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On the scaling of random Tamari intervals and Schnyder woods of random triangulations (with an asymptotic D-finite trick)

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Abstract. We consider a Tamari interval of size n (i.e., a pair of Tamari-comparable Dyck paths) chosen uniformly at random. We show that the typical height of points on the paths scales as $n^{3/4}$, with an explicit limit law. By the Bernardi-Bonichon bijection, this also applies to Schnyder trees of random plane triangulations. The exact solution of the model is based on polynomial equations with one and two catalytic variables. To deduce convergence in law, we use a simple analytic method based on D-finiteness, which is essentially automatic.

1 Introduction and main results

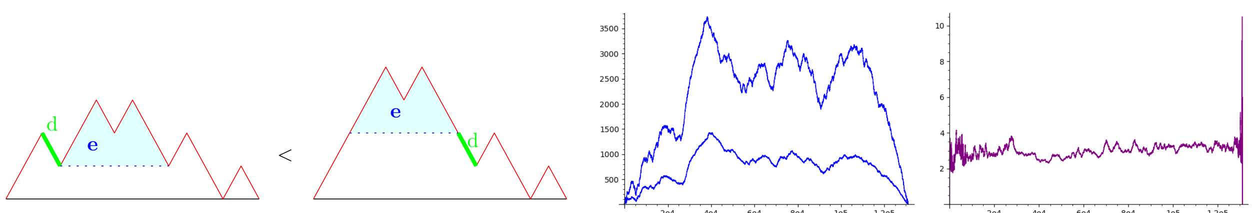

Figure 1: Left: The covering relation defining the Tamari lattice. Right: a uniform random Tamari interval (P_n, Q_n) of size $n = 65536$ (blue) generated with a Python code generously provided by Wenjie Fang, and the plot of Q_n/P_n (purple).

For $n \geq 0$, a *Dyck path of size* n is a lattice path made of n up-steps and n-down steps, starting (and ending) at height 0, and whose height stays always nonnegative. The set $\mathcal{D}_n$ of Dyck paths of size n is endowed with the *Tamari partial order*, whose covering relation is described as follows. Let $\mathfrak{p}$ be a Dyck path, and let d be a down-step in $\mathfrak{p}$, followed by an up-step. Let $\mathfrak{e}$ be the shortest excursion following d in $\mathfrak{p}$ (an excursion is a path staying higher than its starting point except for its last point). Then the Dyck path

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$\mathfrak{q}$ obtained from $\mathfrak{p}$ by exchanging d and $\mathfrak{e}$ is declared *larger* than $\mathfrak{p}$ for the Tamari order. The reflexive transitive closure of this relation defines the Tamari lattice. See Figure 1.

The Tamari lattice plays an important role in many facets of algebraic combinatorics and discrete mathematics, in relation with polytopes (the associahedron), flip graphs, hyperbolic geometry, or mixing times of Markov chains.

This paper is motivated by another famous connection, between Tamari intervals and planar maps. An *interval* of size n in the Tamari lattice is a pair $(\mathfrak{p}, \mathfrak{q}) \in (\mathcal{D}_n)^2$ such that $\mathfrak{p} \leq \mathfrak{q}$ (for the Tamari partial order). We let $\mathcal{I}_n$ be the set of Tamari intervals of size n. In a famous paper, Chapoton proved that the number of elements of $\mathcal{I}_n$ was given by $\frac{2}{n(n+1)}\binom{4n+1}{n-1}$, which is also the number of rooted planar triangulations of size n [13]. An elegant, and deep, direct bijective proof has been given by Bernardi and Bonichon in [1]. Since Chapoton's discovery, the analogy between Tamari intervals and planar maps has been much developped, from the existence of refined product formulas [2] strongly resembling the ones appearing in enumerative geometry, to numerous coincidences between the enumeration of special families of intervals and planar maps (e.g., [8]). These phenomena are still very partially understood and are the subject of active research.

Although large random planar maps have been intensely studied in the last decades (see e.g. [6, 10]), it seems that the behaviour of random Tamari intervals has not been studied, and we are not sure to know why. It is however natural to ask this question:

What does a large uniformly random Tamari interval look like?

In this paper we give a first answer to this question. If $P \in \mathcal{D}_n$ and $i \in [0..2n]$, we write $P(i)$ for the height of the point of P lying at abscissa i. We show:

Theorem 1.1 (Main result)**.** *Let (P, Q) be a Tamari interval of size n, chosen uniformly at random in $\mathcal{I}_n$. Let $I \in [0..2n]$ be an integer chosen uniformly at random. Then we have the convergence in law*

$$\frac{Q_n(I)}{n^{3/4}} \longrightarrow Z = (XY)^{1/4} \tag{1.1}$$

when n goes to infinity, where $X \sim \beta(\frac{1}{3}, \frac{1}{6})$ and $Y \sim \Gamma(\frac{2}{3}, \frac{1}{2})$ are independent random variables. In fact, we have the convergence of all moments: for integer $k \geq 0$,

$$\mathbf{E}\left[\left(\frac{Q_n(I)}{n^{3/4}}\right)^k\right] \longrightarrow \frac{\sqrt{3} \cdot 2^{-\frac{k}{4}-1}}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{4}k + \frac{1}{3})\Gamma(\frac{1}{4}k + \frac{2}{3})}{\Gamma(\frac{1}{4}k + \frac{1}{2})}. \tag{1.2}$$

For the lower path we have similarly, with again the convergence of all moments,

$$\frac{P_n(I)}{n^{3/4}} \longrightarrow \frac{Z}{3}. \tag{1.3}$$

We recall that the random variables $\beta(a, b)$ and $\Gamma(\ell, \theta)$ have respective densities $\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}$ on $(0, 1)$ and $\frac{x^{k-1}e^{-x/\theta}}{\Gamma(k)\theta^k}$ on $\mathbb{R}_+$. Their respective k-th moments are

$\frac{\Gamma(a+b)\Gamma(a+k)}{\Gamma(a)\Gamma(a+b+k)}$, and $\frac{\theta^k\Gamma(l+k)}{\Gamma(\ell)}$, so it is direct to check that (1.2) with k substituted by $4k$ is indeed equal to the k-th moment of XY. Interestingly, the random variable Z^4 already appears in a (seemingly unrelated) physics context, see [11] or OEIS:A064352.

In view of Theorem 1.1 and simulations (Figure 1) it is natural to suspect that $P_n(I)$ is close to $\frac{1}{3}Q_n(I)$ with high probability. Unfortunately our methods based on functional equations make it hard to track the joint law of $(P_n(I), Q_n(I))$. However, they can handle the joint law of $(\tilde{P}_n(J), \tilde{Q}_n(J))$ where J is uniform on $[1, n]$ and where for a path P we write $\tilde{P}(j)$ for the height of the j-th up-step of P. In the long version of this paper we will prove with similar methods[1] that indeed, in probability,

$$\tilde{P}_n(J) = \frac{1}{3}\tilde{Q}_n(J) + o(\tilde{Q}_n(J)). \tag{1.4}$$

To prevent confusion, we mention that the *individual* convergence of $\tilde{P}_n(J)/n^{3/4}$ and $\tilde{Q}_n(J)/n^{3/4}$ to Z and $Z/3$ follow from Theorem 1.1, using a coupling between I and J.

As we said, Bernardi and Bonichon [1] provided an explicit bijection between intervals in $\mathcal{I}_n$ and rooted plane triangulations of size n. Such a triangulation can always be equipped with a *canonical Schnyder wood*, which is a partition of its internal edges into three trees (say red, blue, green) with certain conditions. See Figure 2. It is then not too hard to deduce the following from (1.3):

Corollary 1.2. *Let T_n be a rooted plane triangulation of size n, chosen uniformly at random, and let $(T_n^{(1)}, T_n^{(2)}, T_n^{(3)})$ be its canonical Schnyder wood, that is to say, the one associated to its minimal orientation in the sense of [1]. Let V be a uniform random internal vertex of T_n and let $H_n^{(i)}$ be the height of the vertex V in the tree $T_n^{(i)}$. Then, for any $i \in [3]$ we have*

$$\frac{H_n^{(i)}}{n^{3/4}} \longrightarrow \frac{1}{3}Z.$$

The proof of each half of Theorem 1.1 (namely (1.1) and (1.3)) has two parts: the first one consists in solving "exactly" the model, by obtaining an algebraic equation for the generating function $f(t, s)$ of intervals having a marked abscissa, with control on the size n, and on the lower or upper height. In each case this requires to solve an equation with catalytic variables. The second part is to deduce the asymptotic of moments from there, which is a problem of analytic combinatorics in 2 variables, for which we need to find good tools. We describe below a simple method that will do the trick.

[1] One can write a functional equation for the generating function of intervals (P, Q) of size n with a marked $j \in [n]$ with control on the parameter $\tilde{Q}(j) - 3\tilde{P}(j)$, very similarly to Equation (3.1). The resolution of this equation is very similar to what is done in Section 3.2, with the difference that the equation with catalytic variable to solve in the second step is now quadratic – and can be solved with the quadratic method. At the time of writing, we have not performed the full asymptotics of moments, but from the solution it is immediate to show that $\mathbf{E}[(\tilde{Q}_n(J) - 3\tilde{P}_n(J))^2] = O(n)$, which is enough to conclude (1.4) by the Chebyshev inequality. For the impatient reader, we have already included all details in [5].

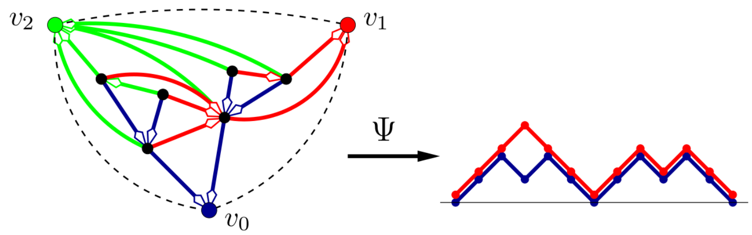

Figure 2: A rooted planar triangulation equipped with its minimal Schnyder-wood, and its image (a Tamari interval) by the Bernardi-Bonichon bijection. The lower path is nothing but the contour function of the blue tree. Figure taken from [1].

1.1 A method to prove moment convergence from algebraic equations

Assume that we have a combinatorial class $\mathcal{A}$ equipped with a size function $|\cdot|$ and an integer statistic μ, and consider the random variable $X_n = \mu(A_n)$ where A_n is an object of size n in $\mathcal{A}$ chosen uniformly at random. Consider the generating function

$$f(t,s) = \sum_{n\geq 0}\sum_{p\in\mathbb{Z}} a_{n,p}t^n s^p = \sum_{n\geq 0}\sum_{p\in\mathbb{Z}} a_n \mathbf{P}[X_n = p]t^n s^p, \tag{1.5}$$

with $a_{n,p} = |\{\alpha \in \mathcal{A}, |\alpha| = n, \mu(\alpha) = p\}|$, and $a_n = |\{\alpha \in \mathcal{A}, |\alpha| = n\}|$, and assume that the generating function $f(t,s)$ is algebraic. For $k \geq 0$ we consider the generating function of factorial moments

$$f^{(k)} \equiv f^{(k)}(t) := \left(\left(\frac{d}{ds}\right)^k f(t,s)\right)\Bigg|_{s=1} = \sum_{n\geq 0} a_n \mathbf{E}[(X_n)_{(k)}]t^n,$$

with $(X_n)_{(k)} := X_n(X_n-1)\dots(X_n-k+1)$. To study the convergence of the random variable X_n, a standard way called the method of moments consists in studying the asymptotics for fixed k of its k-th moment $\mathbf{E}[(X_n)^k]$ (or factorial moment $\mathbf{E}[(X_n)_{(k)}]$). In our setting, this is equivalent to studying the asymptotics of the coefficient $[t^n]f^{(k)}(t)$, for fixed k. Now, since all the functions $f^{(k)}(t)$ are algebraic, they are amenable to singularity analysis in the sense of [9]. Therefore, to study the asymptotics of their coefficients it is enough to determine the nature of the dominant singularity(ies) of $f^{(k)}$ for each $k \geq 0$.

Here comes the main trick: since the function $f(t,s)$ is algebraic, it is also D-finite, i.e. the coefficients of its expansion in any variable satisfy a linear equation with polynomial coefficients (see e.g. [7]). We will apply this to the coefficients of the expansion[2] in the variable r such that $s = r+1$. These coefficients are, up to a factorial factor, the functions

[2]The function $f(t, 1+r)$ is algebraic, therefore it is D-finite in the variable r. No notion of convergence is required to say this. Of course, one has to be careful about which branch of this function one considers when performing actual calculations.

$f^{(k)}$. It follows that one can compute the $f^{(k)}$ by induction, with a recurrence of the form

$$f^{(k)}(t) = \sum_{d=1}^{L} h_d(t,k) f^{(k-d)}(t) \quad , \quad k \geq L, \tag{1.6}$$

for some $L > 0$, where for each $d \in [L]$, $h_d(t,k)$ is a rational function in k whose coefficients are algebraic functions of t (we could assume that the h_k are rational in t, but for applications this weaker asumption is convenient: it will enable us to work under some algebraic change of variables). This leads us to:

Method 1.3 (D-finite trick for moment pumping)**.** Given a bivariate algebraic function f, obtain a linear equation for its derivatives $f^{(k)}$ of the form (1.6) using standard computer algebra tools. Then, use it to determine the asymptotic of $f^{(k)}$ near $t = \rho$ by induction on k. Deduce the asymptotics of $[t^n]f^{(k)}$ using the transfer theorem.

The idea of this method is quite general, but of course one has to be careful to carry the analytical details in the induction. We give below a simple framework of application whose proof is essentially immediate. For a function g and $\alpha, c \in \mathbb{R}$, $\rho > 0$ we write $g(t) \hat{\sim} c(1-t/\rho)^{\alpha}$ if $g(t)$, as an analytic function, has no singularity on the closed circle of radius ρ except maybe at $t = \rho$, and if $g(t)$ has a Puiseux expansion of the form $g(t) = P(t) + c(1-t/\rho)^{\alpha} + o((1-t/\rho)^{\alpha})$ near $t = \rho$, where P is a polynomial.

Theorem 1.4 (D-finite trick for moment pumping, an instance)**.** *Let $f(t,s)$ be a generating function of the form (1.5), and assume that $f(t,s)$ is an algebraic function. Then f is D-finite in the variable $s-1$, and it satisfies an equation of the form (1.6), with $L \geq 1$ and where for $d \in [L]$, $h_d(t,k)$ is a rational function of k whose coefficients are algebraic functions of t. Assume that:*
(i) *There is $\beta > 0$ such that for $d \in [L]$, $h_d(t,k)(1-t/\rho)^{\beta d} \to a_d(k)$ for $t \to \rho$, with possibly $a_d(k) = 0$. Moreover, $h_d(k)$ has no singularity other than ρ on the circle of radius ρ.*
(ii) *"Initial conditions": there is $\alpha \in \mathbb{R} \setminus \mathbb{N}$ and numbers c_ℓ, with $c_0 > 0$, such that $f^{(\ell)} \hat{\sim} c_\ell (1-t/\rho)^{\alpha-\beta\ell}$ for all $0 \leq \ell \leq \ell_0$, with $\ell_0 := L + \max(\lfloor \alpha/\beta \rfloor, -1)$. Moreover, $c_\ell = 0$ if $\alpha - \beta\ell \in \mathbb{N}$.*

Then one has $f^{(k)} \hat{\sim} c_k(1-t/\rho)^{\alpha-\beta k}$ for any $k \geq 0$, where c_k is given by the recurrence

$$c_k = \sum_{d=1}^{L} a_d(k) c_{k-d} \quad , \quad k > \ell_0, \tag{1.7}$$

where the values of c_k for $k \leq \ell_0$ are given by the initial conditions. Moreover, for each $k \geq 0$,

$$\frac{\mathbf{E}[(X_n)^k]}{n^{\beta k}} \to \frac{c_k \Gamma(-\alpha)}{c_0 \Gamma(\beta k - \alpha)}. \tag{1.8}$$

The proof is ommitted in this extended abstract. It consists in showing that $f^{(k)} \hat{\sim} c_k(1-t/\rho)^{\alpha-\beta k}$ for any $k \geq 0$, which in fact follows from a direction induction, and applying the standard theorems for algebraic functions [9]. The reason why the recurrence starts only at ℓ_0 is because we want $\alpha - \beta\ell$ to be negative for all ℓ such that c_ℓ appears in (1.7), so that the symbol $\hat{\sim}$ is in fact a true equivalent, in order for the induction to work.

Remark 1.5. We do not try to provide minimal hypotheses in this theorem: the only requirement for the method to work is that dominant singularities of $f^{(k)}$ are "not too hard" to track by induction from (1.6). Also, it is conceivable to use this technique in more than bivariate examples, provided the corresponding multivariate generating function is algebraic.

We insist that Theorem 1.4 if applicable, is essentially automatic. Indeed, computer algebra softwares (e.g. the Maple package GFUN [12]) are able to provide a recurrence of the form (1.6), and to check the initial conditions, automatically from an algebraic equation for f. Apart from this, the major interest of this method is that it allows a wide variety of limit laws (including non Gaussian) as our main applications show.

To conclude this discussion, let us recall that using D-finiteness to compute coefficients of algebraic functions in the univariate case is a well-known trick going back at least to Comtet [7]. We are only recycling this idea in the context of bivariate asymptotics.

Example. As a simple application, we invite our reader to rediscover the well known limit law for the height H_n of a uniform random point on a uniform random Dyck path of size n. Using standard path decompositions one easily obtains a quadratic equation for the corresponding generating function $f(t,s)$, and from there it is immediate, with Maple, to check that Theorem 1.4 applies (with $-\alpha = \beta = \frac{1}{2}$). The recurrence (1.7) becomes $c_k = \frac{1}{4}k(k-1)c_{k-2}$, and one thus directly shows that $H_n/\sqrt{n}$ converges to a Rayleigh law, of k-th moment $\Gamma(\frac{k}{2}+1)$. All calculations and checks are available in [5]!

Plan of the paper. In Section 2 we solve the exact counting problem which underlies the first half of Theorem 1.1 (Equation (1.1) by studying the classical equation with one catalytic variable for Tamari intervals, which we enrich by an extra variable marking the height. See Theorem 2.3. In Section 3 we solve the exact counting problem which underlies the second half (Equation (1.3)) see Theorem 3.2. Our proof is more technical, as we only manage to write an equation with *two* catalytic variables for this problem (enriched, once again, by an extra variable for the height). In Section 4, we sketch the proof of Theorem 1.1, which given these results directly follows from Theorem 1.4 and computer algebra calculations.

All the calculations supporting our results (including results whose proofs are not fully presented in this abstract) are available in the accompanying Maple worksheet [5].

2 Upper path: exact solution

2.1 The classical equation and its solution, after [4, 3]

Following [3], we call *contact* of a path a vertex of that path lying on the x-axis. We now present a recursive decomposition of Tamari intervals based on contacts of the lower path, following [4, 3].

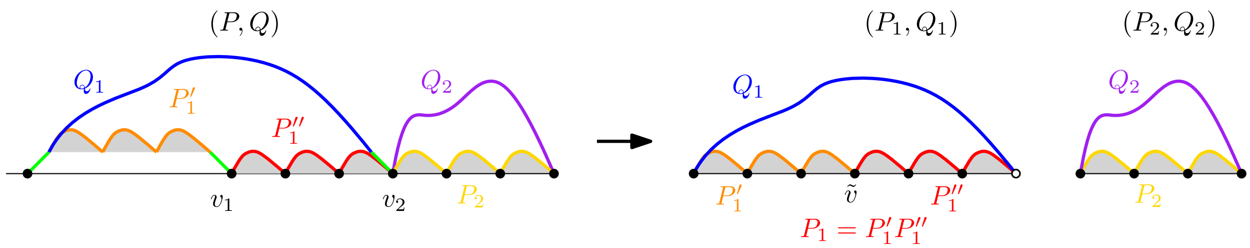

Figure 3: The classical decomposition of Tamari intervals. To the left, an interval of size $n+1$, where v_1, v_2 are the first contacts of the lower and upper path, respectively. The decomposition gives rises, to the right, to two Tamari intervals of total size n, the first of which has a marked contact, called here $\tilde{v}$. This construction is bijective.

Let $(P, Q) \in \mathcal{I}_{n+1}$ be a Tamari interval, for $n \geq 0$, and let v_1 and v_2 be respectively the leftmost contact of P and Q different from the origin. By deleting the first up step of P and Q, and the downstep of P and Q preceding respectively v_1 and v_2, one obtains two paths that can be naturally seen as the concatenation of two pairs of paths (P_1, Q_1) and (P_2, Q_2) as on Figure 3. It is proved in [4, 3] that (P_1, Q_1) and (P_2, Q_2) are two Tamari intervals. Moreover, this operation is a *bijection* (!) between intervals of size $n+1$ and pairs of intervals of total size n, such that the first interval of the pair has a distinguished contact on its lower path – inherited from the vertex v_1. To translate this recursive construction into an equation for enumeration, one needs to introduce a two-parameter generating function:

$$F(x) \equiv F(t, x) := \sum_{n \geq 0} t^n \sum_{(P,Q) \in \mathcal{I}_n} x^{\mathrm{contact}(P)}$$

where $\mathrm{contact}(P)$ is the number of contacts of the lower path P.

Note that, if a path $\tilde{P}_1$ has k contacts, there are k possible ways to mark a contact in $\tilde{P}_1$, thus decomposing it as $\tilde{P}_1 = \tilde{P}_1'\tilde{P}_1''$. When going through these k choices, the number of *non-final* contacts of the path $\tilde{P}_0 := u\tilde{P}_1'd\tilde{P}_1''$ goes through the values $1, 2, \ldots, k$, see Figure 4(up). At the level of generating functions, the corresponding operator is

$$\Delta_x : x^k \longmapsto x^k + \cdots + x^1 = x\frac{x^k - 1}{x - 1} \;\; , \;\; \Delta_x : f(x) \longmapsto x\frac{f(x) - 1}{x - 1}. \tag{2.1}$$

This observation being made, the recursive decomposition immediately translates into the following functional equation [4, 3]

$$F(x) = x + xtF(x)\frac{F(x) - F(1)}{x - 1}. \tag{2.2}$$

Indeed, the first term accounts for the empty path (of size $n = 0$), and in the second term the factor $F(x)$ is the contribution of the interval (P_2, Q_2), and the factor $x\frac{F(x)-F(1)}{x-1} = \Delta_x F(x)$ is the contribution of the interval (P_1, Q_1).

The solution can be written especially nicely with a rational parametrization.

Proposition 2.1 ([3, Thm. 10]). *The functions $F(x)$ and $F(1)$ are given by*

$$F(x) = \frac{1+u}{(1+zu)(1-z)^3}(1-2z-z^2u) \quad , \quad F(1) = \frac{1-2z}{(1-z)^3}, \tag{2.3}$$

with

$$t = z(1-z)^3 \;\;,\;\; x = \frac{1+u}{(1+zu)^2}. \tag{2.4}$$

2.2 The enriched equation and its solution

We now study intervals in which the upper path carries a marked point. We let

$$H(x,s) \equiv H(t,x,s) := \sum_{n\geq 0} t^n \sum_{(P,Q)\in\mathcal{I}_n} x^{\mathrm{contact}(P)} \sum_{i=0}^{2n} s^{Q(i)}.$$

It is clear that the height of the marked point can be tracked in the decomposition above. This leads to the functional equation:

Proposition 2.2. *The generating function $H(x,s)$ is solution of the equation:*

$$H(x,s) = F(x) + sxt\frac{H(x,s)-H(1,s)}{x-1}F(x) + xt\frac{F(x)-F(1)}{x-1}H(x,s). \tag{2.5}$$

Proof. This follows directly from the combinatorial decomposition of Figure 3, applied to intervals with a mark abscissa. The first term accounts for the case where the marked abscissa (and height) is zero. The second term accounts for the case where the marked abscissa appears before vertex v_2. Through the decomposition the corresponding vertex of the upper path becomes a marked vertex of the path Q_1, and its height is shifted by 1, hence a contribution of $s\Delta_x H(s,x)$ for the interval (P_1,Q_1), while the interval (P_2,Q_2) has contribution $F(x)$ as before. The third term accounts for the case where the marked abscissa appears at v_2 of after, in which case there is no shift in the height and the second interval has contribution $H(s,x)$, while the first has contribution $\Delta_x F(x)$ as before. □

Equation (2.5) is easily solved via the kernel method (see e.g. [9, p. 508]). First, let us work under the change of variables $(t,x) \leftrightarrow (u,z)$ of (2.4), so we can consider $F(1)$ and $F(x)$ known. We can then write (2.5) in "kernel" form $K(x,s)H(x,s) = R.H.S.$ where the kernel $K(x,s)$ is an explicit rational function of z and u. One easily checks that there is a unique series $U = U(z)$ cancelling the kernel, given by

$$s = U(1-z)^3/[z(1+U)^2(1-Uz^2-2z)]. \tag{2.6}$$

Solving the R.H.S. for $H(1)$ we obtain (see [5] for full calculations and checks)

Theorem 2.3. *The series $H(1) \equiv H(t,s,1)$ of Tamari intervals with a marked abscissa, where t marks the size, and s the upper height at this abscissa, has the following rational parametrisation:*

$$H(1) = (1-2z-Uz^2)^2(1+U)/(1-z)^6 \tag{2.7}$$

with the change of variables $(t,s) \leftrightarrow (z,U)$ given by $t = z(1-z)^3$ and by (2.6).

3 Lower path: exact solution

We will now apply the same decomposition as in the previous sections, but keep track of the height of points on the lower path P. In order to do this, we will have to treat differently the contacts of P which appear before or after the marked abscissa, which will force us to work with *two* catalytic variables. We write $\mathrm{contact}_{<i}(P), \mathrm{contact}_{\geq i}(P)$ for the number of contacts of P strictly before, or weakly after, abscissa i, respectively. We introduce the generating function

$$G(x,y) \equiv G(t,x,y,w) := \sum_{n\geq 0} t^n \sum_{(P,Q)\in\mathcal{I}_n} \sum_{i=0}^{2n} w^{P(i)} x^{\mathrm{contact}_{<i}(P)} y^{\mathrm{contact}_{\geq i}(P)}.$$

3.1 The enriched equation

Proposition 3.1. *The generating function $G(x,y) \equiv G(t,x,y,w)$ of Tamari intervals with a marked abscissa where w marks the lower height is solution of the equation:*

$$\begin{aligned} G(x,y) = F(y) + txw\frac{G(1,y)-G(1,1)}{y-1}F(y) + tx\frac{F(y)-yF(1)}{y-1}F(y) \\ + t\frac{x^2}{y}\frac{G(x,y)-\frac{y}{x}F(x)-G(1,y)+yF(1)}{x-1}F(y) + tx\frac{F(x)-F(1)}{x-1}G(x,y). \end{aligned} \tag{3.1}$$

Idea of the proof. Given an interval (P,Q) of size $n+1$ with a marked abscissa, we apply again the decomposition of Figure 3. We let i_1, i_2 be the abscissa of the first nonzero contact of P and Q respectively, and i the marked abscissa. We will distinguish fives cases depending on the fact that i belongs to $\{0\}, [1, i_1-1], \{i_1\}, [i_1+1, i_2-1], [i_2, 2n]$. These correspond (from left to right) to the five terms in (3.1).

In this extended abstract, we will only address the second case, which illustrates in a simple way why we need two catalytic variables. If $i \in [1, i_1-1]$, in the decomposition of Figure 3, the corresponding vertex of P becomes a marked vertex of P_1 with a shift of 1 in the height, hence a contribution of w. Moreover, configurations in this case are obtained by applying the construction of Figure 4(top) but restricting it to contacts appearing after the marked abscissa, see Figure 4(center). Therefore, an interval $(\tilde{P}_1, \tilde{Q}_1)$ having a marked abscissa, with respectively k and ℓ contacts strictly before, and weakly

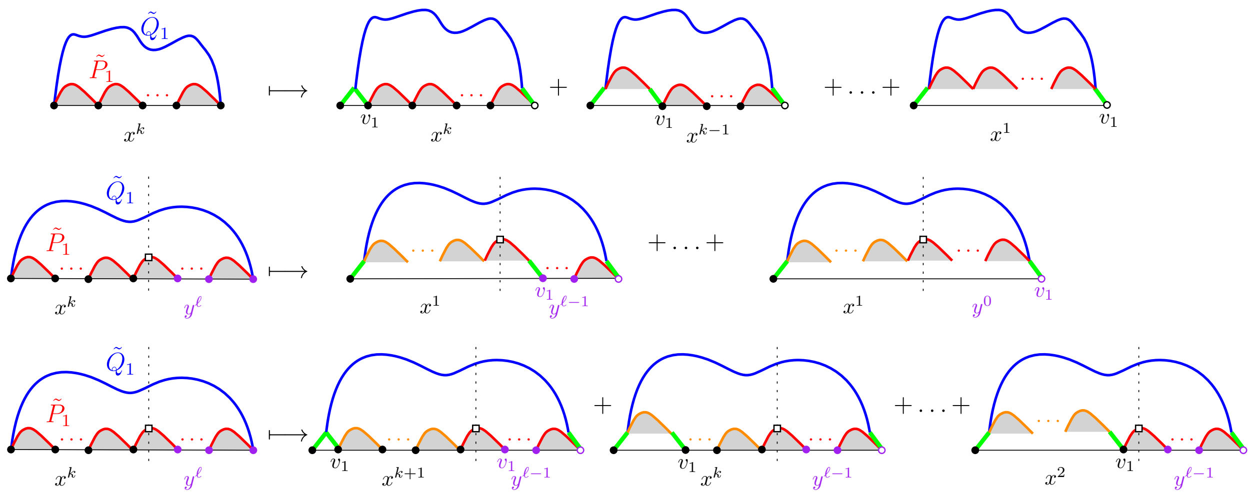

Figure 4: Top: How the divided difference operator appears. On the left, the power of x marks all contacts, while on the right it only marks contacts which are not the last one. Center and Bottom: Refinement to distinguish the case where the marked abscissa is before (upper figure) or after (lower figure) the contact v_1.

after this abscissa (thus having a contribution of $x^k y^\ell$ in $G(x,y)$) gives rise to ℓ intervals contributing to this case, with a contribution of $x^1y^{\ell-1} + x^1y^{\ell-2} + \cdots + x^1y^0 = x\frac{y^\ell-1}{y-1}$. In total, the contribution for the first interval is thus $x\frac{G(1,y)-G(1,1)}{y-1}$. The contribution of the second interval (P_2, Q_2) is just $F(y)$, since all corresponding contacts appear after the marked abscissa. In total, this gives the second term in (3.1).

We omit other cases, but we point out that the fourth case requires a similar discussion where now the catalytic variable x plays the main role, see Figure 4(bottom). Some care is needed since the last vertex of the path $\tilde{Q}_1$ cannot be marked in this case, hence the slightly more complicated numerator in this term. □

3.2 Solution

Although equations with two catalytic variables are notoriously difficult, Equation (3.1) is of a very particular kind, as it involves $G(x,y)$, $G(1,y)$, and $G(1,1)$, but not $G(x,1)$. This will enable us to treat (3.1) as two nested equations, each having only one catalytic variable. In what follows we sketch the resolution, see [5] for more details.

First step: eliminating the variable x (or u). We will work under the change of variables (2.4), with a new variable v which is to y what u is to x, namely

$$t = z(1-z)^3 \;\; , \;\; x = \frac{1+u}{(1+zu)^2} \;\; , \;\; y = \frac{1+v}{(1+zv)^2}. \tag{3.2}$$

We write respectively $\tilde{G}(u,v), \tilde{G}_1(v), \tilde{G}_{11}$ for the quantities $G(x,y), G(1,y), G(1,1)$ expressed in the variables z,u,v after the substitutions (3.2).

Making the substitutions (3.2) and using the known expressions of $F(x)$, $F(y)$, Equation (3.1) takes the form

$$\tilde{K}_2(z,u,v)\tilde{G}(u,v) = \tilde{L}_2(z,u,v,\tilde{G}_1(v),\tilde{G}_{11})$$

for some rational functions $\tilde{K}_2, \tilde{L}_2$ that can be written explicitly [5]. One easily checks that the kernel $\tilde{K}_2$ has a unique root $U_0 \equiv U_0(z,v)$ which is a power series in z. Substituting $u = U_0$ in the last equation, we cancel the left-hand side, hence we also cancel the right-hand side. We are thus left with the following polynomial equation:

$$\tilde{L}_2(z,U_0(z,v),v,\tilde{G}_1(v),\tilde{G}_{11}) = 0. \tag{3.3}$$

The numerator of this equation has 91 terms, but it has only degree one in $\tilde{G}_1(v)$. At this stage, we have eliminated the unknown $\tilde{G}(u,v)$ and the variable u.

Second step: eliminating the variable y (or v). The last equation, (3.3), is nothing but an equation with one catalytic variable, which is now the variable v (or x)! Since it is linear in $\tilde{G}_1(v)$, we can just use the kernel method again: one first checks that there is a unique series $V_0(z)$ cancelling the kernel, thus giving two equations: the linear and constant coefficient in $\tilde{G}_1(v)$ in (3.3), which both vanish when $v = V_0$. Eliminating V_0, we obtain a polynomial equation for the function $\tilde{G}_{11}$ which is not even so big. We obtain:

Theorem 3.2. *The function $G(1,1)$ after the change of variables $z \leftrightarrow t$ given by (2.4) satisfies the polynomial equation $C(G(1,1),z,w) = 0$ with*

$$\begin{aligned}C(h,z,w) &= wz(-1+z)^9h^3 + (-1+z)^6(2w^2z^2 - w^2z + 2z^2 + w - z)h^2 + 4wz^2 - 4wz + w\\ &\quad -(-1+z)^3(w^2z^3 - 3w^2z^2 - 2wz^3 + w^2z - 2wz^2 + z^3 + 5wz - 3z^2 - 2w + z)h.\end{aligned} \tag{3.4}$$

4 Asymptotics of moments

The two parts of Theorem 1.1 are direct applications of Theorem 1.4, up to computer algebra calculations done in [5]. In both cases we have $\rho = \frac{27}{256}$, corresponding to $z = \frac{1}{4}$.

In the case of the upper path, we start from (2.7) in Theorem 2.3, which tells us that the function $f(t,s) = H(1,s)$ is algebraic. With GFUN [12], we directly obtain [5] a recurrence formula for the derivatives at $s = 1$ of the form (1.6) with $L = 6$, where the $h_d(t,k)$ for $d = 1..6$ are Laurent polynomials in $\delta = \sqrt{1-4z}$ and rational functions of k. It is automatic to check that the hypotheses of Theorem 1.4 holds with $\beta = \frac{3}{4}$ and $\alpha = \frac{1}{2}$. One can explicitly check the values of the corresponding constants $a_d(k)$, which are nothing but the coefficients of δ^{-3d} in h_d, up to a scaling factor. They are given [5] by

$$a_1(k), \ldots, a_6(k) = 0, \frac{\sqrt{6}}{96}(3k-4)(3k-8), 0, 0, 0, 0, \tag{4.1}$$

so the main recurrence formula (1.7) becomes

$$c_k = \frac{\sqrt{6}(3k-4)(3k-8)}{96} c_{k-2} \quad , \quad k > 6.$$

The initial conditions require to estimate the main singularity of $f^{(1)}, \ldots, f^{(6)}$, which is easily done automatically, and it is then a direct check that the solution is given by

$$c_k = 16/(27\pi)\, \Gamma\left(\frac{k}{2} + \frac{1}{3}\right) \Gamma\left(\frac{k}{2} - \frac{1}{3}\right) \sqrt{2} \cdot 4^{-k} 6^{3/4\, k}.$$

for all $k \geq 0$. Applying Theorem 1.4, we directly obtain (1.2), and (1.1) follows for example from the Carleman criterion. See [5] for full calculations.

The proof of the second half (i.e. (1.3)) follows similarly from Theorem 3.2, this time with $\alpha = \frac{1}{2}, \beta = \frac{3}{4}$, and $L = 9$. See [5] again.

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