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Dyck combinatorics in p-Kazhdan–Lusztig theory

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Abstract. We survey some recent advances in combinatorial modular representation theory in type A through the lens of p-Kazhdan–Lusztig theory.

1 Introduction

The diagrammatic Hecke category has provided the intuition and tools necessary to cut through the most famous conjectures of Lie theory: the Lusztig and Kazhdan–Lusztig positivity conjectures. These conjectures place the Kazhdan–Lusztig polynomials (associated to parabolic Coxeter systems) centre-stage in the (modular) representation theory of Lie theoretic objects.

Kazhdan–Lusztig polynomials encode a great deal of character-theoretic and indeed cohomological information about cell modules. We further know that Kazhdan–Lusztig polynomials often carry information about the radical layers of indecomposable projective and cell modules. Given the almost ridiculous level of detail these polynomials encode, it is natural to ask *"what are the limits to what p-Kazhdan–Lusztig combinatorics can tell us about the structure of the Hecke category?"*

The family of ordinary Kazhdan–Lusztig polynomials which are combinatorially best understood are those for maximal parabolics of finite symmetric groups $\mathfrak{S}_m \times \mathfrak{S}_n \leqslant \mathfrak{S}_{m+n}$. These polynomials can be calculated in terms of the combinatorics of Dyck tilings [9]. The starting point of this project was to extend this to the modular case by proving that the p-Kazhdan–Lusztig polynomials of $\mathfrak{S}_m \times \mathfrak{S}_n \leqslant \mathfrak{S}_{m+n}$ are entirely independent of $p \geqslant 0$. We also find that there is a wealth of extra, richer combinatorial information which can be encoded into the Dyck tilings. Instead of looking only at the sets of Dyck tilings (which enumerate these Kazhdan–Lusztig polynomials) we look at the relationships for passing between these Dyck tilings. In fact, this "meta-Kazhdan–Lusztig

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combinatorics" is sufficiently rich as to completely determine the full structure of our Hecke categories. In this extended abstract, we discuss how this allows us to provide a complete combinatorial description of the submodule lattices of the cell modules for these categories.

We also proved in [2] that the Hecke categories of $\mathfrak{S}_m \times \mathfrak{S}_n \leqslant \mathfrak{S}_{m+n}$ control the structure of parabolic Verma modules for Lie algebras [4, 8, 9]; the representation category of the general linear supergroups [3]; arc algebras from categorified knot theory [5]; walled Brauer algebras [6]; and the combinatorics of attracting cells for torus fixed points in Springer fibers [11]. This makes the cell modules of these categories some of the most well-understood representations in all of non-semisimple Lie theory.

2 Kazhdan–Lusztig polynomials

Let (W, S_W) be a Coxeter system: W is the group generated by the finite set S_W subject to the relations $(\sigma\tau)^{m_{\sigma\tau}} = 1$ for $\sigma, \tau \in S_W$, $m_{\sigma\tau} \in \mathbb{N} \cup \{\infty\}$ satisfying $m_{\sigma\tau} = m_{\tau\sigma}$, and $m_{\sigma\tau} = 1$ if and only if $\sigma = \tau$. Let $\ell : W \to \mathbb{N}$ be the corresponding length function. Consider $S_P \subseteq S_W$ a subset and (P, S_P) its corresponding Coxeter system. We say that P is the parabolic subgroup corresponding to $S_P \subseteq S_W$. Let ${}^PW \subseteq W$ denote a set of minimal coset representatives in $P\backslash W$. For $\underline{w} = \sigma_1\sigma_2 \cdots \sigma_\ell$ an expression, we define a subword to be a sequence $\underline{t} = (t_1, t_2, \ldots, t_\ell) \in \{0,1\}^\ell$ and set $\underline{w}^{\underline{t}} := \sigma_1^{t_1}\sigma_2^{t_2} \cdots \sigma_\ell^{t_\ell}$. We let $\leqslant$ denote the strong Bruhat order on ${}^PW$: namely $y \leqslant w$ if for some reduced expression $\underline{w}$ there exists a subword $\underline{t}$ and a reduced expression $\underline{y}$ such that $\underline{w}^{\underline{t}} = \underline{y}$. We denote the Hasse diagram of this poset by $\mathcal{G}_{(W,P)}$ and we refer to it as the *Bruhat graph* of the pair (W, P). Explicitly, the vertices of $\mathcal{G}_{(W,P)}$ are labelled by the elements of ${}^PW$ and for $\lambda \in {}^PW$ we have a directed edge $\lambda \to \lambda s_i$ if $\lambda < \lambda s_i \in {}^PW$ for some $s_i \in S_W$. We denote by $\varnothing$ (for the empty word in the generators) the minimal coset representative for the identity coset P.

We define the *extended Bruhat graph* $\widehat{\mathcal{G}}_{(W,P)}$ to be the directed graph having the same set of vertices as $\mathcal{G}_{(W,P)}$ but replacing each edge in $\mathcal{G}_{(W,P)}$ between λ and λs_i for $\lambda < \lambda s_i$ by four "up" and "down" directed edges

$$\lambda \xrightarrow{i} \lambda s_i, \quad \lambda \xrightarrow{i} \lambda, \quad \lambda s_i \xrightarrow{i} \lambda \quad \lambda s_i \xrightarrow{i} \lambda s_i, \tag{2.1}$$

which we denote U_i^1, U_i^0, D_i^1, D_i^0 respectively. We assign a degree to each edge in $\widehat{\mathcal{G}}_{(W,P)}$ by setting

$$\deg(\lambda \xrightarrow{i} \lambda s_i) = \deg(\lambda s_i \xrightarrow{i} \lambda) = 0 \qquad \deg(\lambda \xrightarrow{i} \lambda) = \begin{cases} 1 & \text{if } \lambda s_i > \lambda \\ -1 & \text{if } \lambda s_i < \lambda \end{cases}$$

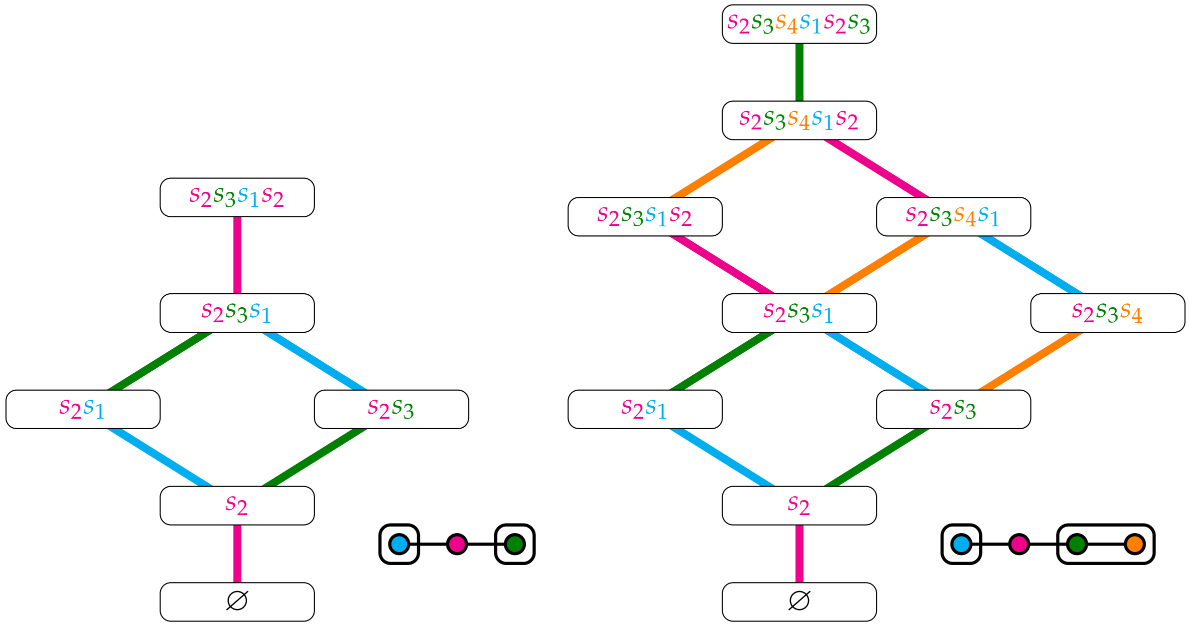


Figure 1: The graph $\mathcal{G}_{(W,P)}$ for $(W,P) = (\mathfrak{S}_4, \mathfrak{S}_2 \times \mathfrak{S}_2)$ and $(\mathfrak{S}_5, \mathfrak{S}_2 \times \mathfrak{S}_3)$ respectively.

Given a path (or "Bruhat stroll") on $\widehat{\mathcal{G}}_{(W,P)}$

$$\mathsf{T} : \lambda_1 \xrightarrow{i_1} \lambda_2 \xrightarrow{i_2} \lambda_3 \xrightarrow{i_3} \dots \xrightarrow{i_{k-1}} \lambda_k,$$

we say that the *degree* $\deg(\mathsf{T})$ is the sum of the degrees of each edge in T. (The degree is also sometimes known as the "Deodhar defect".) We also define the weight of T, denoted by $w(\mathsf{T})$ to be the expression

$$w(\mathsf{T}) := s_{i_1}s_{i_2}s_{i_3}\dots s_{i_{k-1}}.$$

Given $\lambda \in {}^PW$, we let $\mathrm{Path}(\lambda)$ denote the set of all paths from $\varnothing$ and ending at λ in the extended Bruhat graph.

Definition 2.1. We say that a path $\mathsf{T} \in \mathrm{Path}(\mu)$ is *reduced* if it is a path of shortest possible length from $\varnothing$ to μ.

Throughout the paper we will fix one reduced path, $\mathsf{T}^\mu \in \mathrm{Path}(\mu)$, for each $\mu \in {}^PW$. For a fixed λ, we denote the set of all paths $\mathsf{T} \in \mathrm{Path}(\lambda)$ with $w(\mathsf{T}) = \mathsf{T}^\mu$ by $\mathrm{Path}(\lambda, \mathsf{T}^\mu)$.

Examples are given in Figure 2.

Definition 2.2. Given (W, P) a parabolic Coxeter system, we define the *matrix of light-leaves polynomials*

$$\Delta^{(W,P)} := (\Delta_{\lambda,\mu}(q))_{\lambda,\mu\in {}^PW} \qquad \Delta_{\lambda,\mu}(q) = \sum_{\mathsf{S}\in \mathrm{Path}(\lambda,\mathsf{T}^\mu)} q^{\deg(\mathsf{S})}$$

which is a (square) lower uni-triangular matrix. This matrix can be factorised *uniquely* as a product of lower uni-triangular matrices

$$N^{(W,P)} := (n_{\lambda,\nu}(q))_{\lambda,\nu\in {}^PW} \qquad B^{(W,P)} := (b_{\nu,\mu}(q))_{\nu,\mu\in {}^PW}$$

such that $n_{\lambda,\nu}(q) \in q\mathbb{Z}[q]$ for $\lambda \neq \nu$ and $b_{\nu,\mu}(q) \in \mathbb{Z}[q + q^{-1}]$. The polynomials $n_{\lambda,\nu}(q)$ are the anti-spherical Kazhdan–Lusztig polynomials of (W, P).

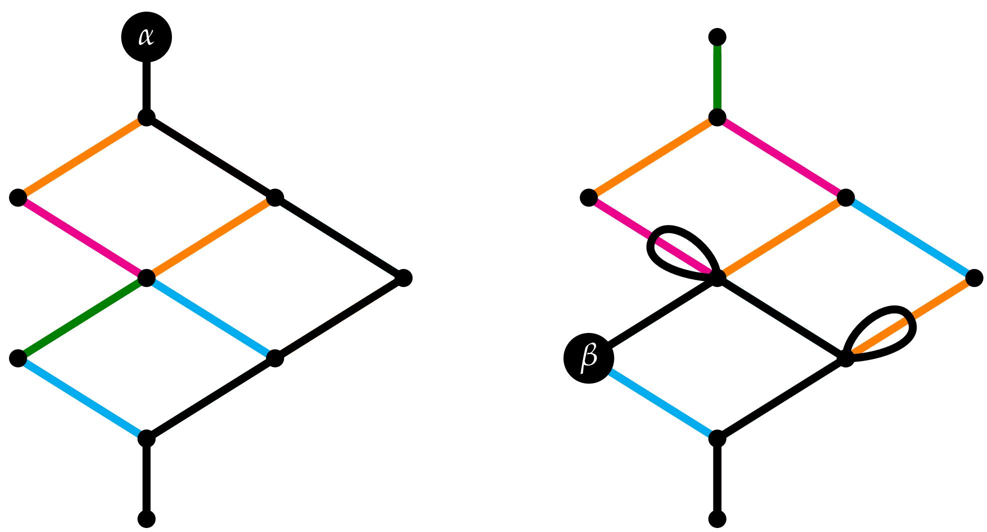


Figure 2: On the left we depict a path T^α and on the right we depict the unique element $\mathsf{S} \in \mathrm{Path}(\beta, \mathsf{T}^\alpha)$ for $\alpha = s_2s_3s_4s_1s_2s_3$ and $\beta = s_2s_1$. These are paths on $\widehat{\mathcal{G}}_{(\mathfrak{S}_5,\mathfrak{S}_2\times\mathfrak{S}_3)}$ (also known as "Bruhat strolls") but we depict only the edges in $\mathcal{G}_{(\mathfrak{S}_5,\mathfrak{S}_2\times\mathfrak{S}_3)}$ (for readability).

Example 2.3. The matrix $\Delta^{\Bbbk}$ in type $(\mathfrak{S}_4, \mathfrak{S}_2 \times \mathfrak{S}_2)$ is depicted below.

$\Delta^{\Bbbk}$	$s_2s_1s_3s_2$	$s_2s_1s_3$	s_2s_1	s_2s_3	s_2	$\varnothing$
$s_2s_1s_3s_2$	1	·	·	·	·	·
$s_2s_1s_3$	q	1	·	·	·	·
s_2s_1	·	q	1	·	·	·
s_2s_3	·	q	·	1	·	·
s_2	q	q^2	q	q	1	·
$\varnothing$	q^2	·	·	·	q	1

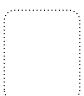
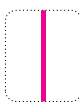
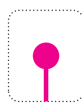
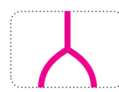
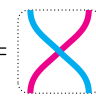
The factorisation of this matrix is trivial, with $N = \Delta^{\Bbbk}$ and $B = \mathrm{Id}_{6\times 6}$ the identity matrix.

The Hecke category (over the complex field) gives a categorification of this matrix factorisation.

3 Hecke categories and p-Kazhdan–Lusztig polynomials

Hecke categories provide the interface between Lie theory and Kazhdan–Lusztig theory. We begin by lifting the "folded paths" of the previous section to provide (what will be) a basis of the Hom-spaces of the Hecke category.

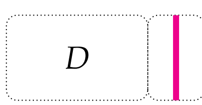
In this section, we will only explicitly discuss the generators and relations for $\mathscr{H}_{(W,P)}$, the category algebra of the Hecke category, when $W = \mathfrak{S}_{n+m}$ is a finite symmetric group and P is a maximal parabolic $P = \mathfrak{S}_m \times \mathfrak{S}_n$, as this simplifies the definitions considerably, whilst still illustrating the important points of the general case. We define the *Soergel generators* to be the framed graphs

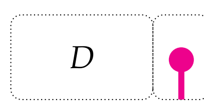
$1_{\emptyset} =$  $1_{\sigma} =$  $\mathsf{spot}_{\sigma}^{\emptyset} =$  $\mathsf{fork}_{\sigma\sigma}^{\sigma} =$  $\mathsf{braid}_{\sigma\tau}^{\tau\sigma} =$ 

associated to any pair $\sigma, \tau \in S_W$ with $m_{\sigma\tau} = 2$. We define the northern/southern reading word of any diagram obtained from horizontal and vertical concatenation of Soergel generators to be the word in the alphabet S_W which records the colours along the northern/southern edge of the frame respectively. We let $\otimes$ to be horizontal concatenation of diagrams, the algebra multiplication $\circ$ will be given by vertical concatenation in the usual manner for diagram algebras. We let $*$ denote the anti-involution which flips a diagram through the horizontal axis.

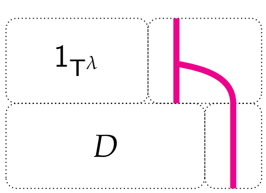
Definition 3.1. We define up and down operators on diagrams as follows

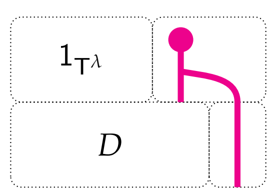
- Suppose that D has northern colour sequence T^{λ} with $\lambda\sigma > \lambda$. We define

$\mathsf{U}_{\sigma}^{1}(D) =$ 

$\mathsf{U}_{\sigma}^{0}(D) =$ 

- Now suppose that D has northern colour sequence $\mathsf{T}^{\lambda} \otimes \sigma$ with $\lambda\sigma > \lambda$. We define

$\mathsf{D}_{\sigma}^{0}(D) =$ 

$\mathsf{D}_{\sigma}^{1}(D) =$ 

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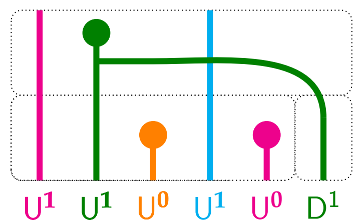
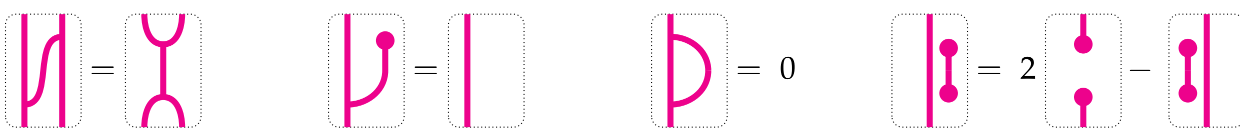
We do not emphasise the braids in our construction/notation since it will not matter if we pre- or post-multiply (at any stage of this construction) with a braid generator.

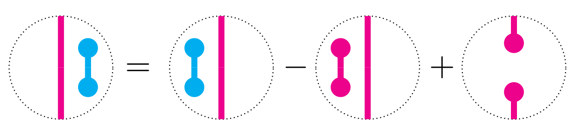
Definition 3.2. For $\mathsf{S} \in \mathrm{Path}(\lambda)$ we construct a Soergel diagram by performing the up and down operators of Definition 3.1 as we encounter any of the four up/down steps in the path. We denote the resulting diagram by c_{S}. The Soergel diagram corresponding to the path on the right hand side of Figure 2 is given below.

$c_{\mathsf{S}} =$ 

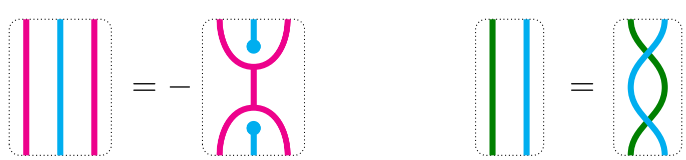
Definition 3.3. (B.-D.-H.-N. [1], Libedinsky–Williamson [10]) Let $(W, P) = (\mathfrak{S}_{n+m}, \mathfrak{S}_n \times \mathfrak{S}_m)$. The algebra $\mathscr{H}_{(W,P)}$ has a graded cellular basis given by $\{c^*_{\mathsf{S}} c_{\mathsf{T}} \,:\, \mathsf{S}, \mathsf{T} \in \mathrm{Path}(\lambda), \lambda \in \mathscr{P}_{m,n}\}$ with $\deg(c^*_{\mathsf{S}} c_{\mathsf{T}}) = \deg(\mathsf{S}) + \deg(\mathsf{T})$ with respect to the poset $(\mathscr{P}_{m,n}, \leqslant)$ and the anti-involution $*$. The multiplication is given by vertical concatenation subject to the following local relations together with their horizontal and vertical flips:



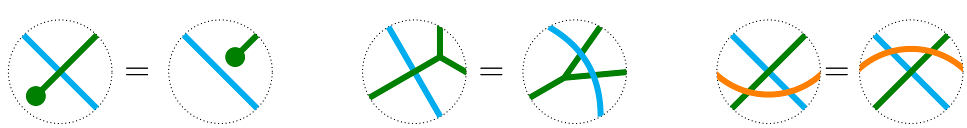
and for $m_{\sigma\tau} = 3$ we have the *2-colour barbell relation*,



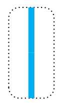
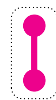
and for $m_{\sigma\tau} = 3$ and $m_{\tau\rho} = 2$ we have the *Temperley–Lieb relations*,



and for $m_{\tau\rho} = m_{\tau\pi} = m_{\rho\pi} = 2$ the *commutativity relations*,



Finally, we have the non-local *cyclotomic relations,*

 $\otimes 1_{\underline{w}} = 0$  $\otimes 1_{\underline{w}} = 0$

for $\sigma \in S_W$, $\tau \in S_P$, and $\underline{w}$ an arbitrary word for some $w \in W$.

The following theorems will hold true in the setting of arbitrary parabolic Coxeter systems (W, P). Thus we state them in that language (lifting the combinatorics from Section 2) despite the fact that we have only provided the (much simplified!) relations of the case $(W, P) = (\mathfrak{S}_{n+m}, \mathfrak{S}_n \times \mathfrak{S}_m)$.

For $\lambda \in {}^PW$ we let $\mathscr{H}^{<\lambda}_{(W,P)}$ denote the span of all diagrams $c^*_{\mathsf{S}} c_{\mathsf{T}}$ with $\mathsf{S}, \mathsf{T} \in \mathrm{Path}(\mu)$ with $\mu < \lambda$.

Theorem 3.4 (The light leaves basis [10])**.** *For each $\lambda \in {}^PW$ the graded cell module $\Delta^{\Bbbk}(\lambda)$ has a basis given by*

$$\{c_{\mathsf{S}} + \mathscr{H}^{<\lambda}_{(W,P)} \mid \mathsf{S} \in \mathrm{Path}(\lambda)\}$$

This module has a unique proper maximal submodule, $\mathrm{rad}(\Delta^{\Bbbk}(\lambda))$*, with simple quotient*

$$L^{\Bbbk}(\lambda) = \Delta^{\Bbbk}(\lambda)/\mathrm{rad}(\Delta^{\Bbbk}(\lambda))$$

Moreover, the set $\{L^{\Bbbk}(\lambda)\langle k\rangle \mid \lambda \in {}^PW, k \in \mathbb{Z}\}$ provides a complete set of pairwise non-isomorphic graded simple modules for $\mathscr{H}_{(W,P)}$.

Theorem 3.5 (The Kazhdan–Lusztig positivity conjecture, Elias–Williamson [7])**.** *Let $\Bbbk$ be a field of characteristic $p \geqslant 0$. The p-Kazhdan–Lusztig polynomials are defined to be the graded composition factor multiplicities*

$${}^pn_{\lambda,\mu}(q) = \sum_{k \in \mathbb{Z}} [\Delta^{\Bbbk}(\lambda) : L^{\Bbbk}(\mu)\langle k\rangle] q^k.$$

For $p = 0$ we have that the ${}^pn_{\lambda,\mu}(q)$ specialise to the classical Kazhdan–Lusztig polynomials of Section 2 and thus the classical Kazhdan–Lusztig polynomials have non-negative coefficients.

4 Partitions and their Dyck combinatorics

Formally, a *partition* λ of ℓ is defined to be a weakly decreasing sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ which sum to ℓ. We call $\ell(\lambda) := \ell = \sum_i \lambda_i$ the length of the partition λ. We define the Young diagram of a partition to be the collection of tiles

$$[\lambda] = \{[r, c] \mid 1 \leqslant c \leqslant \lambda_r\}$$

depicted in Russian style with rows at 135° and columns at 45° (as in Figure 3). We identify a partition with its Young diagram and we write $\lambda \subseteq \mu$ if every box of λ is contained in μ (that is $\lambda_i \leqslant \mu_i$ for all $i \geqslant 1$). We let λ^t denote the transpose partition given by reflection of the Russian Young diagram through the vertical axis. Given $m, n \in \mathbb{N}$ we let $\mathscr{P}_{m,n}$ denote the set of all partitions which fit into an $m \times n$ rectangle, that is

$$\mathscr{P}_{m,n} = \{\lambda \mid \lambda_1 \leqslant m, \lambda_1^t \leqslant n\}.$$

For $\lambda \in \mathscr{P}_{m,n}$, the x-coordinate of a tile $[r,c] \in \lambda$ is equal to $r - c + m \in \{1, 2, \ldots, m+n\}$ and we define this x-coordinate to be the "colour" or "content" of the tile and we write $\mathrm{c}ont[r,c] = r - c + m$. It is well-known that a partition is uniquely determined by the contents of its boxes and this can be seen as the main ingredient in the following result:

Proposition 4.1. *For $(W, P) = (\mathfrak{S}_{n+m}, \mathfrak{S}_n \times \mathfrak{S}_m)$ there is a poset isomorphism between $({}^PW, \leqslant)$ (the minimal coset representatives under the Bruhat ordering) and $(\mathscr{P}_{m,n}, \leqslant)$ (the partitions in an $(m \times n)$-rectangle ordered by inclusion), sending the identity coset to $\varnothing$ and the longest element to (m^n) (see Figures 1 and 3).*

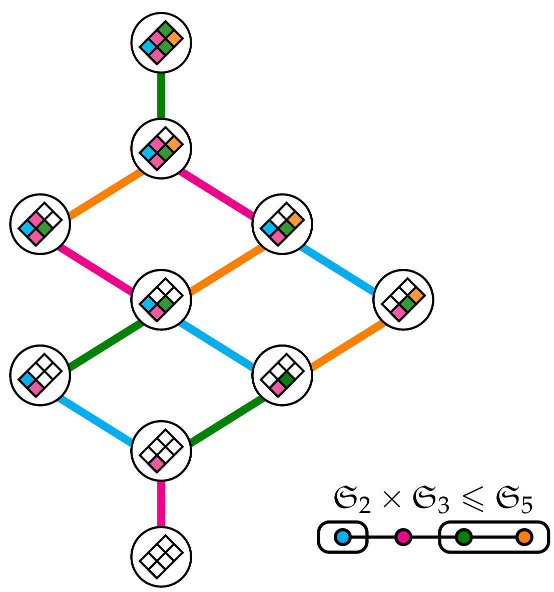


Figure 3: The partitions in a (2×3)-rectangle, ordered by inclusion. At the bottom we depict the empty partition inside a (2×3)-grid and at the top we depict the unique partition of maximal size, namely the rectangle (2^3). Compare this poset with the rightmost poset depicted in Figure 1.

Having encoded the Bruhat order in terms of partition combinatorics, we ask whether it is possible compute the Kazhdan–Lusztig polynomials in a similar fashion. The answer is yes, and makes use of the idea of Dyck paths. We define a path on λ to be a finite non-empty set P of tiles that are ordered $[r_1, c_1] \in \lambda, \dots, [r_s, c_s] \in \lambda$ for some $s \geqslant 1$ such that for each $1 \leqslant i \leqslant s-1$ we have $[r_{i+1}, c_{i+1}] = [r_i + 1, c_i]$ or $[r_i, c_i - 1]$. Note that the set $\mathsf{cont}(P)$ of contents of the tiles in a path P form an interval of integers. We say that P is a Dyck path if

$$\min\{r_i + c_i \,:\, 1 \leqslant i \leqslant s\} = r_1 + c_1 = r_s + c_s,$$

that is the minimal height of the path is achieved at the start and end of the path, see the leftmost diagram in Figure 4 for an example of a single Dyck path on a partition. We say that P and Q are *adjacent* if and only if the multiset given by the disjoint union $\mathsf{cont}(P) \sqcup \mathsf{cont}(Q)$ is an interval (see the central diagram in Figure 4 for an example).

Definition 4.2. Let $\lambda \subseteq \mu \in \mathscr{P}_{m,n}$. A Dyck tiling of the skew partition $\mu \setminus \lambda$ is a set $\{P^1, \dots, P^k\}$ of Dyck paths such that

$$\mu \setminus \lambda = \bigsqcup_{i=1}^{k} P^i$$

and for each $i \neq j$ we have P^i and P^j are not adjacent. If such a Dyck tiling exists, we call (λ, μ) a Dyck pair. Dyck tilings for a given $\mu \setminus \lambda$ are not unique. However, it can be shown that if we have two Dyck tilings $\mu \setminus \lambda = \sqcup_{i=1}^{k} P^i = \sqcup_{j=1}^{l} Q^j$ then we must have $k = l$ and there is a bijection $\{P^i\} \to \{Q^{j_i}\}$ satisfying $\mathsf{cont}(P^i) = \mathsf{cont}(Q^{j_i})$ for all $1 \leqslant i \leqslant k$. Thus it makes sense to define the degree of the Dyck pair (λ, μ) to be $\deg(\lambda, \mu) = k$.

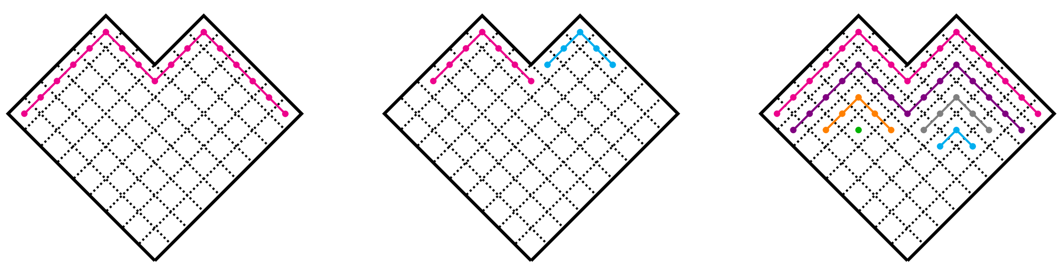


Figure 4: On the left we depict a Dyck path on $(9^6, 6^3)$. The centre diagram depicts two adjacent Dyck paths (and so $(9^6, 6^3) \setminus (9^2, 8^3, 5^3, 3)$ does not admit a Dyck tiling). On the right we depict a Dyck tiling of $(9^6, 6^3) \setminus (9, 7, 6, 5, 4, 2, 1^2)$ of degree 6.

We are now ready to provide a closed combinatorial interpretation for the p-Kazhdan–Lusztig polynomials of $(\mathfrak{S}_{n+m}, \mathfrak{S}_n \times \mathfrak{S}_m)$. This generalises existing results of Lascoux–Schutzenberger to arbitrary fields.

Theorem 4.3 (B.–D.–H.–Norton [1]). *Let $(W,P) = (\mathfrak{S}_{n+m}, \mathfrak{S}_n \times \mathfrak{S}_m)$ and $p \geqslant 0$. We have that*

$$ {}^{p}n_{\lambda,\mu}(q) = \begin{cases} q^{\deg(\lambda,\mu)} & \text{if } (\lambda,\mu) \text{ is a Dyck pair;} \\ 0 & \text{otherwise.} \end{cases} $$

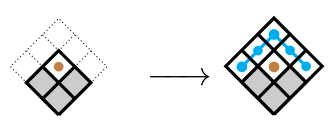
Proof. By Definition 3.3 we know that $\mathscr{H}_{(W,P)}$ has basis indexed by pairs of paths in the weak Bruhat graph of ${}^{P}W$. In [1] we provide a graded bijection between $\mathrm{Path}(\lambda, \mathsf{T}^{\mu})$ and Dyck tilings of shape $\mu \setminus \lambda$. Any Dyck tiling $\mu \setminus \lambda$ is manifestly of *positive degree*, unless $\lambda = \mu$ in which case we obtain a unique (trivial) Dyck tableau of degree zero. Now, since any graded simple module is fixed by the anti-involution $*$ we deduce that it must have graded dimension belonging to $\mathbb{Z}_{\geqslant 0}[q + q^{-1}]$. Putting together the above facts, we deduce that the simple modules are 1-dimensional (concentrated in degree zero) regardless of the characteristic of the field and the result follows. $\square$

5 Submodule lattices of cell modules

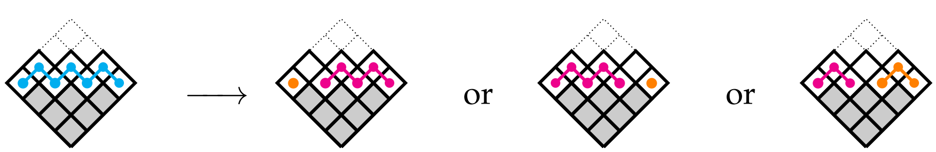
We are now ready to discuss one of the main results of [2]. Namely, we will provide the full submodule lattice of the cell modules for $\mathscr{H}_{(W,P)}$ when $(W,P) = (\mathfrak{S}_{n+m}, \mathfrak{S}_n \times \mathfrak{S}_m)$ over any field $\Bbbk$. We prove in [2] that (the basic algebra of) $\mathscr{H}_{(W,P)}$ is generated in degrees 0 and 1 and hence the grading gives a submodule filtration of $\Delta^{\Bbbk}(\lambda)$. Thus to determine whether there is an extension between two composition factors $L^{\Bbbk}(\mu)$ and $L^{\Bbbk}(\nu)$ within $\Delta^{\Bbbk}(\lambda)$ (where (λ,μ) and (λ,ν) are Dyck pairs, by Theorem 4.3) it is enough to consider pairs of adjacent degree , that is where $\deg(\lambda,\nu) = \deg(\lambda,\mu) + 1$. Using the presentations of [2, Theorem B] we are able to fully determine these extensions combinatorially as follows:

Definition 5.1. Let (λ,μ) and (λ,ν) be Dyck pairs of degree k and $k+1$ respectively. We write $(\lambda,\mu) \to (\lambda,\nu)$ if either:

- ν is obtained from μ by adding a Dyck path.



- ν is obtained from μ by removing a Dyck path, splitting some Dyck path in the tiling of $\mu \setminus \lambda$ into two distinct Dyck paths:



We extend this to a partial ordering, $\prec$, by taking the transitive closure of $\to$.

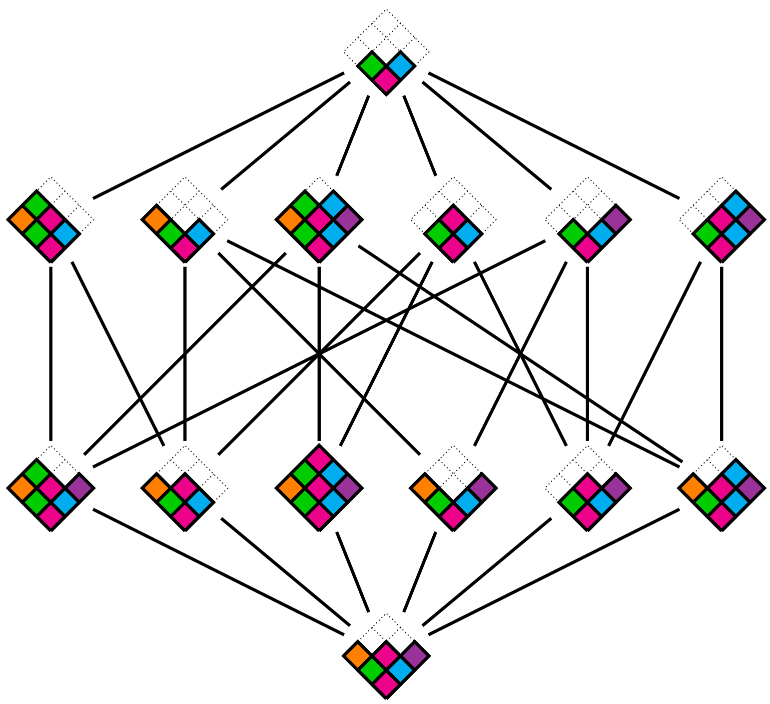
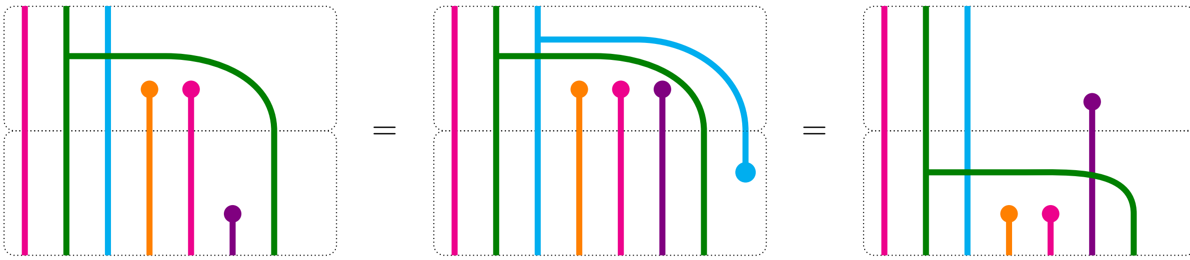


Figure 5: The submodule lattice of $\Delta^{\Bbbk}(2,1)$ for $m = n = 3$.

An example of the lattice on $\Delta^{\Bbbk}(\lambda)$ for $\lambda = (2,1)$ and $m = n = 3$ is depicted in Figure 5. With a little more work, one can prove that there is a unique Dyck pair (λ, α) of maximal degree (and that the submodule lattice is a bonafide lattice in the combinatorial sense!). Indeed we have the following:

Theorem 5.2 (B.-D.-H.-S. [2])**.** *Fix $\lambda \in {}^{P}W$ for $(W,P) = (\mathfrak{S}_{n+m}, \mathfrak{S}_n \times \mathfrak{S}_m)$. The module $\Delta^{\Bbbk}(\lambda)$ has a unique simple submodule, it is rigid (its socle and radical layers coincide) and the full submodule lattice of $\Delta(\lambda)$ is given by the partial ordering $\prec$.*

Proof. We first provide a full quiver and relations presentation of $\mathscr{H}_{(W,P)}$ and then use this to analyse the submodule structures of $\Delta^{\Bbbk}(\lambda)$. For example let $\mu = (3^2, 1)$ as in the leftmost vertex of the penultimate layer of the module of the module $\Delta^{\Bbbk}(2,1)$ depicted in Figure 5. The composition factor $L^{\Bbbk}(3^2,1)$ has three distinct paths leading into it; these come from the simple modules labelled by (3^2), $(3^2,2)$, and $(2,1^2)$ respectively. These three paths can be seen to be equal using the fork-spot relations as follows:



The remaining cases also follow by fork-spot relations, but with a little more thought required. One must then show that these relations are exhaustive — this requires the full quiver and relations presentation of $\mathscr{H}_{(W,P)}$ alluded to above. □

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