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Eulerian Polynomials for Digraphs

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Abstract. Given an n-vertex digraph D and a labeling $\sigma : V(D) \to [n]$, we say that an arc $u \to v$ of D is a descent of σ if $\sigma(u) > \sigma(v)$. Foata and Zeilberger introduced a generating function $A_D(t)$ for labelings of D weighted by descents, which simultaneously generalizes both Eulerian polynomials and Mahonian polynomials. Motivated by work of Kalai, we look at problems related to -1 evaluations of $A_D(t)$. In particular, we give a combinatorial interpretation of $|A_D(-1)|$ in terms of "generalized alternating permutations" whenever the underlying graph of D is bipartite.

Keywords: Eulerian polynomial, alternating permutations, combinatorial reciprocity

1 Introduction

Descents and inversions are two of the oldest and most well-studied *permutation statistics* dating back to work of MacMahon [15, 14]. A *descent* of a permutation $\sigma \in \mathfrak{S}_n$ on the set $[n] := \{1, 2, \dots, n\}$ is an index $i \in [n-1]$ such that $\sigma(i) > \sigma(i+1)$, and an inversion is a pair of integers (i, j) with $1 \le i < j \le n$ such that $\sigma(i) > \sigma(j)$. The number of descents and inversions of σ are denoted by $\operatorname{des}(\sigma)$ and $\operatorname{inv}(\sigma)$, respectively.

The generating functions

$$A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\operatorname{des}(\sigma)} \quad M_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\operatorname{inv}(\sigma)}$$

are called the *Eulerian* and *Mahonian* polynomials respectively. Both of these polynomials are important objects of study in many branches of combinatorics and have been generalized in many different ways. In this paper, we consider a polynomial due to Foata and Zeilberger [9] which generalizes both the Eulerian and Mahonian polynomials via directed graphs.

A *permutation* of an n-vertex digraph $D = (V, E)$ is a bijection $\sigma : V \to [n]$. We will use the notation $\mathfrak{S}_D$, $\mathfrak{S}_V$, or $\mathfrak{S}_n$ to denote the set of permutations of D. For a given

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directed graph $D = (V, E)$ and a permutation σ of D, a *D-descent* (or just *descent* when D is understood) is an arc $u \to v$ such that $\sigma(u) > \sigma(v)$. The total number of D-descents of a permutation σ is denoted by $\mathrm{des}_D(\sigma)$; see Figure 1 for an example.

Figure 1: Two labelings $\pi : V(D) \to [5]$ where descent arcs are marked by red dashed lines.

These statistics generalize both of des and inv as Figure 2 shows.

(a) $\mathrm{des}(23154) = 2$ (b) $\mathrm{inv}(23154) = 3$

Figure 2: The Eulerian polynomial $A_D(t)$ generalizes both descents and inversions.

With all this in mind, we can now define the central object of study for this paper: the *Eulerian polynomial* of a digraph $D = (V, E)$ is the generating function

$$A_D(t) = \sum_{\sigma \in \mathfrak{S}_D} t^{\mathrm{des}_D(\sigma)}. \tag{1.1}$$

In particular, we have $A_{\overrightarrow{P_n}}(t) = A_n(t)$ and $A_{\overrightarrow{K_n}}(t) = M_n(t)$.

This polynomial can be seen in other work: as a *weighted-inversion* generating function as in [11, 5]; as an *Eulerian polynomial for a (particular) family* $\mathfrak{B}_n$ *of digraphs* [1]; as a specialization of the *chromatic quasisymmetric function for digraph* [6] and *B-polynomial* [2]. There are also a myriad of other objects generalizing Eulerian polynomials which are related by varying degrees to ours.

The primary objective of this extended abstract is to study evaluations of $A_D(t)$ at -1. See [4] for the full paper. This is a problem in the area of *combinatorial reciprocity*, which studies combinatorial polynomials evaluated at negative integers. For example, the classical Eulerian and Mahonian polynomials both have good combinatorial interpretations for their evaluation at -1: the former being the number of *alternating permutations* [8] and the latter being the number of *correct proofs of the Riemann hypothesis*[1]. Many more results on combinatorial reciprocity can be found in the book by Beck and Sanyal [3].

[1] As of the time of writing.

Kalai [12, Section 8.1] makes a critical observation about $A_D(-1)$.

Proposition 1.1. *If D, D' are orientations of the same graph G, then $|A_D(-1)| = |A_{D'}(-1)|$.*

With Proposition 1.1 in mind, for any graph G we can define

$$\nu(G) := |A_D(-1)|,$$

where D is any orientation of G. The problem of studying $\nu(G)$ was first introduced by Kalai [12] due to its relation with the Condorcet paradox in social choice theory, and a few basic properties of $\nu(G)$ were established by Even-Zohar [7]. Outside of this, nothing seems to be known about $\nu(G)$ despite Kalai raising the problem over 20 years ago.

In this extended abstract, we prove three types of results related to $\nu(G)$: we give combinatorial interpretations for $\nu(G)$ for a large class of graphs G, we determine the maximum and minimum values achieved by $\nu(G)$ amongst n vertex trees, and we consider the refined problem of determining the multiplicity of -1 as a root of $A_D(t)$.

2 Combinatorial Interpretations for $\nu(G)$

A classical result of Foata and Schützenberger [8] (see also [16, Exercise 135]) states that for odd n the Eulerian polynomial $A_n(t)$ evaluated at $t = -1$ is equal (up to sign) to the number of *alternating permutations* of length n, i.e. the number of permutations σ and $\sigma(1) < \sigma(2) > \sigma(3) < \cdots > \sigma(n)$. Because $A_n(t) = A_{\overrightarrow{P}_n}(t)$ for $\overrightarrow{P}_n$ the directed path, this result implies $\nu(P_n)$ is equal to the number of alternating permutations of size n.

Given this observation, it is natural to expect $\nu(G)$ to count "alternating permutations for graphs" for some generalized notion of alternating permutations. There are many such generalizations one could consider, for example, one could force every maximal path of G to be an alternating permutation. However, it turns out that the definition we will want to consider is the following (non-obvious) generalization.

Definition 2.1. Given an n-vertex graph G, we say that an ordering $\pi = (\pi_1, \ldots, \pi_n)$ of the vertex set $V(G)$ is an *even sequence* if each of the subgraphs $G[\pi_1, \ldots, \pi_i]$ induced by the first i vertices of π have an even number of edges for all $1 \le i \le n$. We let $\eta(G)$ denote the number of even sequences of G.

Lemma 2.2. *For any graph G,*

(a) $\nu(G) \le \sum_{v \in V(G)} \nu(G - v)$.

(b) If G has an odd number of edges, $\eta(G) = 0$. Otherwise, $\eta(G) = \sum_{v \in V(G)} \eta(G - v)$.

(c) $\nu(G) \le \eta(G)$.

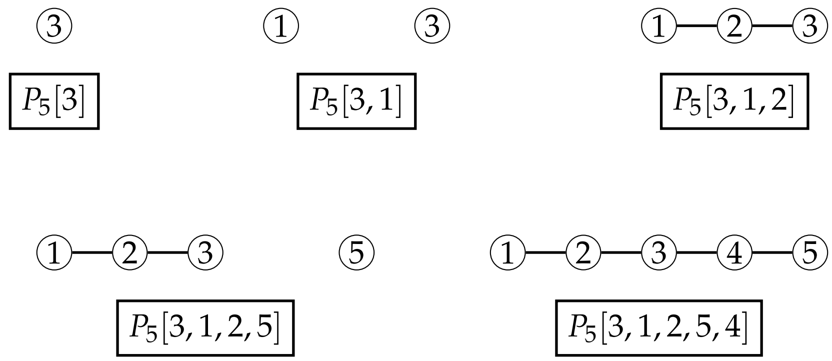


Figure 3: A depiction of the induced subgraphs $P_5[\pi_1, \ldots, \pi_i]$ for the ordering $\pi = (3, 1, 2, 5, 4)$ of the path graph P_5. Note that π is an even sequence since each of these induced subgraphs have an even number of edges. We also observe that $\pi^{-1} = (2, 3, 1, 5, 4)$ is an alternating permutation.

One can verify that even sequences for the path graph P_n with vertex set $[n]$ are exactly inverses of alternating permutations of size n, so $\nu(P_n) = \eta(P_n)$ in this case. Our main result shows that this equality holds for a substantially larger class of graphs.

To state this result, we remind the reader that a graph is *complete multipartite* if one can partition its vertices into sets $V_1, \ldots, V_r$ such that u and v are adjacent if and only if $u \in V_i, v \in V_j$ for some $i \neq j$. We say that a graph is a *blowup of a cycle* if one can partition its vertices into sets $V_1, \ldots, V_r$ such that u and v are adjacent if and only if $u \in V_i$ and $v \in V_{i+1}$ for some i (with the indices written mod r).

Theorem 2.3. *If G is a graph which is either bipartite, complete multipartite, or a blowup of a cycle, then $\nu(G) = \eta(G)$.*

The proofs for each of these cases follows the same basic strategy: We first show that for some "natural" orientation D of G, we can easily predict the sign of $A_D(-1)$. From this we deduce $\nu(G) = \sum \nu(G - v)$, and hence that $\nu(G) = \eta(G)$ since the statistics ν, η satisfy the same recurrence relation. Accordingly, we will only discuss the proof for bipartite graphs in this extended abstract, leaving the other classes of graphs for the full paper.

Lemma 2.4. *Let D be a digraph such that one can partition its vertex set into $U \cup V$ such that every arc $u \to v$ of D has $u \in U$ and $v \in V$. Then*

$$A_D(-1) \geq 0,$$

and if D has an even number of arcs, then

$$A_D(-1) = \sum_{v \in V(D)} A_{D-v}(-1).$$

Corollary 2.5. *If G is a bipartite graph with an odd number of edges, then $\nu(G) = 0$, and otherwise $\nu(G) = \sum_v \nu(G - v)$.*

Proof of Theorem 2.3. We aim to show that $\nu(G) = \eta(G)$ whenever G is bipartite, complete multipartite, or a blowup of a cycle. We first consider the case that G is bipartite. We prove this result by induction on $|V(G)|$, the base case $\nu(K_1) = \eta(K_1) = 1$ being trivial. By Corollary 2.5 and Lemma 2.2, if G has an odd number of edges then $\nu(G) = \eta(G) = 0$, and otherwise

$$\nu(G) = \sum_{v \in V(G)} \nu(G - v) = \sum_{v \in V(G)} \eta(G - v) = \eta(G),$$

where the middle equality used the inductive hypothesis (and that $G - v$ is bipartite whenever G is). □

It is tempting to try to generalize this approach by finding "natural" orientations of other graphs in order to show $\nu(G) = \sum \nu(G - v)$; see for example Conjecture 5.2. However, the following theorem shows that the inductive proof of Theorem 2.3 can not be extended beyond the class of graphs which are bipartite, complete multipartite, or blowups of cycles.

Theorem 2.6. *If G is a connected graph such that $\nu(G') = \eta(G')$ for all induced subgraphs $G' \subseteq G$, then G is either bipartite, complete multipartite, or a blowup of a cycle.*

Our proof of Theorem 2.6 relies on a structural graph theory result which may be of independent interest. The *odd pan graph* C^*_{2k+1} is defined to be the graph obtained by taking the odd cycle C_{2k+1} and then adding a new vertex u adjacent to exactly one vertex of C_{2k+1}. We say that a graph G is *odd pan-free* if it contains no induced subgraph which is isomorphic to C^*_{2k+1} for any $k \geq 1$.

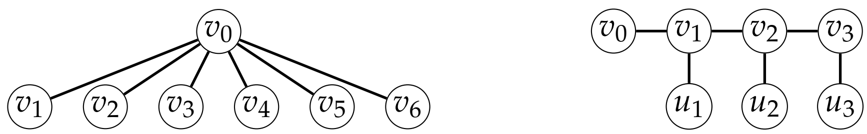
Proposition 2.7. *If G is a connected graph, then G is odd pan-free if and only if it is either bipartite, complete multipartite, or a blowup of a cycle.*

While we do not have a full understanding of $\nu(G)$ for arbitrary graphs, we are able to prove several other results regarding $\nu(G)$, such as the general bound $\nu(G) \leq \eta(G)$ in the full paper.

3 Upper and lower bounds of $\nu(G)$ and $\eta(G)$

We next turn to the extremal problem of studying the largest and smallest possible values of $\nu(G)$ and $\eta(G)$. For arbitrary n-vertex graphs this is an uninteresting problem, since $\nu(\overline{K_n}) = \eta(\overline{K_n}) = n!$ and $\nu(K_n) = \eta(K_n) = 0$ for $n \geq 2$ are easily seen to achieve the maximum and minimum possible values. However, this problem becomes non-trivial when one looks at smaller classes of graphs. To this end, we consider these extremal problems for trees.

To state our result, we recall that a tree is a *star* $K_{1,n}$ if there is a single-non leaf vertex; see Figure 4a. We say that a tree is a *hairbrush* if it consists of a path $v_0 v_1 \cdots v_n$ such that each vertex v_i with $i \geq 1$ is adjacent to a leaf u_i; see Figure 4b.



(a) The star $K_{1,6}$ (b) The hairbrush H_3

Theorem 3.1. *If T is a tree on $2n+1$ vertices, then*

$$n!2^n \leq \nu(T) = \eta(T) \leq (2n)!$$

Moreover, equality holds in the lower bound if and only if T is a hairbrush, and equality holds in the upper bound if and only if T is a star.

To aid with our proofs, given a tree T, we define

$$\widetilde{X}(T) = \{x \in V(T) : \text{each component of } T - x \text{ has an even number of edges}\},$$

and we will denote this simply by $\widetilde{X}$ whenever T is understood. Our motivation for this definition is the following.

Lemma 3.2. *If T is a tree with an even number of edges, then*

$$\nu(T) = \sum_{x \in \widetilde{X}} \nu(T - x).$$

With this lemma in mind, the idea for the proofs of the upper and lower bounds is as follows: we first apply Lemma 3.2 and then use induction to bound each of the terms $\nu(T-x)$ in the sum. Finally, we bound our total sum in terms of $|\widetilde{X}|$ and show that equality can only occur when $|\widetilde{X}| = 1$. In particular, we can show that

$$n!2^n \leq \nu(T - x) \leq \frac{1}{2|\widetilde{X}| - 1}(2n)! \tag{3.1}$$

for all trees with an even number of edges and $x \in \tilde{X}$ and so the result follows.

Remark 3.3. Our proofs yield slightly stronger bounds on $\nu(T)$ whenever $\widetilde{X}$ is large. For example, (3.1) gives the lower bound $\nu(T) \geq |\widetilde{X}| n! 2^n$. Bounds of this form are known as *stability results* in extremal graph theory, which roughly are results saying that bounds for a graph T can be substantially improved if T is "far" from a unique extremal construction. Here, T being "far" from H_n and $K_{1,2n}$ is measured by having $|\widetilde{X}|$ large.

4 Multiplicity of Roots

Lastly, we consider the problem of determining the multiplicity of -1 as a root of $A_D(t)$, and we denote this quantity by $\mathrm{mult}(A_D(t), -1)$.

One of the first questions one might ask in this setting is how large $\mathrm{mult}(A_D(t), -1)$ can be amongst all n-vertex digraphs? Trivially, $\mathrm{mult}(A_D(t), -1) \leq e(D)$ (since the degree of $A_D(t)$ is at most $e(D)$), which implies $\mathrm{mult}(A_D(t), -1) \leq \binom{n}{2}$ if D has n vertices. We prove a substantially stronger upper bound which turns out to be sharp.

Theorem 4.1. *If D is an n-vertex digraph, then*

$$\mathrm{mult}(A_D(t), -1) \leq n - s_2(n),$$

where $s_2(n)$ denotes the number of 1's in the binary expansion of n. Moreover, for all n, there exist n-vertex digraphs D with $\mathrm{mult}(A_D(t), -1) = n - s_2(n)$.

The upper bound can be achieved with the following construction. Given digraphs D_1, D_2, and a root vertex $v \in D_2$, the *rooted product digraph*, denoted $D_1 \circ_v D_2$, is obtained by gluing a copy of D_2 at v to each vertex of D_1, see Figure 5 for an example.

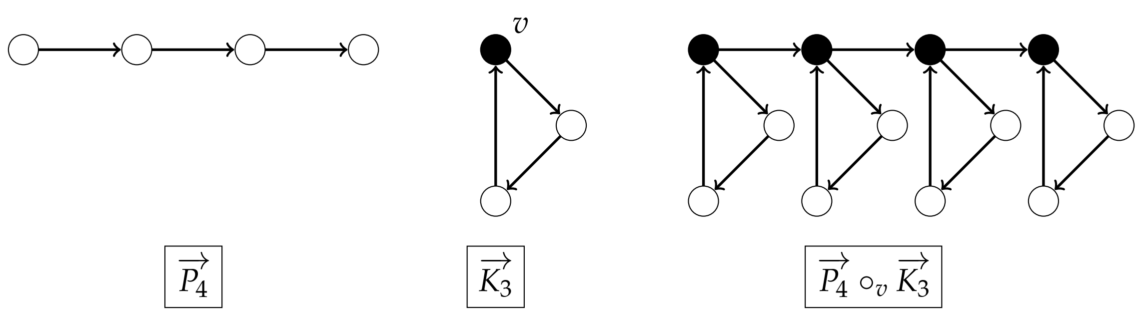


Figure 5: The rooted product digraph $\overrightarrow{P_4} \circ_v \overrightarrow{K_3}$ with the vertex v highlighted in black.

This product was first defined by Godsil and McKay [10], and it turns out that this operation plays very nicely with the Eulerian polynomial.

Proposition 4.2. *Let D_1 and D_2 be two digraphs on m and n vertices respectively. If $v \in D_2$, then*

$$A_{D_1 \circ_v D_2}(t) = \frac{1}{m!}\binom{mn}{n, \ldots, n} A_{D_1}(t) A_{D_2}(t)^m.$$

In particular, the polynomial is the same for any choice of root $v \in D_2$.

Remark 4.3. The last line of the statement implies that there are non-isomorphic digraphs with the same Eulerian polynomial.

With this, we first consider the case when $n = 2^m$ for some $m \geq 1$. Let P_2 be the graph on vertices v_1, v_2 with a single arc $v_1 \to v_2$. Define a sequence of digraphs $\{L_m\}_{m \in \mathbb{N}}$ by

$$L_1 = P_2 \quad \text{and} \quad L_{m+1} = L_m \circ_{v_1} P_2.$$

We observe that L_m has 2^m vertices and $2^m - 1$ arcs. Then from Proposition 4.2, we have

$$A_{L_m}(t) = (2^m)! \left(\frac{1+t}{2}\right)^{2^m-1}.$$

Since $s_2(2^m) = 1$, this gives the desired construction when n is a power of two. For arbitrary n, we let $a_1, \ldots, a_\ell$ be the indices of nonzero powers of 2 in the binary expansion of n and then define D to be the disjoint union of the digraphs $L_{a_1}, \ldots, L_{a_\ell}$. Then $A_D(t)$ gives the desired upper bound.

We also obtain a general lower bound on $\operatorname{mult}(A_D(t), -1)$.

Proposition 4.4. *Let D be an orientation of an n-vertex graph G. If every matching in the complement of G has size at most m, then* $\operatorname{mult}(A_D(t), -1) \geq \lfloor \frac{n}{2} \rfloor - m$.

Roughly speaking, Proposition 4.4 says that if G is "dense" (i.e. if the complement of G contains small only matchings), then $\operatorname{mult}(A_D(t), -1)$ will be large. While Proposition 4.4 is not tight in general, it turns out to be tight if D is an orientation of the complete graph as we show now.

Let $OP(\alpha)$ denote the set of all ordered set partitions of type α, and let $SP(\lambda)$ denote the set of all unordered set partitions with type λ. For two sets S, T of vertices in a digraph D, let $e_D(S, T)$ be the number of edges which start in S and end in T. For a digraph D and an ordered set partition $P = (P_1, \ldots, P_k)$ of the vertices of D of length k and $i \in [k]$, define the *i-th forward sequence number* of P to be

$$FS_{D,P}(i) = \sum_{j=i+1}^{k} e_D(P_i, P_j)$$

and the *i-th reverse sequence number* of P to be

$$RS_{D,P}(i) = \sum_{j=i+1}^{k} e_D(P_j, P_i)$$

where we set $FS_{D,P}(k) = 0$ and $RS_{D,P}(1) = 0$.

With this notation in hand, we can factor the Eulerian polynomial.

Lemma 4.5. *If D is a tournament on the vertex set $[n]$ and α is the integer composition (2^k) of n if n is even and $(1, 2^k)$ if n is odd, then*

$$A_D(t) = (1+t)^k \frac{1}{2^k} \sum_{P \in OP(\alpha)} \prod_{i=1}^{k} t^{FS_{D,P}(i)} + t^{RS_{D,P}(i)}.$$

A parity argument shows that the sum in the lemma does not have -1 as a root. Therefore, we obtain the following.

Theorem 4.6. *If D is a tournament on n vertices, then* $\mathrm{mult}(A_D(t), -1) = \lfloor \frac{n}{2} \rfloor$.

More generally, we suspect that Proposition 4.4 is tight for orientations of complete multipartite graphs; see Conjecture 5.4 for more.

Given Theorem 4.6 and the fact that $|A_D(-1)| = |A_{D'}(-1)|$ whenever D, D' are orientations of the same graph, it is perhaps natural to guess that $\mathrm{mult}(A_D(t), -1)$ depends only on the underlying graph of D. This turns out to be false; see Figure 6 for a counterexample.

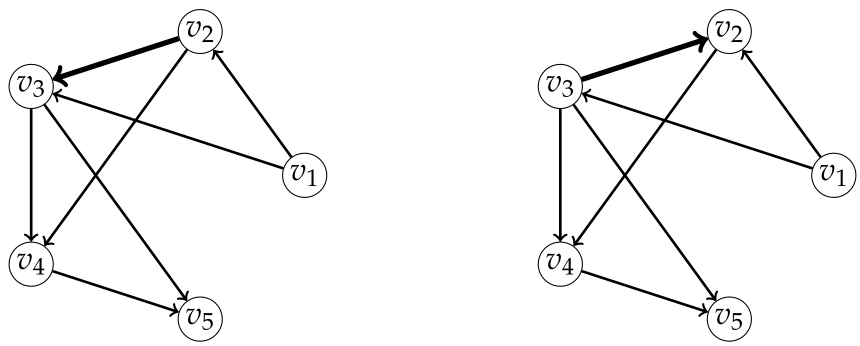


Figure 6: Two orientations of the same graph with different -1 multiplicities. The digraph on the left has $A_{D_1}(t) = (1+t)^3(1+t+11t^2+t^3+t^4)$ while the one on the right has $A_{D_2}(t) = (1+t)(1+5t+16t^2+16t^3+16t^4+5t^5+t^6)$.

5 Concluding Remarks and Open Problems

In this extended abstract, we studied a notion of Eulerian polynomials $A_D(t)$ for digraphs D and proved a number of results related to evaluations at $t = -1$. We conclude by listing a number of remaining open problems themed around interpreting $\nu(G)$ and multiplicities of -1 as a root of $A_D(t)$.

Interpretations for $\nu(G)$. Recall that for any graph G we define $\nu(G) = |A_D(-1)|$ where D is any orientation of G. While Theorem 2.3 provides a combinatorial interpretation for $\nu(G)$ when G is bipartite, complete bipartite, or a blowup of a cycle, we are still far from understanding this quantity for general graphs, which we leave as the main open problem for this paper.

Question 5.1. *Can one give a combinatorial interpretation for $\nu(G)$ for arbitrary graphs G?*

In view of Theorem 2.3 and the bound $\nu(G) \leq \eta(G)$ from Lemma 2.2(a), we suspect that in general $\nu(G)$ should count even sequences of G with some special properties, but what these properties should be remains a mystery.

To answer Question 5.1, it might be useful to establish which graphs G satisfy $\nu(G) = \sum_v \nu(G-v)$, as recurrences of this form were a key step in proving Theorem 2.3. In particular, computational evidence suggests that the following could hold, where here we recall that a graph is *Eulerian* if all of its degrees are even.

Conjecture 5.2. *If G is an Eulerian graph, then $\nu(G) = \sum_v \nu(G-v)$.*

We note that an Eulerian graph has a "natural" orientation via orienting each edge according to an Eulerian tour. Given that e.g. our proof of Corollary 2.5 relied on "natural" orientations of bipartite graphs, it is plausible that this natural orientation for Eulerian graphs could be used to prove Conjecture 5.2.

Our proof of Theorem 2.3 is non-combinatorial, and it would be interesting to have a more direct combinatorial proof of this fact, say for bipartite graphs.

Problem 5.3. *For any bipartite graph $G = ([n], E)$ and orientation D of G, construct an explicit involution $\varphi : \mathfrak{S}_n \to \mathfrak{S}_n$ such that*

(a) The set of fixed points $\mathcal{F}_\varphi$ of φ is the set of (inverses of) even sequences of G, and

(b) $(-1)^{\mathrm{des}_D(\sigma)} = -(-1)^{\mathrm{des}_D(\varphi(\sigma))}$ for all $\sigma \notin \mathcal{F}_\varphi$.

Such an involution is known to exist when $G = P_n$ (i.e. when inverses of even sequences are exactly alternating permutations), but this involution is somewhat complex; see [16, Exercise 135] for more.

Multiplicity of Roots. In Theorem 4.6 we showed every n vertex tournament D has -1 as a root of $A_D(t)$ with multiplicity exactly $\lfloor \frac{n}{2} \rfloor$. A natural generalization of this result would be the following.

Conjecture 5.4. *If D is the orientation of a complete multipartite graph which has r parts of odd size, then* $\mathrm{mult}(A_D(t), -1) = \lfloor \frac{r}{2} \rfloor$.

Observe that the bound $\mathrm{mult}(A_D(t), -1) \geq \lfloor \frac{r}{2} \rfloor$ follows from Proposition 4.4, so the difficulty lies in proving the upper bound.

Another direction is to look at the more general quantity $\mathrm{mult}(A_D(t), \alpha)$, which is defined to be the multiplicity of α as a root of $A_D(t)$. For example, it is not difficult to see that $\mathrm{mult}(A_D(t), 0)$ is equal to the minimum number of arcs that one must remove in D to obtain an acyclic digraph. Such a set of arcs is known as a *minimum feedback arc set*, and determining the size of such a set is well known to be an NP-Complete problem [13].

This connection to feedback arc sets, together with the results of this paper, establishes a number of results for $\mathrm{mult}(A_D(t), \alpha)$ when $\alpha \in \{0, -1\}$, and it is natural to ask what can be said about other integral values of α. An immediate obstacle to this is the following.

Question 5.5. *Does there exist a digraph D such that $A_D(t)$ has an integral root which is not equal to either 0 or -1?*

We have verified that no such digraph exists on at most 5 vertices. We also note that there exist digraphs with real roots of magnitude larger than 2, so the obstruction to finding these integral roots is not that their magnitudes are too large.

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