré polynomial, $\mathsf{Poin}(P;y) = \mathsf{Poin}_{\{\hat{0}\}}(P;y)$. The ranks of a given chain $\mathcal{C}$ is given by

$$\mathsf{Rank}(\mathcal{C}) = \{\mathsf{rank}(\mathcal{C}_i) \mid 1 \leq i \leq k\}.$$

We often consider polynomials in noncommuting variables $\mathbf{a}$ and $\mathbf{b}$ with coefficients being polynomials in $\mathbb{Z}[y]$. For a subset $S \subseteq \{i, i+1, \ldots, j\}$, we write $\mathsf{m}_S = m_i \ldots m_j$ for the monomial with $m_k = \mathbf{b}$ if $k \in S$ and $m_k = \mathbf{a}$ if $k \notin S$ and we similarly write $\mathsf{wt}_S = w_i \ldots w_j$ for the polynomial with

$$w_k = \begin{cases} \mathbf{b} & \text{if } k \in S, \\ \mathbf{a} - \mathbf{b} & \text{if } k \notin S. \end{cases} \tag{1.1}$$

The supersets $\{i, i+1, \ldots, j\}$ are understood from the context as the set of all indices that can possibly be contained in the set S. In case of ambiguity, we in addition identify the considered superset. For a chain $\mathcal{C}$ in P, we also set $\mathsf{m}_{\mathcal{C}} = \mathsf{m}_{\mathsf{Rank}(\mathcal{C})}$ and $\mathsf{wt}_{\mathcal{C}} = \mathsf{wt}_{\mathsf{Rank}(\mathcal{C})}$. The following is the main object of study of this paper.

Definition 1.1. The (***Poincaré-****)extended* **ab-*index*** of P is

$$_{\mathsf{ex}}\Psi(P; y, \mathbf{a}, \mathbf{b}) = \sum_{\mathcal{C} \text{ chain in } P \backslash \{\hat{1}\}} \mathsf{Poin}_{\mathcal{C}}(P; y) \cdot \mathsf{wt}_{\mathcal{C}} \ \in \mathbb{Z}[y]\langle \mathbf{a}, \mathbf{b} \rangle\,,$$

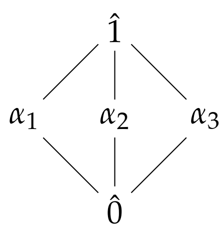
where $\mathsf{wt}_{\mathcal{C}} = w_0 \cdots w_{n-1}$ is given in Equation (1.1).

Since P has a unique minimum, we always have $\mathsf{Poin}(P; 0) = 1$, implying

$$_{\mathsf{ex}}\Psi(P; 0, \mathbf{a}, \mathbf{b}) = \sum_{\mathcal{C} \text{ chain in } P \backslash \{\hat{1}\}} \mathsf{wt}_{\mathcal{C}}\,.$$

This recovers the **ab-*index*** $\Psi(P; \mathbf{a}, \mathbf{b}) = {}_{\mathsf{ex}}\Psi(P; 0, \mathbf{a}, \mathbf{b})$.

Example 1.2. We compute the extended **ab**-index of the poset $\mathcal{L}$ drawn below on the left.



$\mathcal{C}$	$\mathsf{Poin}_{\mathcal{C}}(\mathcal{L}; y)$	$\mathsf{Rank}(\mathcal{C})$	$\mathsf{wt}_{\mathcal{C}}$
$\{\}$	1	$\{\}$	$(\mathbf{a}-\mathbf{b})^2$
$\{\hat{0}\}$	$1+3y+2y^2$	$\{0\}$	$\mathbf{b}(\mathbf{a}-\mathbf{b})$
$\{\alpha_i\}$	$1+y$	$\{1\}$	$(\mathbf{a}-\mathbf{b})\mathbf{b}$
$\{\hat{0} < \alpha_i\}$	$(1+y)^2$	$\{0,1\}$	$\mathbf{b}^2$

The extended **ab**-index and its specialization to the **ab**-index are thus

$$\begin{aligned} {}_{\mathsf{ex}}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b}) &= (\mathbf{a}-\mathbf{b})^2 + (1+3y+2y^2)\mathbf{b}(\mathbf{a}-\mathbf{b}) + 3\cdot(1+y)(\mathbf{a}-\mathbf{b})\mathbf{b} + 3\cdot(1+y)^2\mathbf{b}^2 \\ &= \mathbf{a}^2 + (3y+2y^2)\mathbf{ba} + (2+3y)\mathbf{ab} + y^2\mathbf{b}^2, \\ \Psi(\mathcal{L}; \mathbf{a}, \mathbf{b}) &= \mathbf{a}^2 + 2\mathbf{ab}\,. \end{aligned}$$

Remark 1.3. Taking chains $\mathcal{C}$ in $P \setminus \{\hat{1}\}$, rather than in P, is a harmless reduction in the definition of the extended **ab**-index since $\mathsf{Poin}_{\mathcal{C}}(P; y) = \mathsf{Poin}_{\mathcal{C} \cup \{\hat{1}\}}(P; y)$. If we consider both $\mathcal{C}$ and $\mathcal{C} \cup \{\hat{1}\}$ separately as summands of ${}_{\mathsf{ex}}\Psi(P; y, \mathbf{a}, \mathbf{b})$, we would need to consider weights $\mathsf{wt}^+_{\mathcal{C}} = w_0 \cdots w_n$ taking also the n-th position into account. We would have the two terms $\mathsf{Poin}_{\mathcal{C}}(P; y) \cdot \mathsf{wt}^+_{\mathcal{C}}$ and $\mathsf{Poin}_{\mathcal{C} \cup \{\hat{1}\}}(P; y) \cdot \mathsf{wt}^+_{\mathcal{C} \cup \{\hat{1}\}}$, differing only in the last entry of the weight, so their sum is $\mathsf{Poin}_{\mathcal{C}}(P; y) \cdot \mathsf{wt}_{\mathcal{C}} \cdot \mathbf{a}$. This holds for all chains, proving

$$_{\mathsf{ex}}\Psi(P; y, \mathbf{a}, \mathbf{b}) \cdot \mathbf{a} = \sum_{\mathcal{C} \text{ chain in } P} \mathsf{Poin}_{\mathcal{C}}(P; y) \cdot \mathsf{wt}^+_{\mathcal{C}}. \tag{1.2}$$

The fact that $\hat{1}$ is included in every chain in the computation of the chain Poincaré polynomial is inspired by the setting of hyperplane arrangements; see [1, 22] for more details. A (central, real) ***hyperplane arrangement*** $\mathcal{A}$ is a finite collection of hyperplanes in $\mathbb{R}^d$, all of which have a common intersection. The ***lattice of flats*** $\mathcal{L}$ of $\mathcal{A}$ is the poset of subspaces of $\mathbb{R}^d$ obtained from intersections of subsets of the hyperplanes, ordered by reverse inclusion. The open, connected components of the complement $\mathbb{R}^d \setminus \mathcal{A}$ are called (open) ***chambers***. The set of (closed) ***faces*** Σ is the set of *closures* of chambers of $\mathcal{A}$, together with all possible intersections of closures of chambers (ignoring intersections which are empty). This set comes equipped with a natural partial order by reverse inclusion, and for this reason we refer to Σ as the ***face poset*** of $\mathcal{A}$. There is an order-preserving, rank-preserving surjection $\mathsf{supp} : \Sigma \twoheadrightarrow \mathcal{L}$ sending a face to its affine span [8, Proposition 4.1.13]. This map extends to chains, and the fiber sizes are given, for $\mathcal{C} = \{\mathcal{C}_1 < \cdots < \mathcal{C}_k\} \subseteq \mathcal{L}$, by

$$\#\mathsf{supp}^{-1}(\mathcal{C}) = \prod_{i=1}^{k} \mathsf{Poin}([\mathcal{C}_i, \mathcal{C}_{i+1}]; 1) = \mathsf{Poin}_{\mathcal{C}}(P; 1), \tag{1.3}$$

with $\mathcal{C}_{k+1} = \hat{1}$; see [8, Proposition 4.6.2]. This is the key motivation for the next definition.

Definition 1.4. The ***pullback* ab*-index*** of P is

$$\Psi_{\mathsf{pull}}(P; \mathbf{a}, \mathbf{b}) = {}_{\mathsf{ex}}\Psi(P; 1, \mathbf{a}, \mathbf{b}).$$

Let Σ be the face poset and $\mathcal{L}$ the lattice of flats of a real central hyperplane arrangement. Since Σ may not have a unique minimum element, we formally add a minimum element $\hat{0}$ and let $\Sigma \cup \{\hat{0}\}$ be the resulting poset. Now, Equation (1.3) relates the **ab**-index of the face poset and the pullback **ab**-index of the lattice of flats by

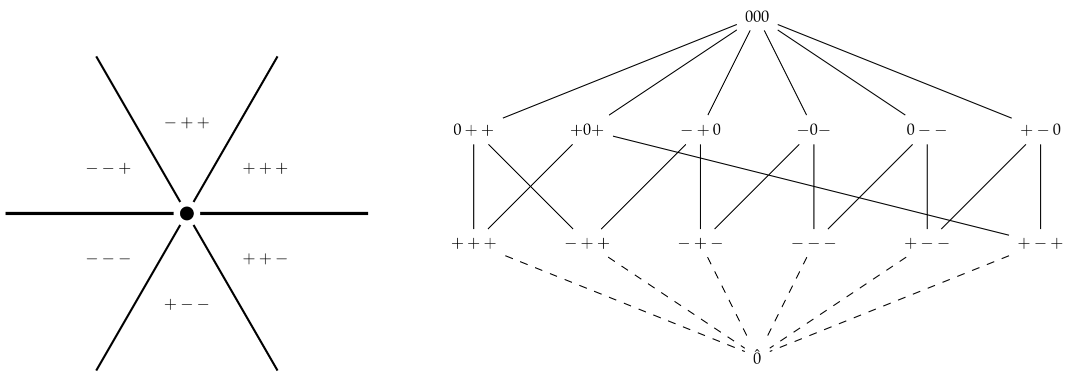
$$\Psi(\Sigma \cup \{\hat{0}\}; \mathbf{a}, \mathbf{b}) = \mathbf{a} \cdot \Psi_{\mathsf{pull}}(\mathcal{L}; \mathbf{a}, \mathbf{b})\,. \tag{1.4}$$

Note that this relates the evaluation of ${}_{\mathsf{ex}}\Psi(\Sigma \cup \{\hat{0}\}; y, \mathbf{a}, \mathbf{b})$ at $y = 0$ to the evaluation of ${}_{\mathsf{ex}}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b})$ at $y = 1$. Equation (1.3) and thus also Equation (1.4) hold indeed in the more general context of oriented matroids.

Example 1.5. The pullback **ab**-index of the poset from Example 1.2 is

$$\Psi_{\mathsf{pull}}(\mathcal{L}; \mathbf{a}, \mathbf{b}) = {}_{\mathsf{ex}}\Psi(\mathcal{L}; 1, \mathbf{a}, \mathbf{b}) = \mathbf{a}^2 + 5\mathbf{ba} + 5\mathbf{ab} + \mathbf{b}^2\,.$$

Consider the arrangement of three lines in the plane through a common intersection as shown below on the left in a way that emphasizes its face structure. Its lattice of flats is the poset $\mathcal{L}$ from Example 1.2. To the right, we draw its face poset Σ with $\hat{0}$ included.



The **ab**-index of $\Sigma \cup \{\hat{0}\}$ can be computed as

$$\mathbf{a}^3 + 5\mathbf{aba} + 5\mathbf{a}^2\mathbf{b} + \mathbf{ab}^2 = \mathbf{a}(\mathbf{a}^2 + 5\mathbf{ba} + 5\mathbf{ab} + \mathbf{b}^2) = \mathbf{a} \cdot \Psi_{\mathsf{pull}}(\mathcal{L}; \mathbf{a}, \mathbf{b})\,.$$

1.2 Main results

The main results of this paper concern *R-labeled posets*. These form a large family of posets including *distributive lattices*, *geometric lattices*, and *semimodular lattices*. In order to state Theorem 1.6, we introduce a combinatorial statistic on maximal chains of these posets and use this to describe the extended **ab**-index. In Section 2, we briefly discuss this combinatorial statistic for general edge labeled graded posets.

A function λ from the set of cover relations $X \lessdot Y$ in P into the positive integers is an ***R*-*labeling*** of P if, for every interval $[X, Y]$ in P, there is a unique maximal chain $X = \mathcal{M}_i \lessdot \mathcal{M}_{i+1} \lessdot \cdots \lessdot \mathcal{M}_j = Y$ such that

$$\lambda(\mathcal{M}_i, \mathcal{M}_{i+1}) \leq \lambda(\mathcal{M}_{i+1}, \mathcal{M}_{i+2}) \leq \cdots \leq \lambda(\mathcal{M}_{j-1}, \mathcal{M}_j).$$

We say a poset P is ***R*-*labeled*** if it is finite, graded, and admits an R-labeling. Throughout this section, we consider R-labeled posets with a fixed R-labeling λ.

The first result is a combinatorial statistic describing the coefficients of the extended **ab**-index which witnesses their nonnegativity. It generalizes [6, Corollary 7.2] and also reproves it using purely combinatorial arguments. For a maximal chain $\mathcal{M} = \{\mathcal{M}_0 \lessdot \mathcal{M}_1 \lessdot \cdots \lessdot \mathcal{M}_n\}$ in P, define the monomial $\mathsf{u}(\mathcal{M}) = u_1 \cdots u_n$ in $\mathbf{a}, \mathbf{b}$ given by $u_1 = \mathbf{a}$ and for $i \in \{2, \ldots, n\}$ by

$$u_i = \begin{cases} \mathbf{a} & \text{if } \lambda(\mathcal{M}_{i-2}, \mathcal{M}_{i-1}) \leq \lambda(\mathcal{M}_{i-1}, \mathcal{M}_i)\,, \\ \mathbf{b} & \text{if } \lambda(\mathcal{M}_{i-2}, \mathcal{M}_{i-1}) > \lambda(\mathcal{M}_{i-1}, \mathcal{M}_i)\,. \end{cases} \tag{1.5}$$

Now, let $E \subseteq \{1, \ldots, n\}$, viewed as a subset of the cover relations in the chain $\mathcal{M}$. Define the monomial $\mathsf{u}(\mathcal{M}, E) = v_1 \ldots v_n$ in $\mathbf{a}, \mathbf{b}$ to be obtained from $\mathsf{u}(\mathcal{M})$ by

- replacing all variables $\mathbf{a}$ by $\mathbf{b}$ at positions $i \in \{1, \ldots, n\}$ if $i \in E$ and

- replacing all variables $\mathbf{b}$ by $\mathbf{a}$ at positions $i \in \{2, \ldots, n\}$ if $i-1 \in E$.

In particular, we have $\mathsf{u}(\mathcal{M}, \varnothing) = \mathsf{u}(\mathcal{M})$, and $v_1 = \mathbf{b}$ if and only if $1 \in E$.

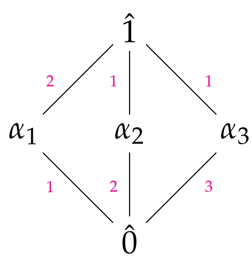
Theorem 1.6. *Let P be an R-labeled poset of rank n. Then*

$$_{\mathsf{ex}}\Psi(P; y, \mathbf{a}, \mathbf{b}) = \sum_{(\mathcal{M}, E)} y^{\#E} \cdot \mathsf{u}(\mathcal{M}, E)$$

where the sum ranges over all maximal chains $\mathcal{M}$ in P and all subsets $E \subseteq \{1, \ldots, n\}$.

When P is a geometric lattice, setting $y = 0$ in Theorem 1.6 recovers [6, Corollary 7.2]. Specifically $\Psi(P; \mathbf{a}, \mathbf{b}) = \sum_{\mathcal{M}} \mathsf{u}(\mathcal{M})$, where the sum ranges over all maximal chains $\mathcal{M} = \{\mathcal{M}_0 \lessdot \cdots \lessdot \mathcal{M}_n\}$.

Example 1.7. The poset from the previous examples admits the R-labeling given below on the left. On the right, we collect the relevant data to compute the combinatorial description of the extended $\mathbf{ab}$-index.



E	$y^{\#E}$	$\hat{0} \lessdot \alpha_1 \lessdot \hat{1}$	$\hat{0} \lessdot \alpha_2 \lessdot \hat{1}$	$\hat{0} \lessdot \alpha_3 \lessdot \hat{1}$
$\{\}$	1	**aa**	**ab**	**ab**
$\{1\}$	y	**ba**	**ba**	**ba**
$\{2\}$	y	**ab**	**ab**	**ab**
$\{1,2\}$	y^2	**bb**	**ba**	**ba**

Then $_{\mathsf{ex}}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b}) = \mathbf{aa} + (3y + 2y^2)\mathbf{ba} + (2 + 3y)\mathbf{ab} + y^2\mathbf{bb}$.

Corollary 1.8. *For an R-labeled poset P, we have*

$$_{\mathsf{ex}}\Psi(P; y, \mathbf{a}, \mathbf{b}) = \omega\big(\Psi(P; \mathbf{a}, \mathbf{b})\big)$$

where the substitution ω replaces all occurrences of $\mathbf{ab}$ with $\mathbf{ab} + y\mathbf{ba} + y\mathbf{ab} + y^2\mathbf{ba}$ and then simultaneously replaces all remaining occurrences of $\mathbf{a}$ with $\mathbf{a} + y\mathbf{b}$ and $\mathbf{b}$ with $\mathbf{b} + y\mathbf{a}$.

Using Corollary 1.8, the monomials $\mathsf{u}(\mathcal{M}, E)$ in Theorem 1.6 capture the same information as the *generalized descent sets* on *réseaux* as defined by Bergeron, Mykytiuk, Sottile, and van Willigenburg in [5, Section 7] in the context of quasisymmetric functions. The next corollary can be seen as a refinement of [27, Proposition 2.2] and of [5, Theorem 7.2], stated in terms of $\mathbf{ab}$-indices rather than quasisymmetric functions. Both can be seen as the special case for the pullback $\mathbf{ab}$-index: the first for *enriched P-partitions* and the second for general edge-labeled graded posets, compare with Section 2. We start by describing their relevant combinatorics in the present notation. Let $\mathcal{M}$ be a maximal chain with $\mathsf{u}(\mathcal{M}) = u_1 \ldots u_n$, and let

$$\mathsf{Peak}(\mathcal{M}) = \big\{ i \in \{2, \ldots, n\} \mid u_{i-1} = \mathbf{a}, u_i = \mathbf{b} \big\}$$

denote its ***peak set***. A set $S \subseteq \{1,\dots,n\}$ is then $\mathcal{M}$***-peak-covering*** if

$$\mathsf{Peak}(\mathcal{M}) \subseteq S \cup \{i+1 \mid i \in S\}\,.$$

For $\mathsf{u}(\mathcal{M},S) = v_1 \cdots v_n$, let b-out$(\mathcal{M},S)$ be the number of positions $i \in \{1,\dots,n\} \setminus S$ where $v_i = \mathbf{b}$.

Corollary 1.9. *For an R-labeled poset P of rank n, we have*

$$_{\mathsf{ex}}\Psi(P;y,\mathbf{a},\mathbf{b}) = \sum_{(\mathcal{M},S)} (1+y)^{\#S} \cdot y^{\text{b-out}(\mathcal{M},S)} \cdot \mathsf{wt}_S\,,$$

where the sum ranges over all maximal chains $\mathcal{M}$ and all $\mathcal{M}$-peak-covering subsets $S \subseteq \{1,\dots,n\}$ and where $\mathsf{wt}_S = w_1 \dots w_n$ as given in Equation (1.1).

Another consequence of Corollary 1.8 is that the Poincaré polynomial of P is in fact encoded in its **ab**-index. To see this, we define another substitution ι, which deletes the first letter from every **ab**-monomial, so $\iota(\mathbf{a}^3\mathbf{ba} + (1+y)\mathbf{ba}) = \mathbf{a}^2\mathbf{ba} + (1+y)\mathbf{a}$ for example. This gives us a way to obtain the Poincaré polynomial from the **ab**-index, a result which is similar in spirit to [6, Proposition 5.3].

Corollary 1.10. *For an R-labeled poset P of rank n, the Poincaré polynomial is the coeffcient of $\mathbf{a}^{n-1}$ in $\iota\big(\omega\big(\Psi(P;\mathbf{a},\mathbf{b})\big)\big)$.*

Corollary 1.8 generalizes [6, Theorem 3.1] relating the **ab**-index of the lattice of flats of an oriented matroid with the **ab**-index of its face poset. As a consequence, we see that $_{\mathsf{ex}}\Psi(P;y,\mathbf{a},\mathbf{b})$ is akin to a refinement of a **cd**-index. We make this observation precise in the following corollary.

Corollary 1.11. *For an R-labeled poset P, there exists a polynomial $\Phi(P;\mathbf{c}_1,\mathbf{c}_2,\mathbf{d})$ in noncommuting variables $\mathbf{c}_1,\mathbf{c}_2,\mathbf{d}$ such that*

$$_{\mathsf{ex}}\Psi(P;y,\mathbf{a},\mathbf{b}) = \Phi(P;\ \mathbf{a}+y\mathbf{b},\ \mathbf{b}+y\mathbf{a},\ \mathbf{ab}+y\mathbf{ba}+y\mathbf{ab}+y^2\mathbf{ba}).$$

In particular, the pullback **ab***-index $\Psi_{\mathsf{pull}}(P;\mathbf{a},\mathbf{b})$ is a polynomial in noncommuting variables $\mathbf{c} = \mathbf{a}+\mathbf{b}$ and $2\mathbf{d} = 2(\mathbf{ab}+\mathbf{ba})$.*

Remark 1.12 (The synthetic **cd**-index)**.** Recall that the **cd**-index of a poset exists if the **ab**-index can be written as a polynomial in $\mathbf{c} = \mathbf{a}+\mathbf{b}$ and $\mathbf{d} = (\mathbf{ab}+\mathbf{ba})$. Bayer and Klapper proved a conjecture of Fine that a poset satisfies the *generalized Dehn-Sommerville relations* if and only if its **cd**-index exists and has integer coefficients [4, Theorem 4]. The **cd**-index of an Eulerian poset always exists (see [3, Theorem 2.1]) and has nonnegative coefficients when it comes from the face poset of a shellable regular *CW* sphere like the face poset of a convex polytope [26, Theorem 2.2] (or, more generally, from a Gorenstein* poset [17, Theorem 1.3]).

In [6], Billera, Ehrenborg, and Readdy give an elegant alternative proof of the non-negativity of the **cd**-index of the face poset of an oriented matroid. They use the support map from Equation (1.3) to relate the **ab**-index of the lattice of flats to the **ab**-index of the face poset. In our language, they interpret (using posets and polytopes) the extended **ab**-index of an oriented matroid at $y = 0$ and $y = 1$. Every matroid admits an extended **ab**-index, and the evaluation at $y = 0$ is the **ab**-index of its lattice of flats. This raises the natural question whether there is a geometric or poset-theoretic interpretation of the $y = 1$ evaluation of the extended **ab**-index. For this reason, we call the $y = 1$ evaluation of the extended **ab**-index rewritten in terms of **c** and **d** the ***synthetic* cd*-index***.

Example 1.13 (The Fano matroid)**.** Setting $y = 1$ and then $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$ in the extended **ab**-index of the *Fano matroid* [8, Example 6.6.2(1)] gives the synthetic **cd**-index of the Fano matroid: $12\mathbf{cd} + 28\mathbf{dc} + \mathbf{c}^3$. A convex 3-polytope with this **cd**-index would have 30 vertices and 14 facets; see [21]. Thus its polar dual polytope would have 14 vertices and 30 facets, contradicting the the Upper Bound Theorem [20, p.180].

Example 1.14 (The Mac Lane matroid)**.** We compute the synthetic **cd**-index of the *Mac Lane matroid*; see [9, page 114] and [29, Section 2]. We get the synthetic **cd**-index $18\mathbf{cd} + 32\mathbf{dc} + \mathbf{c}^3$, which is the **cd**-index of a polytope!

Remark 1.15 (Oriented interval greedoids)**.** The argument used for oriented matroids and their lattices of flats also applies to *oriented interval greedoids*, where the analogue of Equation (1.3) is given in [24, Theorem 6.8]. Since the lattice of flats of an interval greedoid is a semimodular lattice, it admits an R-labeling; see [7, Theorem 3.7]. Applying Corollary 1.8 and setting $y = 1$ gives [24, Corollary 6.12].

Remark 1.16 (Distributive lattices & r-signed Birkhoff posets)**.** Ehrenborg discussed an ω-like substitution for arbitrary distributive lattices [11]. Remarkably, that substitution is equivalent to the substitution in Corollary 1.8 for $y = r - 1 \in \mathbb{N}$. In that case of distributive lattices, the parameter r is a fixed integer (rather than a variable) carrying information about the fiber sizes of a certain support map. For a (not necessarily graded) finite poset P, the *r-signed Birkhoff poset* $J_r(P)$ is the collection of pairs (F, f) where F is an *order ideal* in P and f is a map from the maximal elements in F to the set $\{1, \dots, r\}$, with order relation given by

$$(F, f) \leq (G, g) \quad \iff \quad G \subseteq F \text{ and } f(x) = g(x) \text{ for all } x \in \max(F) \cap \max(G)\,.$$

These posets were defined in [15, 11] and studied in connection to the Birkhoff lattice $J(P) = J_1(P)$. The map $z : J_r(P) \to J(P)$ with $(F, f) \mapsto F$ is an order- and rank-preserving poset surjection for which the fiber size of a chain $\mathcal{C}$ in $J(P)$ can—in the notation from the previous sections—be computed by $\#z^{-1}(\mathcal{C}) = \mathsf{Poin}_{\mathcal{C}}(J(P); r - 1)$, see [11, Proposition 5.2]. Since distributive lattices are modular, they admit R-labelings; see [7, Theorem 3.7]. Thus, applying Corollary 1.8 for $y = r - 1$ gives the first part of [11, Theorem 4.2].

We next turn toward the coarse flag Hilbert–Poincaré series introduced and studied in [19]. The numerator of this rational function is defined in [19, Equation (1.13)], and we extend this definition to graded posets via

$$\mathsf{Num}(P;y,t) = \sum_{\mathcal{C} \text{ chain in } P\setminus\{\hat{0},\hat{1}\}} \mathsf{Poin}_{\{\hat{0}\}\cup\mathcal{C}}(P;y) \cdot t^{\#\mathcal{C}}(1-t)^{n-1-\#\mathcal{C}} \in \mathbb{Z}[y,t]\,.$$

By removing the first letter of every **ab** monomial and then specializing via $\mathbf{a} \mapsto 1$ and $\mathbf{b} \mapsto t$ we obtain a proof of [19, Conjecture E] and its generalization to R-labeled posets:

Corollary 1.17. *For an R-labeled poset P, the coefficients of $\mathsf{Num}(P;y,t)$ are nonnegative.*

Together with Corollary 1.10, we obtain $\mathsf{Poin}(P;y) = [t^0]\ \mathsf{Num}(P;y,t)$. The substitutions in the previous corollaries show that Theorem 1.6 also gives analogous combinatorial interpretations for the coefficients of $\iota\big({}_{\mathsf{ex}}\Psi(P;y,\mathbf{a},\mathbf{b})\big)$ and of $\mathsf{Num}(P;y,t)$.

Remark 1.18 (Geometric semilattices)**.** Note that [19, Conjecture E] concerns all hyperplane arrangements (central and affine). While the intersection posets of central hyperplane arrangements are geometric lattices and, thus, admit R-labelings [7, Example 3.8], the intersection posets of affine arrangements are part of a more general family called *geometric semilattices*, first explicitly studied by Wachs and Walker in [28]. A theorem of Ziegler shows that if $\mathcal{L}$ is a geometric semilattice, then $\mathcal{L} \cup \{\hat{1}\}$ admits an R-labeling [30, Theorem 2.2]. Thus Theorem 1.6 holds for intersection posets of affine arrangements.

Remark 1.19 (Implications for other zeta functions)**.** The coarse flag Hilbert–Poincaré polynomial of a poset P comes from a natural specialization of its flag Hilbert–Poincaré series. The flag Hilbert–Poincaré series is a rational function in $\mathbb{Q}[y](t_x \mid x \in P)$ given by

$$\mathsf{fHP}_P(y,\mathbf{t}) = \sum_{\mathcal{C} \text{ chain in } P\setminus\hat{0}} \mathsf{Poin}_{\mathcal{C}}(P;y) \prod_{x\in\mathcal{C}} \frac{t_x}{1-t_x}\,.$$

The coarse flag Hilbert–Poincaré polynomial $\mathsf{Num}(P;y,t)$ is obtained by setting all the t_x equal to t and considering $(1-t)^{\mathsf{rank}(P)}\mathsf{fHP}_P(y,t)$. Different specializations of $\mathsf{fHP}_P(y,\mathbf{t})$ yield other well-studied zeta functions like local Igusa zeta functions of hyperplane arrangements [10], motivic zeta functions of matroids from [16], and the conjugacy class counting zeta functions of certain group schemes defined in [23]. Moreover, each of these is obtained from $\mathsf{fHP}_P(y,\mathbf{t})$ by a monomial substitution of the form $y = -p^{-1}$ and $t_x = p^{\lambda_x}t^{\mu_x}$ for some integers λ_x and μ_x, where p is a prime and $t = p^{-s}$ for a complex variable s; see [19, Remark 1.3].

The specialization of $\mathsf{Num}(P;y,t)$ at $y = 1$ was studied further for matroids and oriented matroids by the second author and Kühne in [18], who showed $\mathsf{Num}(P;1,t)$ is the sum of h-polynomials of simplicial complexes related to the chambers if P is the lattice of flats of a real central hyperplane arrangement. The following corollary proves a generalized version of the conjectured lower bound from [18, Conjecture 1.4].

Corollary 1.20. *Let P be an R-labeled poset of rank n. The coefficient of t^k in $\mathsf{Num}(P;1,t)$ is bounded below by $\binom{n-1}{k} \cdot \mathsf{Poin}(P;1)$.*

2 Connection to quasisymmetric functions

Theorem 1.6 shows that the extended **ab**-index of an R-labeled poset has nonnegative coefficients. Nonnegativity may fail, however, for posets that do not admit R-labelings. For example, the weak order for the symmetric group $\mathfrak{S}_3$ (the hexagon poset) does *not* admit an R-labeling and has extended **ab**-index

$$\begin{aligned}\mathbf{aaa} + (-1+2y)\mathbf{aab} + (1+2y)\mathbf{aab} + y(2+y^2)\mathbf{baa} + (2y^2-1)\mathbf{abb} \\ + (-y^3+2y^2)\mathbf{bab} + y^2(2+y)\mathbf{bba} + y(3y^2+2y-2)\mathbf{bbb}\,.\end{aligned}$$

Using the right-hand side in Theorem 1.6, we define the (***combinatorial***) ***extended* ab*-index*** of a finite edge-labeled graded poset P, which is *manifestly positive*, via

$${}_{\mathsf{cx}}\Psi(P;y,\mathbf{a},\mathbf{b}) = \sum_{(\mathcal{M},E)} y^{\#E} \cdot \mathsf{u}(\mathcal{M},E) \ \in \mathbb{N}[y]\langle \mathbf{a},\mathbf{b}\rangle\,.$$

While ${}_{\mathsf{cx}}\Psi$ is in general not linked to the Poincaré polynomial, the proofs of Corollaries 1.8 and 1.9 still hold. In particular, ${}_{\mathsf{cx}}\Psi(P;y,\mathbf{a},\mathbf{b})$ is a polynomial in $\mathbf{c}_1 = \mathbf{a}+y\mathbf{b}, \mathbf{c}_2 = \mathbf{b}+y\mathbf{a}$ and $\mathbf{d} = \mathbf{ab}+y\mathbf{ba}+y\mathbf{ab}+y^2\mathbf{ba}$. This means that $2\cdot {}_{\mathsf{cx}}\Psi(P;1,\mathbf{a},\mathbf{b})$ is an **ab**-analogue of the *peak enumerator* from [5, Definition 7.1]. The remainder of this section is devoted to presenting a conjecture inspired by this specialization.

Let $S = \{s_1 < \cdots < s_k\}$ be a subset of $\{1,\ldots,n\}$. The ***monomial quasisymmetric function*** M_S is the power series

$$M_S = \sum_{i_1<i_2<\cdots<i_k<i_{k+1}} x_{i_1}^{s_1} x_{i_2}^{s_2-s_1} \cdots x_{i_k}^{s_k-s_{k-1}} x_{i_{k+1}}^{n+1-s_k} \quad \in \mathbb{Q}[[x_1,x_2,x_3,\ldots]]\,.$$

Note that M_S is homogeneous of degree $n+1$ and—although we surpress it in the notation—implicitly depends on n. The ring of ***quasisymmetric functions*** QSym is the (linear) span of $M_\bullet = 1$ and all M_S for $n \geq 0$. Following [12, Section 3], we define a vector space isomorphism $\Xi : \mathbb{Q}\langle \mathbf{a},\mathbf{b}\rangle \longrightarrow \mathsf{QSym}$ defined by sending wt_T to M_T. Using the isomorphism Ξ, we can view the map ω from Corollary 1.8 as a map from QSym to $\mathsf{QSym} \otimes \mathbb{Q}[y]$ given by $F_S \mapsto \omega(F_S) = \Xi\big(\omega(\mathsf{m}_S)\big)$, where F_S is given in [13, Equation 2]. In [27, Equation (1.8)], Stembridge shows how to obtain (skew) Schur functions as P-partition enumerators of certain posets given in [27, Section 1.3]. The following conjecture[1] concerning the Schur functions has been verified for all integer partitions of size at most 11 using `SageMath`.

[1]This conjecture was exhibited at the *90th Séminaire Lotharingien de Combinatoire* in Bad Boll, Germany in September 2023 in collaboration with Darij Grinberg.

Conjecture 2.1. *For any partition $\lambda \vdash n$, the quasisymmetric function $\omega(s_\lambda)$ is symmetric and Schur positive. Specifically, for each $\mu \vdash n$, there exist $c_\lambda^\mu(y) \in \mathbb{N}[y]$ such that*

$$\omega(s_\lambda) = \sum_{\mu \vdash n} c_\lambda^\mu(y) \cdot s_\mu \,.$$

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