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On the sum of the entries in a character table

Arvind Ayyer*[1], Hiranya Kishore Dey†[1], and Digjoy Paul‡[1]

[1] *Department of Mathematics, Indian Institute of Science, Bangalore 560012, India*

Abstract. In 1961, Solomon proved that the sum of all the entries in the character table of a finite group does not exceed the cardinality of the group. We state a different and incomparable property here – this sum is at most twice the sum of dimensions of the irreducible characters. We establish the validity of this property for all finite irreducible Coxeter groups. The main tool we use is that the sum of a column in the character table of such a group is given by the number of square roots of the corresponding conjugacy class representative. We then show that the asymptotics of character table sums is the same as the number of involutions in symmetric, hyperoctahedral and demihyperoctahedral groups. Finally, we derive generating functions for the character table sums for these latter groups as well as generalized symmetric groups as infinite products of continued fractions.

Keywords: finite group, irreducible Coxeter group, character table, symmetric group, hyperoctahedral group, demihyperoctahedral group, absolute square roots, generalized symmetric group, asymptotics, continued fractions

1 Introduction

For any finite group, it is natural to consider the sum of the entries of the character table. Solomon [18] proved that this is always a nonnegative integer by proving something stronger, namely that all row sums are nonnegative integers. He did so by showing that the sum of a row indexed by an irreducible representation is the multiplicity of that representation in the group algebra with respect to the conjugacy action. He then deduced that the sum of the entries in the character table of a finite group is at most the cardinality of the group.

In this extended abstract, we take a different approach to estimating the sum of the entries of the character table by considering column sums instead. It is well known that the column sums are always integers, though not necessarily non-negative [10, Proposition 3.14]. However, for groups whose irreducible characters are real, the column sums

*arvind@iisc.ac.in. AA thanks the DST FIST program - 2021 [TPN - 700661] and SERB Core grant CRG/2021/001592 for support.

†hiranyadey@iisc.ac.in. HKD thanks the DST FIST program - 2021 [TPN - 700661] and SERB-NPDF fellowship PDF/2021/001899 for support.

‡digjoypaul@iisc.ac.in DP thanks the DST FIST program - 2021 [TPN - 700661] for support.

are given by the number of square roots of conjugacy class representatives by a classical result by Frobenius and Schur [11]. Weyl groups are well-known examples of such groups. However, this is not the case for generalized symmetric groups $G(r,1,n), r \geq 3$. For $G(r,1,n)$, column sums are given by so-called absolute square roots [2].

From extensive computations, we observe the following upper bound for the sum of the entries of the character table for many but not all groups.

Property S . *The sum of the the entries of the character table of a finite group is at most twice the sum of dimensions of its irreducible representations.*

We know Property S will not hold in general, but it seems to hold for a large class of natural groups. The smallest counterexamples are of order 64. Our main result is for the following important class of finite groups.

Theorem 1.1. *Property S holds for all finite irreducible Coxeter groups.*

Note that this also settles the issue for Weyl groups. The proof of Theorem 1.1 follows by a case analysis. By analysing the square roots, it is easy to prove the result for dihedral groups. We will prove Property S for the symmetric, hyperoctahedral and demihyperoctahedral groups in the later sections. By explicit computations, we have verified the result for exceptional irreducible finite Coxeter groups. Details will appear in [4]. It is tempting to believe that Property S holds for all finite simple groups. We have not yet done a systematic study in that direction, but we certainly believe the following.

Conjecture 1.2. *Property S holds for all alternating groups.*

Property S holds for abelian groups H because the orthogonality of rows in a character table leads to the vanishing of row sums of all representations except the trivial one. Using this fact, we prove that $G \times H$ satisfies the property if it is true for G. It turns out that Property S holds for any finite group whose all irreducible representations have dimensions at most 2. This class includes generalized dihedral groups and generalized quaternion groups.

It is natural to consider the sequence of these sums for the infinite familes of irreducible Coxeter groups. In Section 2, we consider this sum s_n for the symmetric group S_n. We compute its generating function in Section 2.1. In Section 2.2, we sketch the proof of Property S for S_n and show that the asymptotics of s_n is the same as the number of involutions in S_n. We state similar results for the hyperoctahedral groups B_n in Section 3 and for the demihyperoctahedral groups D_n in Section 4. Since the main ideas are similar, we only state the results. We then extend the generating function result to the generalized symmetric groups $G(r,1,n)$ in two ways. We give generating functions for the sum of the number of square roots as well as column sums for conjugacy class representatives in Section 5. The proofs of these results will appear in an upcoming article [4].

2 Symmetric groups

2.1 Generating function for the total sum of character table

Let S_n be the symmetric group on n letters. The set of irreducible representations and the conjugacy classes of S_n are indexed by the set of integer partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$ of n, denoted $\lambda \vdash n$. Write a partition in frequency notation as

$$\lambda = \langle 1^{m_1}, \dots, n^{m_n} \rangle, \tag{2.1}$$

where m_i denote the number of parts of length i in λ. We are interested in s_n, the sum of the entries of the character table of S_n [17, Sequence A082733]. The first few terms of (s_n) are given by

$$1, 2, 5, 13, 31, 89, 259, 842, 2810, 10020, 37266, 145373.$$

No formula is given for this sequence. Let Γ_λ be the sum of entries of the column indexed by $\lambda \vdash n$ in the character table of S_n. By applying the following classical result of Frobenius and Schur for the symmetric group, we obtain a formula for column sums in terms of square root counting function.

Theorem 2.1 ([11, Theorem 4.5]). *Given a finite group G, let $\mathrm{Irr}(G)$ denote the set of irreducible characters of G. Then*

$$|\{x \in G \mid x^2 = g\}| = \sum_{\chi \in \mathrm{Irr}(G)} \sigma(\chi)\chi(g) \quad \textit{for each } g \in G,$$

where $\sigma(\chi)$, known as the Frobenius-Schur indicator *of χ, is* 1, 0 *or* -1 *if χ is real, complex or quaternionic, respectively.*

Remark 2.2. *It is a standard fact [9, Section 8.10] that all irreducible characters of any Weyl group (for example, symmetric, hyperoctahedral and demihyperoctahedral groups) have Frobenius-Schur indicator* 1*. Thus, column sums of the character table of any Weyl group are given by the number of square roots of conjugacy class representatives.*

Therefore, $\Gamma_\lambda = |\{x \in S_n : x^2 = w_\lambda\}|$, where w_λ is some fixed element of cycle type λ. Recall that the *double factorial* of an integer n is given by $n!! = n(n-2)\cdots$ ending at either 2 or 1 depending on whether n is even or odd respectively. Define

$$o_r(m) = \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m}{2k} (2k-1)!!\, r^k. \tag{2.2}$$

Proposition 2.3 ([1, Corollary 3.2]). *The column sum Γ_λ is 0 unless m_{2i} is even for all $i \in \{1, \dots, \lfloor n/2 \rfloor\}$. If that is the case,*

$$\Gamma_\lambda = \prod_{i=1}^{\lfloor n/2 \rfloor} (m_{2i} - 1)!!\, (2i)^{m_{2i}/2} \prod_{j=0}^{\lfloor n/2 \rfloor} o_{2j+1}(m_{2j+1}).$$

Let $\mathcal{S}(x)$ be the (ordinary) generating function of the sequence (s_n), i.e.

$$\mathcal{S}(x) = \sum_{n \geq 0} s_n x^n. \tag{2.3}$$

To give a formula for $\mathcal{S}(x)$, we recall that generating functions which are expressed as continued fractions have a long history beginning with the influential work of Flajolet [8]. There are two kinds of continued fractions which appear commonly. The *Stieltjes continued fraction*, or *S-fraction* has linear terms and the *Jacobi continued fraction*, or *J-fraction* has quadratic terms.

Recall that an *involution* in S_n is a permutation w which squares to the identity. Let i_n be the number of involutions in S_n. A well-known result due to Flajolet [8, Theorem 2(iia)] gives the generating function $\mathcal{I}(x)$ of involutions in S_n as the J-fraction

$$\mathcal{I}(x) = \sum_{n \geq 0} i_n x^n = \cfrac{1}{1 - x - \cfrac{x^2}{1 - x - \cfrac{2x^2}{\ddots}}}. \tag{2.4}$$

Flajolet also showed in the same theorem [8, Theorem 2(iib)] that the generating function of odd double factorials is the S-fraction

$$\mathcal{D}(x) = \sum_{n \geq 0} (2n-1)!!\, x^n = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{2x}{\ddots}}}. \tag{2.5}$$

The quantity $o_r(m)$ (defined in (2.2)) and its generalizations have been studied in [12]. Setting $t = 0, m = 0$ and $u_1 = 1$ in the same theorem [8, Theorem 2] we obtain the generating function for $o_r(m)$ as the J-fraction

$$\mathcal{R}_r(x) = \sum_{n \geq 0} o_r(n) x^n = \cfrac{1}{1 - x - \cfrac{rx^2}{1 - x - \cfrac{2rx^2}{\ddots}}}. \tag{2.6}$$

Bessenrodt–Olsson [5] found an explicit bijection between the number of columns in the character table of S_n that have sum zero and the number of partitions of n with at least one part congruent to $2 \pmod 4$. They also computed the generating function for the number partitions whose associated column sum is nonzero.

Let $x, x_1, x_2, \ldots$ be a family of commuting indeterminates. The following result answers a question of Amdeberhan [3].

Theorem 2.4. *The number of square roots of a permutation with cycle type λ written as* (2.1) *is the coefficient of $x_1^{m_1}x_2^{m_2}\ldots x_n^{m_n}$ in*

$$\prod_{i\geq 1}\mathcal{D}(2ix_{2i}^2)\mathcal{R}_{2i-1}(x_{2i-1}).$$

Consequently, the generating function of the character table sum is

$$\mathcal{S}(x)=\prod_{i\geq 1}\mathcal{D}(2ix^{4i})\mathcal{R}_{2i-1}(x^{2i-1}).$$

2.2 Proof of Property S for S_n

Recall that *derangements* are permutations without fixed points. We define another sequence (g_n) by

$$g_n:=\sum_{\substack{\lambda\vdash n\\ m_1(\lambda)=0}}\Gamma_\lambda,\quad n\geq 1\quad\text{and}\quad g_0=1.$$

Then g_n counts the sum of those columns of the character table of S_n which are indexed by the conjugacy classes corresponding to derangements. The next result is a convolution type statement involving s_n, g_n, and i_n.

Proposition 2.5. *For a positive integer n, we have*

$$s_n=\sum_{k=0}^{n}i_k g_{n-k}.$$

We next prove the following lemma which gives us control over the sequence s_n.

Lemma 2.6. *For $n\geq 2$, we have $2i_{n-1}\leq i_n\leq ni_{n-1}$. Further, for $n\geq 4$, we have $i_k g_{n-k}\leq i_{n-1}/(n-2)$ for all $0\leq k\leq n-3$.*

Using Lemma 2.6, we show that $s_n\leq i_n+i_{n-1}$, which helps to prove the following:

Theorem 2.7. *Property S holds for all symmetric groups.*

Using the asymptotics of (i_n) derived by Chowla–Herstein–Moore [6, Theorem 8], we confirm the observation of user Lucia [3].

Corollary 2.8. *The total sum sequence (s_n) grows asymptotically as fast as (i_n) and hence*

$$s_n\sim\left(\frac{n}{e}\right)^{n/2}\frac{e^{\sqrt{n}-1/4}}{\sqrt{2}}.$$

3 Weyl groups of type B

The group $\mathbb{Z}_2 \wr S_n$ is called the *hyperoctahedral group* B_n. It can also be written as the generalized symmetric group $G(2,1,n)$. But following [16], we can define it in a more elementary way. But of course, some of these statements here can be seen directly from Section 5 using that language.

Definition 3.1. *Regard S_{2n} as the group of permutations of the set $\{\pm 1, \ldots, \pm n\}$. For an integer $n \geq 2$, the* hyperoctahedral *group of type B_n is defined as*

$$B_n := \{w \in S_{2n} \mid w(i) + w(-i) = 0, \text{ for all } i,\ 1 \leq i \leq n\}$$

Every element $w \in B_n$ can be uniquely expressed as a product of cycles

$$w = w_1\overline{w_1} \cdots w_r\overline{w_r} v_1 \cdots v_s,$$

where for $1 \leq j \leq r$, $w_j\overline{w_j} = (a_1, \ldots, a_{\lambda_j})(-a_1, \ldots, -a_{\lambda_j})$ for some positive integer λ_j and for $1 \leq t \leq s$, $v_t = (b_1, \ldots, b_{\mu_t}, -b_1, \ldots, -b_{\mu_t})$ for some positive integer μ_t. An element $w_j\overline{w_j}$ is called a *positive* cycle of length λ_j and v_t is called a *negative* cycle of length μ_t. This cycle decomposition of w determines a unique pair of partitions $(\lambda \mid \mu)$ called the cycle type of w, where $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_s)$.

Theorem 3.2 ([16, Theorem 7.2.5])**.** *The set of conjugacy classes of B_n is in natural bijection with the set of ordered pairs of partitions $(\lambda \mid \mu)$ such that $|\lambda| + |\mu| = n$.*

Let s_n^B denote the total sum of the entries of the character table of B_n. The generating function of s_n^B can be obtained from the more general results in Section 5; see Remark 5.9. Let $\Gamma^B_{(\lambda|\mu)}$ be the column sum corresponding to the conjugacy class $(\lambda \mid \mu)$. To find the asymptotics of s_n^B, we define the following

$$g_n^B := \sum_{(\lambda|\mu)}{}' \, \Gamma^B_{(\lambda|\mu)}$$

where the sum runs over all ordered pairs of partitions $(\lambda \mid \mu)$ of total size n such that λ has no part of size 1. Moreover, let i_n^B denote the number of involutions in B_n. Here, we have the following counterpart of Proposition 2.5.

Proposition 3.3. *For positive integers n, we have*

$$s_n^B = \sum_{k=0}^{n} i_k^B g_{n-k}^B.$$

Following similar ideas as in the case of the symmetric group, we prove the next two results, where we use a result of Lin [13, Eq. (5)] for the asymptotics of i_n^B.

Theorem 3.4. *Property S holds for all hyperoctahedral groups.*

Corollary 3.5. *The total sum sequence (s_n^B) grows asymptotically as fast as (i_n^B) and hence*

$$s_n^B \sim \frac{e^{\sqrt{2n}}}{\sqrt{2e}} \left(\frac{2n}{e}\right)^{n/2}.$$

4 Weyl groups of type D

The Weyl group of type D, also known as the *demihyperoctahedral group* D_n, is defined as the following index two subgroup of B_n:

$$D_n := \{w \in B_n \mid w(1) \cdots w(n) > 0\}.$$

Proposition 4.1 ([15, Lemma 2.3])**.** *Let $\pi \in B_n$ have cycle type $(\lambda \mid \mu)$. Then $\pi \in D_n$ if and only if $\ell(\mu)$ is even.*

The following results gives a description of the conjugacy classes in D_n and characterize the existence of square roots.

Proposition 4.2 ([16, Theorem 8.2.1])**.** *Given a pair of partitions $(\lambda \mid \mu)$ of n, if an element $\pi \in D_n$ has cycle type $(\lambda \mid \mu)$, the associated conjugacy class $C_{\lambda,\mu}$ in B_n splits into two D_n conjugacy classes if and only if $\mu = \emptyset$ and all the parts of λ are even. The class $C_{\lambda,\mu}$ remains a D_n conjugacy class if and only if either $\mu \neq \emptyset$ or else one of the parts of λ is odd. In particular, for an odd n, any conjugacy class of B_n does not split.*

Proposition 4.3. *A pair of partitions $(\lambda \mid \mu)$ of n (such that $\ell(\mu)$ is even) is the cycle type of a square element of D_n if and only if the following holds:*

1. *all even parts of λ have even multiplicity,*
2. *all parts of μ have even multiplicity, and*
3. *either λ has an odd part or $4 \mid \ell(\mu)$.*

Using Proposition 4.3, we then obtain the following.

Theorem 4.4. *The generating function for the number of conjugacy classes in D_n with non-zero column sum is*

$$\prod_{i=1}^{\infty} \frac{1}{1-q^{4i}} \left[\left(\prod_{j=1}^{\infty} \frac{1}{1-q^{2j}} \right) \left(\prod_{k=0}^{\infty} \frac{1}{1-q^{2k+1}} - 1 \right) + \frac{1}{2} \left(\prod_{j=1}^{\infty} \frac{1}{1-q^{2j}} + \prod_{k=1}^{\infty} \frac{1}{1+q^{2k}} \right) + 1 \right] - 1.$$

Recall the generating function for double factorials in (2.5). To find the generating function for the sum of the entries in the character table of D_n, we generalize $\mathcal{R}_r(x)$ by the J-fraction

$$\mathcal{R}'_r(x,y) = \cfrac{1}{1-(1+y)x - \cfrac{rx^2}{1-(1+y)x-\cfrac{2rx^2}{\ddots}}}. \tag{4.1}$$

Theorem 4.5. *The generating function of the sum of the entries in the character table of D_n is obtained by setting all even powers of y to 1 and odd powers of y to 0 in the formal power series*

$$\prod_{i\geq 0}\Big(\mathcal{D}(4ix^{4i})\mathcal{D}(2iyx^{2i})\mathcal{R}'_{2i+1}(x^{2i+1},y)\Big) + \prod_{i\geq 0}\mathcal{D}(4ix^{4i}) - 1.$$

Let s_n^D denote the sum of the entries of the character table of D_n and i_n^D denote the number of involutions in D_n. The following lemma relates the quantities s_n^D and i_n^D.

Lemma 4.6. *For positive integers n, $s_n^D \leq i_n^D + (s_n^B - i_n^B) + g_n^B$. Moreover, for odd positive integers n, $i_n^D = i_n^B/2$ and $s_n^B = 2s_n^D$. When n is even, $2i_n^D - i_n^B = 2^{n/2}\,(n-1)!!$. Therefore, for all positive integers n, $i_n^D \leq i_n^B \leq 2i_n^D$.*

The main result here follows now from Lemma 4.6.

Theorem 4.7. *Property S holds for all demihyperoctahedral groups.*

Corollary 4.8. *The total sum sequence (s_n^D) grows asymptotically as fast as (i_n^D) and hence*

$$s_n^D \sim \frac{e^{\sqrt{2n}}}{2\sqrt{2e}}\left(\frac{2n}{e}\right)^{n/2}.$$

5 Generalized symmetric groups

We follow [15, Section 2] for the notational background used in this section. For non-negative integers r, n, let $\mathbb{Z}/r\mathbb{Z} \equiv \mathbb{Z}_r = \{\overline{0}, \overline{1}, \dots, \overline{r-1}\}$ be the additive cyclic group of order r, where we use bars to distinguish these elements from those in the symmetric group. Then define the *generalized symmetric group*

$$G(r,1,n) = \mathbb{Z}_r \wr S_n := \{(z_1,\dots,z_n;\sigma) \mid z_i \in \mathbb{Z}_r, \sigma \in S_n\}.$$

If $\pi = (z_1, z_2, \dots, z_n; \sigma)$ and $\pi' = (z'_1, z'_2, \dots, z'_n; \sigma')$, then their product is given by

$$\pi\,\pi' = (z_1 + z'_{\sigma^{-1}(1)}, \dots, z_n + z'_{\sigma^{-1}(n)}; \sigma\sigma'),$$

where $\sigma\sigma'$ is the standard product of permutations in S_n.

The group $G(r,1,n)$ can also be realized as a subgroup of the symmetric group S_{rn}. In this interpretation $G(r,1,n)$ consists of all permutations π of the set $\{\bar{k}+i \mid 0 \leq k \leq r-1, 1 \leq i \leq n\}$ satisfying $\pi(\bar{k}+i) = \bar{k} + \pi(i)$ for all allowed k and i. For convenience, we identify the letters $\bar{0}+i$ with i for $1 \leq i \leq n$. Given a permutation $\pi \in G(r,1,n)$, its values at $1,\ldots,n$ determine π uniquely.

The two definitions above are identified using the bijective map ϕ defined on the window $[1,\ldots,n]$ by

$$\phi((z_1,\ldots,z_n;\sigma)) = \begin{bmatrix} 1 & 2 & \cdots & n \\ z_{\sigma(1)}+\sigma(1) & z_{\sigma(2)}+\sigma(2) & \cdots & z_{\sigma(n)}+\sigma(n). \end{bmatrix}$$

This map satisfies $\phi(\pi\,\pi') = \phi(\pi) \circ \phi(\pi')$, where $\circ$ is the usual composition of permutations in S_{rn}.

Let $\pi = (z_1,\ldots,z_n;\sigma) \in G(r,1,n)$ and $(u_1),\ldots,(u_t)$ be the cycles of σ. Let $(u_i) = (u_{i,1},\ldots,u_{i,\ell_i})$ where ℓ_i is the length of the cycle (u_i). Define the *color* of the cycle (u_i) as $z(u_i) := z_{u_{i,1}} + z_{u_{i,2}} + \ldots + z_{u_{i,\ell_i}} \in \mathbb{Z}_r$. For $j \in \{0,\ldots,r-1\}$, let λ^j be the partition formed by the lengths of cycles of color j of σ. Note that $\sum_j |\lambda^j| = n$. The r-tuple of partitions $\boldsymbol{\lambda} = (\lambda^0 \mid \lambda^1 \mid \ldots \mid \lambda^{r-1})$ is called the *cycle type* of π. We refer to such an r-tuple of partitions as an *r-partite partition* of size n, denoted $\lambda \models_r n$. For example, the cycle type of the element

$$(\bar{2},\bar{1},\bar{1},\bar{1},\bar{0},\bar{2};\,(123)(45)(6)) \in G(3,1,6)$$

is $(\emptyset \mid (3,2) \mid (1))$. The following theorem asserts that the conjugacy classes of $G(r,1,n)$ are indexed by r-partite partitions of n.

Theorem 5.1. *[14, p. 170] Two elements π_1 and π_2 in $G(r,1,n)$ are conjugate if and only if their corresponding cycle types are equal.*

Recall the function $\mathcal{D}(x)$ from (2.5) and $\mathcal{R}(x)$ from (2.6). The following result generalizes Theorem 2.4.

Theorem 5.2. *The generating function (in n) for the sum of the number of square roots of all the conjugacy class representatives in $G(r,1,n)$ is*

$$\begin{cases} \displaystyle\prod_{i\geq 0}\left(\mathcal{D}(2irx^{4i})^r\,\mathcal{R}_{r(2i+1)}(x^{2i+1})^r\right) & r \text{ odd}, \\ \displaystyle\prod_{i\geq 0}\left(\mathcal{D}(2irx^{4i})^r\,\mathcal{D}((2i+1)rx^{4i+2})^{r/2}\,\mathcal{R}_{\frac{r(2i+1)}{4}}(2x^{2i+1})^{r/2}\right) & r \text{ even}. \end{cases}$$

In contrast with the case of S_n, the square root function does not give column sums for character table of $G(r,1,3)$, $r > 2$ as the group has non-real irreducible characters [2]. Given $\pi = (z_1,z_2,\ldots,z_n;\sigma) \in G(r,1,n)$, define the bar operation as $\overline{\pi} := (-z_1,\ldots,-z_n;\sigma)$. An element $g \in G(r,1,n)$ is said to have an *absolute square root* if there exists $\pi \in G(r,1,n)$ such that $\pi\overline{\pi} = g$. The next result describes columns sum for $G(r,1,n)$ in terms of absolute square roots.

Theorem 5.3. *[2, Theorem 3.4] Let $\{\chi_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda}$ is a r-partite partition of $n\}$ be the set of irreducible characters of $G(r,1,n)$. Then*

$$\sum_{\boldsymbol{\lambda}} \chi_{\boldsymbol{\lambda}}(g) = |\{\pi \in G(r,1,n) \mid \pi\overline{\pi} = g\}| \quad \forall g \in G(r,1,n),$$

where the sum runs over all r-partite partitions $\boldsymbol{\lambda}$.

By analyzing the absolute square roots we will provide generating functions for number of columns with zero sums and the total sum of the character table of $G(r,1,n)$.

Lemma 5.4.
1. *The absolute square of a cycle of odd length d (of any color) is a cycle of the same length of color 0.*
2. *The absolute square of a cycle of even length d (of any color) is a product of two cycles, each of length $d/2$, such that sum of their colors is zero.*

The following results extend Bessenrodt–Olsson's theorems [5] from S_n to $G(r,1,n)$.

Proposition 5.5. *An r-partite partition $\lambda = (\lambda^0 \mid \lambda^1 \mid \ldots \mid \lambda^{r-1})$ is the cycle-type of an absolute square in $G(r,1,n)$ if and only if the following hold:*

1. *each even part in λ^0 has even multiplicity,*
2. *$\lambda^i = \lambda^{r-i}$ for all $i \geq 1$, and*
3. *each part in $\lambda^{r/2}$ has even multiplicity when r is even.*

Theorem 5.6. *The generating function for r-partite partitions which are cycle-types of absolute squares in $G(r,1,n)$ is:*

$$\begin{cases} \left(\displaystyle\prod_{i=0}^{\infty} \frac{1}{1-q^{2i+1}}\right) \left(\displaystyle\prod_{j=1}^{\infty} \frac{1}{1-q^{4j}}\right) \left(\displaystyle\prod_{k=1}^{\infty} \frac{1}{1-q^{k}}\right)^{(r-1)/2} & r \text{ odd}, \\ \left(\displaystyle\prod_{i=0}^{\infty} \frac{1}{1-q^{2i+1}}\right) \left(\displaystyle\prod_{j=1}^{\infty} \frac{1}{(1-q^{4j})(1-q^{2j})}\right) \left(\displaystyle\prod_{k=1}^{\infty} \frac{1}{1-q^{k}}\right)^{(r-2)/2} & r \text{ even}. \end{cases}$$

Using [2, Observation 4.2], we obtain the number of absolute square roots for cycles of a single length and color.

Proposition 5.7. *Given a positive integer r, the following holds.*

1. *The number of absolute square roots of an element of cycle type $\lambda^0 = ((2k)^{2m_{2k}})$ (and all other λ^i is zero) is $(2m_{2k}-1)!!\,(2kr)^{m_{2k}}$.*

2. *The number of absolute square roots of an element of cycle type* $\lambda^0 = ((2k+1)^{m_{2k+1}})$ *(and all other* λ^i *is zero) is*

$$\sum_{j=0}^{\lfloor \frac{m_{2k+1}}{2} \rfloor} \binom{m_{2k+1}}{2j} (2j-1)!!\,(2k+1)^j\, r^{m_{2k+1}-j}.$$

3. *The number of absolute square roots of an element of cycle type* $\lambda^a = \lambda^{r-a} = (k^{m_k})$ *(and all other* λ^i *is zero) is* $m_k!\,(kr)^{m_k}$.

4. *For even* r*, the number of absolute square roots of an element of cycle type* $\lambda^{r/2} = (k^{2m_k})$ *(and all other* λ^t *is zero) is* $(2m_k-1)!!\,(kr)^{m_k}$.

Adin–Postnikov–Roichman [2, Corollary 4.3] also give a formula to count the number of absolute square roots of any element in $G(r,1,n)$. Using Proposition 5.7, we generalize their result to determine the sum of the character table in terms of generating functions. To do so, we also need the classic generating function for the factorials due to Euler [7] given by

$$\mathcal{F}(x) = \sum_{n\geq 0} n!x^n = \cfrac{1}{1-\cfrac{x}{1-\cfrac{x}{1-\cfrac{2x}{1-\cfrac{2x}{\ddots}}}}}. \tag{5.1}$$

Theorem 5.8. *The generating function (in* n*) of the total sum of the character table of* $G(r,1,n)$ *is*

$$\begin{cases} \prod_{i\geq 0} \Big(\mathcal{F}(irx^{2i})^{(r-1)/2}\, \mathcal{D}(2irx^{4i})\, \mathcal{R}_{(2i+1)/r}(rx^{2i+1}) \Big) & r \text{ odd},\\ \prod_{i\geq 0} \Big(\mathcal{F}(irx^{2i})^{(r-2)/2}\, \mathcal{D}(2irx^{4i})\, \mathcal{D}(rix^{2i})\, \mathcal{R}_{(2i+1)/r}(rx^{2i+1}) \Big) & r \text{ even}. \end{cases}$$

Remark 5.9. *When* $r = 2$*, absolute square roots are exactly the usual square roots. Thus the generating function for* (s_n^B) *can be obtained by setting* $r = 2$ *in either Theorem 5.2 or Theorem 5.8.*

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