Séminaire Lotharingien de Combinatoire **91B** (2024)
Article #32, 12 pp.

Proceedings of the 36th Conference on Formal Power Series and Algebraic Combinatorics (Bochum)

Macdonald characters from a new formula for Macdonald polynomials

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Abstract. We introduce a new operator $\mathbf{\Gamma}$ on symmetric functions, which enables us to obtain a creation formula for Macdonald polynomials. This formula makes a connection between the theory of Macdonald operators initiated by Bergeron, Garsia, Haiman and Tesler, and shifted Macdonald polynomials introduced by Knop, Okounkov and Sahi.

We use this formula to introduce a two-parameter generalization of Jack characters. Finally, we provide a change of variables in order to formulate several positivity conjectures related to these generalized characters. Our conjectures extend some important problems on Jack polynomials, including some famous conjectures of Goulden and Jackson.

Keywords: Macdonald polynomials, Macdonald characters, Matchings-Jack conjecture

1 Introduction

1.1 Jack and Macdonald polynomials

Jack polynomials are symmetric functions depending on one parameter α which have been introduced by Jack [13]. The combinatorial analysis of Jack polynomials has been initiated by Stanley [20] and a first combinatorial interpretation has been given by Knop and Sahi in terms of tableaux [14]. A second family of combinatorial objects related to Jack polynomials is given by *maps*, which are roughly graphs embedded in surfaces. This connection has first been observed in the conjectures of Goulden and Jackson [11] and important progress has recently been made in this direction [3, 2] with a first "topological expansion" of Jack polynomials in terms of maps.

Macdonald polynomials are symmetric polynomials introduced by Macdonald in 1989, which depend on two parameters q and t. Jack polynomials can be obtained from

*houcine.ben-dali@univ-lorraine.fr. Supported by the LUE DrEAM project of Université de Lorraine.
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Macdonald polynomials by taking an appropriate limit. Several combinatorial results on Jack polynomials have been generalized to the Macdonald case, in particular, an interpretation in terms of tableaux was established in [12]. However, no connection between Macdonald polynomials and maps is known, even conjecturally. As a first step towards a Macdonald generalization of maps, we introduce in this paper some new tools that make the parallel between the Jack and Macdonald stories more compelling.

First, we prove a creation formula (Equations (1.1) and (1.2)) for Macdonald polynomials inspired from the one used in [2] to connect Jack polynomials to maps. Second, we use this formula to introduce a Macdonald analog of Jack characters (Section 1.4). Finally, we formulate a Macdonald version of some Jack conjectures, including Goulden and Jackson's Matchings-Jack and b-conjectures.

1.2 Preliminaries

For the results of this section we refer to [7, 17]. A *partition* $\lambda = [\lambda_1, ..., \lambda_\ell]$ is a weakly decreasing sequence of positive integers $\lambda_1 \geq ... \geq \lambda_\ell > 0$. We denote by $\mathbb{Y}$ the set of integer partitions. The integer ℓ is called the *length* of λ and is denoted $\ell(\lambda)$. The size of λ is the integer $|\lambda| := \lambda_1 + \lambda_2 + ... + \lambda_\ell$. If n is the *size* of λ, we say that λ is a partition of n and we write $\lambda \vdash n$. The integers λ_1,...,λ_ℓ are called the *parts* of λ. For $i \geq 1$, we denote $m_i(\lambda)$ the number of parts of size i in λ. We then set $z_\lambda := \prod_{i\geq 1} m_i(\lambda)! i^{m_i(\lambda)}$. We identify a partition λ with its *Young diagram*, defined by $\lambda := \{(i,j), 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\}$. Fix a box $\square := (i,j) \in \lambda$. Its *arm*, *leg*, *co-arm* and *co-leg* are respectively given by

$$a_\lambda(\square) := |\{(i,c) \in \lambda, c > j\}|, \qquad \ell_\lambda(\square) := |\{(r,j) \in \lambda, r > i\}|,$$

$$a'_\lambda(\square) := |\{(i,c) \in \lambda, c < j\}|, \text{ and } \ell'_\lambda(\square) := |\{(r,j) \in \lambda, r < i\}|.$$

Finally, n and n' denote respectively the two statistics on Young diagram given by

$$n(\lambda) := \sum_{\square\in\lambda} \ell_\lambda(\square) \quad \text{and} \quad n'(\lambda) := \sum_{\square\in\lambda} a_\lambda(\square).$$

We consider the graded algebra $\Lambda = \oplus_{r\geq 0}\Lambda^{(r)}$ of symmetric functions in the alphabet $(x_1, x_2, \dots)$ with coefficients in $\mathbb{Q}(q,t)$. Let p_λ and h_λ denote the power-sum and the complete symmetric functions in $(x_i)_{i\geq 1}$, respectively. We use here a variable u to keep track of the degree of the functions, and an extra variable v; all the functions considered are in $\Lambda[v][\![u]\!]$. Consider the *Hall scalar product* defined by $\langle p_\mu, p_\nu\rangle = \delta_{\mu,\nu} z_\mu$. Let $f^\perp$ denote the adjoint of multiplication by $f \in \Lambda$ with respect to $\langle , \rangle$.

We will use the *plethystic notation*: if $E(q,t,u,v,x_1,x_2,\dots) \in \Lambda[v][\![u]\!]$ and $f \in \Lambda$ then $f[E]$ is the image of f under the algebra morphism defined by

$$\begin{aligned} \Lambda[v][\![u]\!] &\longrightarrow \Lambda[v][\![u]\!] \\ p_k &\longmapsto E(t^k, q^k, u^k, v^k, x_1^k, \dots). \end{aligned}$$

Set $X := x_1 + x_2 + \dots$. Notice that $f[X] = f(x_1, x_2, \dots)$ for any f. Moreover,

$$p_k\left[X\frac{1-q}{1-t}\right] = \frac{1-q^k}{1-t^k}p_k(x_1, x_2, \dots) \text{ and } p_\lambda[-X] = (-1)^{\ell(\lambda)}p_\lambda(x_1, x_2, \dots).$$

We consider the scalar product $\langle , \rangle_{q,t}$ on Λ defined by

$$\langle p_\mu[X], p_\nu[X]\rangle_{q,t} = \delta_{\mu,\nu} z_\mu(q,t) := \delta_{\mu,\nu} z_\mu p_\mu\left[\frac{1-q}{1-t}\right].$$

The *integral form of Macdonald polynomials* $J_\lambda^{(q,t)}$ can be defined as the unique family of polynomials satisfying a triangularity property in the monomial basis, and orthogonal with respect to $\langle , \rangle_{q,t}$. More precisely $\left\langle J_\lambda^{(q,t)}, J_\rho^{(q,t)} \right\rangle_{q,t} = \delta_{\lambda,\rho} j_\lambda^{(q,t)}$ with

$$j_\lambda^{(q,t)} := \prod_{\square\in\lambda}\left(1 - q^{a_\lambda(\square)+1}t^{\ell_\lambda(\square)}\right)\left(1 - q^{a_\lambda(\square)}t^{\ell_\lambda(\square)+1}\right).$$

For every $r \in \mathbb{N}$, the set $\{J_\lambda^{(q,t)} \mid \lambda \vdash r\}$ is a basis of $\Lambda^{(r)}$.

Finally, let $\mathcal{P}_Z$ be the operator such that $\mathcal{P}_Z \cdot f[X] = \mathrm{Exp}[ZX]f[X]$, i.e. the multiplication by the *plethystic exponential* $\mathrm{Exp}[ZX] := \sum_{n\geq 0} h_n[ZX]$, and let $\mathcal{T}_Z := \sum_{\mu\in\mathbb{Y}} z_\mu^{-1} p_\mu[Z] p_\mu^\perp$ be the translation operator, so that $\mathcal{T}_Z \cdot f[X] = f[X+Z]$. Note that $\mathcal{P}_{Z+Z'} = \mathcal{P}_Z \cdot \mathcal{P}_{Z'}$.

1.3 A new formula for Macdonald polynomials

Consider the operators[1] $\boldsymbol{\nabla}$ and $\boldsymbol{\Delta}_v$ on symmetric functions defined by

$$\boldsymbol{\nabla} \cdot J_\lambda^{(q,t)} = (-1)^{|\lambda|}\left(\prod_{\square\in\lambda} q^{a'(\square)}t^{-\ell'(\square)}\right) J_\lambda^{(q,t)}, \quad \boldsymbol{\Delta}_v \cdot J_\lambda^{(q,t)} = \prod_{\square\in\lambda}\left(1 - v\cdot q^{a'(\square)}t^{-\ell'(\square)}\right) J_\lambda^{(q,t)}.$$

These are integral versions of some known operators on modified Macdonald polynomials (see Section 2.1).

We finally introduce the following operator[2] on $\Lambda[v][\![u]\!]$

$$\boldsymbol{\Gamma}(u,v) := \boldsymbol{\Delta}_{1/v}\mathcal{P}_{\frac{uv(1-t)}{1-q}}\boldsymbol{\Delta}_{1/v}^{-1} \quad .$$

The fact that $\boldsymbol{\Gamma}(u,v)$ is a polynomial in v is a consequence of the Pieri rule.

We can state our new formula for Macdonald polynomials.

Theorem 1.1. *For any partition $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_k]$, we have*

$$\boldsymbol{\Gamma}_{\lambda_1}^{(+)}\boldsymbol{\Gamma}_{\lambda_2}^{(+)}\cdots\boldsymbol{\Gamma}_{\lambda_k}^{(+)} \cdot 1 = J_\lambda^{(q,t)} \quad \textit{where} \quad \boldsymbol{\Gamma}_m^{(+)} := [u^m]\boldsymbol{\nabla}^{-1}\boldsymbol{\Gamma}(u, q^m)\boldsymbol{\nabla}. \tag{1.1}$$

[1] We use boldface symbols to distinguish these operators from their relatives from Section 2.1.

[2] This operator is a close relative of the Theta operator in [7], first introduced in [5].

It turns out that Theorem 1.1 is an easy consequence of the following *creation formula*.

Theorem 1.2. *For any partition $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_k]$, we have*

$$\mathbf{\Gamma}(u, q^{\lambda_1})\mathbf{\Gamma}(t^{-1}u, q^{\lambda_2}) \cdots \mathbf{\Gamma}(t^{-(k-1)}u, q^{\lambda_k}) \cdot 1 = t^{-n(\lambda)} \mathbf{\nabla} \mathcal{T}_{\frac{1}{u(1-t)}} J_\lambda^{(q,t)}[uX]. \tag{1.2}$$

In Section 2.3, we prove an analogous result for modified Macdonald polynomials Theorem 2.1 from which we deduce Theorem 1.2.

In addition to giving a direct construction of Macdonald polynomials, Theorem 1.2 provides a dual approach to study the structure of these polynomials. Indeed, Equation (1.2) allows to think of $J_\lambda^{(q,t)}$ as a function in the partition λ described by the alphabet $(q^{\lambda_1}, q^{\lambda_2}, \dots)$. This dual approach plays a key role in this paper and is used in Section 3 to introduce a q, t-deformation of Jack characters.

1.4 Macdonald characters

Jack characters, introduced by Lassalle in [16], can be seen as a one parameter deformation of the symmetric group characters, and are directly related to the coefficients of Jack polynomials in the power-sum basis. Jack characters have been useful to understand asymptotic behavior of large Young diagram sampled with respect to a Jack deformed Plancherel measure [4, 6]. Moreover, Goulden and Jackson's Mathings-Jack conjecture has a natural interpretation in terms of structure coefficients of Jack characters. Recently, a combinatorial interpretation of Jack characters in terms of maps on non orientable surfaces has been proved in [2], answering a positivity conjecture of Lassalle.

Using the operator $\mathbf{\Gamma}$, we introduce a two parameter deformation $\boldsymbol{\theta}_\mu^{(q,t)}$ of Jack characters. These characters have a structure of shifted symmetric functions and are related to shifted Macdonald polynomials, see [15, 18]. Macdonald characters $\boldsymbol{\theta}_\mu^{(q,t)}$ can be thought of as a natural generalization of the coefficients of Macdonald polynomials in the power-sum basis (see Equation (3.5)) which are hard to guess without the new operator Γ.

In Section 4, we make several positivity conjectures related to the new characters $\boldsymbol{\theta}_\mu^{(q,t)}$. These conjectures suggest that the characters $\boldsymbol{\theta}_\mu^{(q,t)}$ have a combinatorial structure which generalizes the one given by maps and that we hope to investigate in future works.

2 A new creation formula for Macdonald polynomials

2.1 Modified Macdonald polynomials

In [8], Garsia and Haiman introduced a *modified* version of *Macdonald polynomials*

$$\widetilde{H}_\lambda^{(q,t)} = t^{n(\lambda)} J_\lambda^{(q,1/t)} \left[\frac{X}{1 - 1/t} \right].$$

The operators ∇ and Δ_v are defined by

$$\nabla \widetilde{H}_\lambda^{(q,t)} := (-1)^{|\lambda|} \prod_{\square\in\lambda} q^{a'_\lambda(\square)} t^{\ell'_\lambda(\square)} \widetilde{H}_\lambda^{(q,t)}, \qquad \Delta_v \widetilde{H}_\lambda^{(q,t)} := \prod_{\square\in\lambda} \left(1 - v q^{a'_\lambda(\square)} t^{\ell'_\lambda(\square)}\right) \widetilde{H}_\lambda^{(q,t)}.$$

These operators are related by the *five-term relation* of Garsia and Mellit [10]

$$\nabla \mathcal{P}_{\frac{u}{M}} \nabla^{-1} \mathcal{P}_{\frac{uv}{M}} = \Delta_{1/v} \mathcal{P}_{\frac{uv}{M}} \Delta_{1/v}^{-1}\,, \tag{2.1}$$

where $M := (1-q)(1-t)$. Let $B_\lambda := \sum_{\square\in\lambda} q^{a'_\lambda(\square)} t^{\ell'_\lambda(\square)} = \sum_{1\le i\le \ell(\lambda)} t^{i-1} \frac{1-q^{\lambda_i}}{1-q}$, and $D_\lambda := MB_\lambda - 1$. We state another fundamental identity for Macdonald polynomials, due to Garsia, Haiman and Tesler [9]: for any partition λ

$$\nabla \mathcal{P}_{-\frac{u}{M}} \mathcal{T}_{\frac{1}{u}} \widetilde{H}_\lambda[uX] = \operatorname{Exp}\left[-\frac{uXD_\lambda}{M}\right]. \tag{2.2}$$

2.2 Creation formula for modified Macdonald polynomials

We start by proving a modified version of Theorem 1.2. Set $\Gamma(u,v) := \Delta_{1/v} \mathcal{P}_{\frac{uv}{1-q}} \Delta_{1/v}^{-1}$.

Theorem 2.1. *For $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k]$ partition we have*

$$\Gamma(u, q^{\lambda_1}) \Gamma(tu, q^{\lambda_2}) \cdots \Gamma(t^{\ell-1}u, q^{\lambda_\ell}) \cdot 1 = \nabla \mathcal{T}_{\frac{1}{u}} \widetilde{H}_\lambda[uX] = \nabla \widetilde{H}_\lambda[uX+1]. \tag{2.3}$$

Lemma 2.2. *We have*

$$\Gamma(u,v) = \nabla \mathcal{P}_{\frac{u}{M}} \nabla^{-1} \mathcal{P}_{\frac{uv}{1-q}} \nabla \mathcal{P}_{\frac{-tu}{M}} \nabla^{-1}.$$

Proof. The operator Γ can be rewritten as follows

$$\Gamma(u,v) = \left(\Delta_{1/v} \mathcal{P}_{\frac{uv}{M}} \Delta_{1/v}^{-1}\right) \left(\Delta_{1/v} \mathcal{P}_{\frac{-tuv}{M}} \Delta_{1/v}^{-1}\right) = \left(\Delta_{1/v} \mathcal{P}_{\frac{uv}{M}} \Delta_{1/v}^{-1}\right) \left(\Delta_{1/v} \mathcal{P}_{\frac{tuv}{M}} \Delta_{1/v}^{-1}\right)^{-1}.$$

Using the five-term relation (2.1) on each one of the two factors, we obtain

$$\Gamma(u,v) = \nabla \mathcal{P}_{\frac{u}{M}} \nabla^{-1} \mathcal{P}_{\frac{uv}{M}} \mathcal{P}_{\frac{-utv}{M}} \nabla \mathcal{P}_{\frac{-tu}{M}} \nabla^{-1} = \nabla \mathcal{P}_{\frac{u}{M}} \nabla^{-1} \mathcal{P}_{\frac{uv}{1-q}} \nabla \mathcal{P}_{\frac{-tu}{M}} \nabla^{-1}. \quad \square$$

We now prove Theorem 2.1.

Proof of Theorem 2.1. It follows, using Lemma 2.2, that

$$\Gamma(u,v_1)\Gamma(tu,v_2)\cdots\Gamma(t^{k-1}u,v_k)\cdot 1 = \nabla \mathcal{P}_{\frac{u}{M}} \nabla^{-1} \mathcal{P}_{\frac{uv_1}{1-q}} \mathcal{P}_{\frac{utv_2}{1-q}} \cdots \mathcal{P}_{\frac{ut^{k-1}v_k}{1-q}} \nabla \mathcal{P}_{\frac{-ut^k}{M}} \nabla^{-1} \cdot 1\ .$$

Using $\nabla^{-1} \cdot 1 = 1$ and $\nabla \mathcal{P}_{-\frac{z}{M}} \cdot 1 = \mathcal{P}_{\frac{z}{M}} \cdot 1$ (see e.g. [7, Eq. (1.47)] with $k = n$), we get

$$\begin{aligned}\Gamma(u,v_1)\Gamma(tu,v_2)\cdots\Gamma(t^{k-1}u,v_k)\cdot 1 &= \nabla \mathcal{P}_{\frac{u}{M}} \nabla^{-1} \mathcal{P}_{\frac{uv_1}{1-q}} \mathcal{P}_{\frac{utv_2}{1-q}} \cdots \mathcal{P}_{\frac{ut^{k-1}v_k}{1-q}} \mathcal{P}_{\frac{ut^k}{M}} \cdot 1 \\ &= \nabla \mathcal{P}_{\frac{u}{M}} \nabla^{-1} \operatorname{Exp}\left[\frac{ut^k X}{M} + \frac{uX}{1-q} \sum_{1\le i\le k} t^{i-1} v_i\right] \\ &= \nabla \mathcal{P}_{\frac{u}{M}} \nabla^{-1} \operatorname{Exp}\left[\frac{uX}{M} - \frac{uX}{M}(1-t) \sum_{1\le i\le k} t^{i-1}(1-v_i)\right].\end{aligned}$$

Fix now a partition λ. Applying the previous equation, we get

$$\begin{aligned}\Gamma(u,q^{\lambda_1})\Gamma(tu,q^{\lambda_2})\cdots\Gamma(t^{\ell-1}u,q^{\lambda_\ell})\cdot 1 &= \nabla \mathcal{P}_{\frac{u}{M}} \nabla^{-1} \operatorname{Exp}\left[\frac{uX}{M} - \frac{uX}{M}(1-t) \sum_{i\ge 1} t^{i-1}(1-q^{\lambda_i})\right] \\ &= \nabla \mathcal{P}_{\frac{u}{M}} \nabla^{-1} \operatorname{Exp}\left[-\frac{uXD_\lambda}{M}\right],\end{aligned}$$

Now applying Equation (2.2) concludes the proof of the theorem. □

2.3 Proof of Theorem 1.2

In this subsection we deduce Theorem 1.2 from Theorem 2.1.

Consider the transformation ϕ on Λ defined by

$$f = \sum_\mu d^f_\mu(q,t) p_\mu[X] \longmapsto \phi(f) := \sum_\mu d^f_\mu(q,1/t) p_\mu \left[\frac{X}{1-1/t}\right],$$

where d^f_μ are the coefficients of f in the power-sum basis. Notice that ϕ is invertible and

$$\phi^{-1}(f) = \sum_\mu d^f_\mu(q,1/t) p_\mu \left[X(1-t)\right] \text{ for any } f.$$

With this definition, one has $\widetilde{H}^{(q,t)}_\lambda = t^{n(\lambda)} \phi(J^{(q,t)}_\lambda)$. Moreover, $\boldsymbol{\nabla} = \phi^{-1} \cdot \nabla \cdot \phi$ and $\boldsymbol{\Delta}_v = \phi^{-1} \cdot \Delta_v \cdot \phi$. Finally $\phi^{-1} \cdot \mathcal{P}_{\frac{ut^i}{1-q}} \cdot \phi = \mathcal{P}_{\frac{t^{-i}(1-t)u}{1-q}}$, for any $i \ge 0$. We deduce that $\phi^{-1} \cdot \Gamma(t^i u, v) \cdot \phi = \boldsymbol{\Gamma}(t^{-i}u, v)$, for any $i \ge 0$.

On the other hand, one can check that $\phi^{-1} \cdot \mathcal{T}_{\frac{1}{u}} \cdot \phi = \mathcal{T}_{\frac{1}{u(1-t)}}$. Hence, applying ϕ^{-1} on Equation (2.3), we obtain Equation (1.2).

3 A two-parameter generalization of Jack characters

The results of this section will be proved in the long version of the paper.

3.1 Shifted symmetric functions

Definition 3.1. We say that a *polynomial* in k variables $f(u_1,\dots,u_k)$ is *shifted symmetric* if it is symmetric in the variables $u_1, u_2t^{-1},\dots,u_kt^{1-k}$. A *shifted symmetric function* $f(u_1,u_2,\dots)$ is a sequence of polynomials $(f_k)_{k\geq 1}$ of bounded degrees, such that f_k is a shifted symmetric polynomial in k variables for each k, and $f_{k+1}(v_1,\dots,v_k,1) = f_k(v_1,\dots,v_k)$.

If f is a shifted symmetric function, we consider its evaluation on a Young diagram $\lambda = [\lambda_1,\dots,\lambda_k]$ defined by $f(\lambda) := f(q^{\lambda_1},q^{\lambda_2},\dots,q^{\lambda_k},1,1,\dots)$. It is well known that the *space of shifted symmetric functions* Λ^* can be identified with a subspace of the space of functions on Young diagrams through the map $f \longmapsto (f(\lambda))_{\lambda\in\mathbb{Y}}$.

Theorem 3.2 (Shifted Macdonald polynomials)**.** *[18] Let μ be a partition. There exists a unique function $J^*_\mu(v_1,v_2,\dots)$ such that*

1. *J^*_μ is shifted symmetric of degree $|\mu|$.*
2. *$J^*_\mu(\mu) = (-1)^{|\mu|}q^{n'(\mu)}t^{-2n(\mu)}j^{(q,t)}_\mu$ (normalization property).*
3. *for any partition $\lambda \not\supset \mu$ one has $J^*_\mu(\lambda) = 0$ (vanishing property).*

Moreover, the top homogeneous part of J^*_μ is $J^{(q,t)}_\mu(v_1,t^{-1}v_2,t^{-2}v_3,\dots)$.

Since Macdonald polynomials form a basis of Λ, using a triangularity argument it can be deduced that shifted Macdonald polynomials form a basis of Λ^*. As a consequence we can extend the map $J_\mu \longmapsto J^*_\mu$ to a linear isomorphism

$$\begin{array}{ccc} \Lambda & \longrightarrow & \Lambda^* \\ f & \longmapsto & f^*. \end{array} \tag{3.1}$$

3.2 An explicit isomorphism between the spaces of symmetric and shifted-symmetric functions

The main purpose of this subsection is to give two explicit formulas for the isomorphism (3.1). The first one is Equation (3.2), which gives the image of a function f^* as a shifted symmetric function. The second formula is Equation (3.3), which gives the image as a function on Young diagrams. The proof, that we omit, is based on Equation (1.2) and the Pieri rule.

Theorem 3.3. *For any symmetric function f, the following holds*

$$f^*(v_1,\dots,v_k) = \left\langle f, \mathbf{\Gamma}(1,v_1)\mathbf{\Gamma}(t^{-1},v_2)\cdots\mathbf{\Gamma}(t^{-(k-1)},v_k)\cdot 1\right\rangle_{q,t}. \tag{3.2}$$

Equivalently, for any Young diagram λ,

$$f^*(\lambda) = \left\langle \mathcal{P}_{\frac{1}{1-q}} \boldsymbol{\nabla} \cdot f, t^{-n(\lambda)} J_\lambda \right\rangle_{q,t}. \tag{3.3}$$

Remark 1. The isomorphism given in Equation (3.3) has been implicitly described by Lassalle, see [15, Definition 1]. However, the formula of Equation (3.2) seems to be new.

3.3 Macdonald characters

Definition 3.4. The *Macdonald character* $\boldsymbol{\theta}_\mu^{(q,t)}$ is the function defined by

$$\boldsymbol{\theta}_\mu^{(q,t)}(v_1, v_2, \dots) := (p_\mu)^*(v_1, v_2, \dots) = \left\langle p_\mu, \boldsymbol{\Gamma}(1, v_1)\boldsymbol{\Gamma}(t^{-1}, v_2) \cdot \dots \cdot 1 \right\rangle_{q,t}.$$

Moreover, for any Young diagram λ

$$\boldsymbol{\theta}_\mu^{(q,t)}(\lambda) = \begin{cases} \left\langle p_\mu, \boldsymbol{\nabla} h^\perp_{|\lambda|-|\mu|}\left[\frac{X}{1-t}\right] \cdot t^{-n(\lambda)} J_\lambda^{(q,t)} \right\rangle_{q,t} & \text{if } |\mu| \le |\lambda| \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

In particular, when $|\mu| = |\lambda|$ the characters $\boldsymbol{\theta}_\mu^{(q,t)}(\lambda)$ are given by the power-sum expansion of $J_\lambda^{(q,t)}$:

$$(-1)^{|\lambda|} q^{n(\lambda')} t^{-2n(\lambda)} J_\lambda^{(q,t)} = \sum_{\mu \vdash |\lambda|} \frac{\boldsymbol{\theta}_\mu^{(q,t)}(\lambda)}{z_\mu(q,t)} p_\mu. \tag{3.5}$$

We give here a characterization of $\boldsymbol{\theta}_\mu^{(q,t)}$, which has been observed by Féray in the case of Jack polynomials, and proved very useful in practice in this case (see [2, Theorem 2.5]).

Theorem 3.5. *Let μ be a partition. Then $\boldsymbol{\theta}_\mu^{(q,t)}$ is the unique shifted symmetric function degree $|\mu|$ whose top homogeneous part is $p_\mu(v_1, t^{-1}v_2, t^{-2}v_3, \dots)$ and such that $\boldsymbol{\theta}_\mu^{(q,t)}(\lambda) = 0$ for any partition $|\lambda| < |\mu|$.*

4 Macdonald generalization of some Jack conjectures

4.1 A normalization related to Jack polynomials

Jack polynomials can be obtained from the integral form of Macdonald polynomials as follows (see [17, Chapter VI, eq (10.23)])

$$\lim_{t \to 1} \frac{J_\lambda^{(q=1+\alpha(t-1),t)}}{(1-t)^{|\lambda|}} = J_\lambda^{(\alpha)}. \tag{4.1}$$

In the following, we introduce the parameters (α, γ) related to (q,t) by

$$\begin{cases} q = 1+\gamma\alpha \\ t = 1+\gamma \end{cases} \longleftrightarrow \begin{cases} \alpha = \frac{1-q}{1-t} \\ \gamma = t-1 \end{cases}.$$

We consider the following normalization of Macdonald polynomials

$$\mathfrak{J}_\lambda^{(\alpha,\gamma)} := \frac{J_\lambda^{(q,t)}}{(1-t)^{|\lambda|}}.$$

Hence, this normalization is directly related to Jack polynomials by $\mathfrak{J}_\lambda^{(\alpha,\gamma=0)} = J_\lambda^{(\alpha)}$.

Remark 2. Unlike the integral form $J_\lambda^{(q,t)}$, the coefficients of $\mathfrak{J}_\lambda^{(\alpha,\gamma)}$ in the monomial basis are positive in γ and u. This can be seen using the combinatorial interpretation of Macdonald polynomials given in [12, Proposition 8.1].

We observed that the parameterization (α, γ) allows to generalize several conjectures about Jack polynomials to Macdonald polynomials which we now formulate.

4.2 Weak Lassalle's conjecture

We consider the following normalization of Macdonald characters defined in Section 3.3

$$\theta_\mu^{(\alpha,\gamma)}(s_1, s_2, \dots) := \frac{1}{z_\mu(q,t)\gamma^{|\mu|}} \boldsymbol{\theta}_\mu^{(q,t)}(1+\alpha\gamma s_1, 1+\alpha\gamma s_2, \dots).$$

For any partition λ, we denote

$$\theta_\mu^{(\alpha,\gamma)}(\lambda) := \theta_\mu^{(\alpha,\gamma)}\left(\frac{q^{\lambda_1}-1}{\alpha\gamma}, \frac{q^{\lambda_2}-1}{\alpha\gamma}, \dots\right) = \frac{\boldsymbol{\theta}_\mu^{(q,t)}(\lambda)}{z_\mu(q,t)\gamma^{|\mu|}}. \tag{4.2}$$

Jack characters, introduced by Lassalle [16], are obtained from the characters $\theta_\mu^{(\alpha,\gamma)}$ by specializing $\gamma = 0$. We formulate the following conjecture[3], tested for $k \le 3$ and $|\mu| \le 7$.

Conjecture 1. *Fix $k \ge 1$ and a partition μ. Then, $(-1)^{|\mu|} t^{(k-1)|\mu|} z_\mu(q,t) \theta_\mu^{(\alpha,\gamma)}(s_1, s_2, \dots, s_k)$ is a polynomial in $\gamma, b := \alpha - 1, -\alpha s_1, -\alpha s_2, \dots, -\alpha s_k$ with non-negative integer coefficients.*

Computer tests also suggest that the action of the operator $\boldsymbol{\Gamma}$ on the power-sum basis satisfies a positivity property that would imply the positivity part in Conjecture 1.

[3] This is a generalization of a weak version of Lassalle's conjecture on Jack characters, in which we keep one alphabet $(s_1, s_2, \dots)$ instead of two alphabets associated to the multirectangular coordinates of λ.

4.3 Structure coefficients of $\theta_\mu^{(\alpha,\gamma)}$

It follows from the definition of the characters $\boldsymbol{\theta}_\mu^{(q,t)}$ that they form a basis of the space of shifted symmetric functions Λ^*. Moreover, $\theta_\mu^{(\alpha,\gamma)}(\lambda)$ is obtained from $\boldsymbol{\theta}_\mu^{(q,t)}$ by a normalization by a scalar and a change of variables (see Equation (4.2)) and by consequence their structure coefficients are well defined:

$$\theta_\mu^{(\alpha,\gamma)}\theta_\nu^{(\alpha,\gamma)} = \sum_\pi g_{\mu,\nu}^\pi(\alpha,\gamma)\theta_\pi^{(\alpha,\gamma)}. \tag{4.3}$$

The coefficients $g_{\mu,\nu}^\pi(\alpha,\gamma)$ are a two parameter generalization of structure coefficients of Jack characters $\theta_\mu^{(\alpha)}$ introduced by Dołęga and Féray in [6] (see also [19]).

Let f be the function defined on triplets of non-negative integers by

$$f(n_1,n_2,k) := (N-n)(N+n-k)+n(n-1)-(k-N)(k-N-1),$$

where $N := \max(n_1,n_2)$ and $n = \min(n_1,n_2)$. We make the following conjecture which extends [19, Conjecture 2.2].

Conjecture 2. *Let π,μ,ν be three partitions. Then, the coefficients $(1+\gamma)^{f(|\mu|,|\nu|,|\pi|)}z_\mu z_\nu g_{\mu,\nu}^\pi$ are polynomials in $b := \alpha-1$ and γ with non-negative integer coefficients.*

4.4 Generalized Goulden and Jackson's conjectures

We define the coefficients $c_{\mu,\nu}^\pi$ and $h_{\mu,\nu}^\pi$ for partitions π,μ and ν of the same size by

$$\sum_{\lambda\in\mathbb{Y}} u^{|\lambda|}t^{-2n(\lambda)}q^{n'(\lambda)}\frac{\mathfrak{J}_\lambda^{(\alpha,\gamma)}[X]\mathfrak{J}_\lambda^{(\alpha,\gamma)}[Y]\mathfrak{J}_\lambda^{(\alpha,\gamma)}[Z]}{j_\lambda^{(q,t)}\gamma^{-2|\lambda|}} = \sum_{n\geq 0}\sum_{\pi,\mu,\nu\vdash n}\frac{u^n c_{\mu,\nu}^\pi(\alpha,\gamma)}{z_\pi(q,t)}p_\pi[X]p_\mu[Y]p_\nu[Z],$$

$$\log\left(\sum_{\lambda\in\mathbb{Y}} u^{|\lambda|}t^{-2n(\lambda)}q^{n'(\lambda)}\frac{\mathfrak{J}_\lambda^{(\alpha,\gamma)}[X]\mathfrak{J}_\lambda^{(\alpha,\gamma)}[Y]\mathfrak{J}_\lambda^{(\alpha,\gamma)}[Z]}{j_\lambda^{(q,t)}\gamma^{-2|\lambda|}}\right) = \sum_{n\geq 0}\sum_{\pi,\mu,\nu\vdash n}\frac{u^n h_{\mu,\nu}^\pi(\alpha,u)}{\alpha[n]_q}p_\pi[X]p_\mu[Y]p_\nu[Z],$$

where $[n]_q := 1+q+\cdots+q^{n-1}$. By taking $\gamma = 0$, the coefficients $c_{\mu,\nu}^\pi(\alpha,\gamma)$ and $h_{\mu,\nu}^\pi(\alpha,\gamma)$ give the coefficients of the celebrated Matchings-Jack and b-conjectures formulated by Goulden and Jackson in [11]. The coefficients $c_{\mu,\nu}^\pi$ are actually a special case of $g_{\mu,\nu}^\pi$.

Proposition 4.1. *Let π,μ and ν be three partitions of the same size. Then*

$$c_{\mu,\nu}^\pi(\alpha,\gamma) = g_{\mu,\nu}^\pi(\alpha,\gamma).$$

We consider here a Macdonald analog of Goulden and Jackson conjectures.

Conjecture 3 (Macdonald generalization of the Matchings-Jack conjecture)**.** *For any positive integer n and partitions π, μ, ν of n, the quantity $(1+\gamma)^{n(n-1)} z_\mu z_\nu c^{\pi}_{\mu,\nu}(\alpha, \gamma)$ is a polynomial in b and γ with non-negative integer coefficients.*

Conjecture 4 (Macdonald generalization of the b-conjecture)**.** *For any positive integer n and partitions π, μ, ν of n, the quantity $(1+\gamma)^{n(n-1)} z_\pi z_\mu z_\nu h^{\pi}_{\mu,\nu}(\alpha, \gamma)$ is a polynomial in b and γ with non-negative integer coefficients.*

Conjecture 3 has been tested for $n \leq 8$ and Conjecture 4 for $n \leq 9$. Notice that by Proposition 4.1, Conjecture 3 is a special case of Conjecture 2.

Remark 3. Given the integrality result of [1], it is easy to see that Conjecture 3 implies the Matchings-Jack conjecture [11, Conjecture 4.2]. Similarly, Conjecture 4 implies the positivity in the b-conjecture [11, Conjecture 6.2].

4.5 A generalization of Stanley's conjecture

We conclude with a generalization of Stanley's conjecture about the structures coefficients of Jack polynomials. While not directly related to Macdonald characters, this conjecture is also obtained from the new parameterization in α and γ.

Conjecture 5 (Macdonald version of Stanley's conjecture)**.** *Given λ, μ, ν partitions, the quantity $\langle \mathfrak{J}^{(\alpha,\gamma)}_\lambda \mathfrak{J}^{(\alpha,\gamma)}_\mu, \mathfrak{J}^{(\alpha,\gamma)}_\nu \rangle_{q,t}$ is a polynomial in the parameters α and γ with integer non-negative coefficients.*

This conjecture has been tested for $|\nu| \leq 9$. Stanley's conjecture [20, Conjecture 8.5] corresponds to the case $\gamma = 0$ of Conjecture 5.

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