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Combinatorics of Boundary Algebras

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Abstract. Boundary algebras are an important tool in the categorification, by Jensen–King–Su and by Pressland, of cluster structures on positroid varieties, defined by Scott and by Galashin–Lam. Each connected positroid has a corresponding boundary algebra. We give a combinatorial way to recover a positroid from its boundary algebra. We then describe the set of algebras which arise as the boundary algebra of some positroid. Finally, we give the first complete description of the minimal relations in the boundary algebra. We expect this description to be helpful in extending results known for Grassmannian boundary algebras to more general settings.

Keywords: Categorification, Positroids, Cluster Algebras

1 Introduction

An *open positroid variety*, defined by Knutson–Lam–Speyer [8], is the variety of points in the Grassmannian realizing a given positroid. They broaden the positroid stratification of the nonnegative Grassmannian [11] to the full Grassmannian. As conjectured in [10, 9] and proven in [14, 15, 6], the coordinate ring of an open positroid variety has the structure of a *cluster algebra*. Such a cluster structure is a combinatorially rich algebraic structure that in particular interacts well with nonnegativity [5]. For instance, the positivity of the cluster variables in a single cluster is enough to guarantee the positivity of all the other, possibly infinitely many, cluster variables.

Boundary algebras appear in the context of *categorification* of the cluster structure on an open positroid variety. Categorification is a process by which structures from other areas of math are realized using category theory, often through module categories. In 2006, Scott [14] showed that the Grassmannian has a cluster structure, which was categorified by Jensen–King–Su [7] as the category of *Gorenstein-projective modules* over the *circle algebra*. In 2016, Baur–King–Marsh [4] connected this with dimer models by realizing the circle algebra as a completed boundary algebra of a Grassmannian dimer model.

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Pressland [12] extended this setup in 2022 by showing that the cluster structure defined by Galashin–Lam on an arbitrary connected positroid variety [6] is categorified by the Gorenstein-projective category of the completed boundary algebra of an appropriate dimer model.

This categorification has proven useful in understanding the Galashin–Lam cluster structure on positroid varieties. Pressland [13] used it to prove a conjecture of Muller–Speyer that two a priori different cluster are closely related [10]. In particular, he shows that the source-labelled and target-labelled cluster structures on a positroid variety quasi-coincide. Much work is being done on the cluster structures on Grassmannians, including studying the Gorenstein-projective modules over the circle algebra corresponding to rank 2 and 3 cluster variables [2, 3]. This work is difficult to extend to all positroid varieties because there is no generators-and-relations description in the literature for boundary algebras of general positroid varieties. Forthcoming work of the first author and Khrystyna Serhiyenko contains a combinatorial construction of the boundary algebra, but only describes the relations up to an operation called *cancellative closure*. We build on this construction and give a combinatorial description of the boundary algebra of a connected positroid variety, including a minimal set of relations. We isolate combinatorial data which determines the boundary algebra, and call it a **boundary chart**. We characterize boundary charts of connected positroids and provide an explicit bijection between realizable boundary charts and connected positroids. This gives us a new cryptomorphism of connected positroids.

Our new description of boundary algebras gives additional tools for studying the Gorenstein-projective modules over these algebras. We expect our results to be useful in generalizing the work mentioned above from the Grassmannian setting to general positroid varieties.

2 Background

2.1 Positroids

A *positroid* is a special type of realizable matroid which reflects the combinatorial structure of the *totally nonnegative Grassmannian*. See [11] for background on positroids. In this section, we introduce *perfect orientations* and *decorated permutations*, which are two of many equivalent descriptions of positroids.

Definition 2.1. *A* ***plabic graph*** *(planar bi-colored graph) is an undirected planar graph embedded in a disc with n vertices on the boundary, labelled b_i for $i \in [n]$ in clockwise order. Plabic graphs may have additional vertices in the interior of the disc which are each assigned one of two colors, either white (○) or black (●). Boundary vertices must be incident to exactly one edge. We consider plabic graphs modulo homotopy.*

There are moves and reductions that can be applied to plabic graphs which preserve the key combinatorial properties we are interested in. Using these, we may, and will, assume that our plabic graphs are bipartite (for this purpose, we ignore boundary vertices) and *reduced* [11, Def. 12.5]. An example of such a graph is illustrated in Figure 1.

We may define positroids in terms of *perfect orientations* of plabic graphs. These are orientations $\mathcal{O}$ of the edges of the plabic graph such that each white internal vertex is incident to exactly one incoming edge and each black internal vertex is incident to exactly one outgoing edge. The source set of $\mathcal{O}$ is then $\{i \mid b_i \text{ is a source in } \mathcal{O}\}$. Fix a plabic graph G on n boundary vertices. All perfect orientations have source sets of the same size k. We say such a plabic graph is of *type* (k, n). The set consisting of the source sets of all perfect orientations forms a positroid $\mathcal{P}(G)$ of rank k on $[n]$ [11].

Move equivalent plabic graphs give the same positroid. One way to distinguish move equivalence classes of plabic graphs is by using their *trip permutations*. These are permutations π defined as follow: For each $i \in [n]$, construct a trip which starts from b_i and follow the edges of the plabic graph according to the "rules of the road": At each white vertex turn right, and at each black vertex turn left. This trip will end at some boundary vertex b_j and we define $\pi(i) = j$. We obtain a fixed point $\pi(i) = i$ if and only if b_i is connected by a single edge to a leaf. At fixed points, we additionally keep track of the color of this leaf. With this additional data, we refer to π as a *decorated permutation*. Decorated permutations of $[n]$ are in bijection with positroids on $[n]$. We denote by $\mathcal{P}_\pi$ the positroid corresponding to a decorated permutation π. In this abstract, we will be primarily concerned with connected positroids, in which case the decorated permutations are *stabilized-interval-free permutations* [1]. These have no fixed points and so, in particular, are undecorated permutations.

Consider $\mathrm{Gr}_{k,n}$, the Grassmannian of k-planes in $\mathbb{C}^n$, embedded in $\mathbb{CP}^{\binom{n}{k}-1}$ by *Plücker coordinates* Δ_I, where I is a k-element subset of $[n]$. To define *open positroid varieties*, we will need to label faces F of a plabic graph G of type (k, n) by the set of $i \in [n]$ such that F lies to the left of the trip terminating at i. One can show that each such label will have size k and, if $\mathcal{P}(G) = \mathcal{P}(G')$, then the boundary faces of G and G' will have the same labels [10].

Definition 2.2. *Fix a positroid $\mathcal{P}$ of rank k on $[n]$. Let G be a reduced plabic graph such that $\mathcal{P} = \mathcal{P}(G)$. Label the faces of G as above. The* ***open positroid variety*** $\Pi^\circ_{\mathcal{P}}$ *is the subset of* $\mathrm{Gr}_{k,n}$ *where $\Delta_I = 0$ for all $I \notin \mathcal{P}$ and $\Delta_I \neq 0$ for all I which label a boundary face of G.*

Finally, we construct the *quiver* of a plabic graph.

Definition 2.3. *A* ***quiver*** *$Q = (Q_0, Q_1)$ is a directed graph with vertices Q_0 and arrows Q_1, with no loops or oriented 2-cycles. Some vertices $F_0 \subset Q_0$ may be marked as frozen.*

For a bipartite plabic graph G, define the quiver $Q(G)$ as follows: Place a vertex at each internal face of the plabic graph. Faces incident to the boundary contain frozen

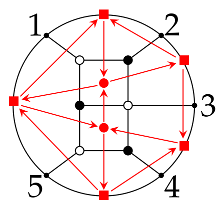

Figure 1: A plabic graph on 5 boundary vertices, with its quiver indicated in red, where frozen vertices are squares and unfrozen vertices are circles. The decorated permutation, obtained by following the rules of the road, is $\pi = 45123$.

vertices. For each edge e of G which is incident to a white vertex, add an arrow α to the quiver between the faces on either side of e such that the white endpoint of e is to the right of α. The only other edges of G are those which connect boundary vertices to black vertices. In this case, add an arrow α such that the black vertex is to the left of α. The quiver of a plabic graph is illustrated in Figure 1.

Galashin and Lam have shown that the coordinate ring of an open positroid variety $\Pi^\circ_{\mathcal{P}(G)}$ is isomorphic to the cluster algebra with quiver $Q(G)$ [6].

2.2 Boundary Algebras

We now introduce the *boundary algebra*, which is important for the categorification of the cluster structure on an open positroid variety introduced by Pressland [12].

Definition 2.4. *For a quiver Q, the **path algebra** $\mathbb{C}Q$ is spanned by finite paths in Q, including empty paths at each vertex. The product in the algebra between paths p and q is the concatenation of p and q, if p ends at the start of q, and 0 otherwise.*

Fix a stabilized-interval-free permutation π. Let G be any plabic graph with trip permutation π, and let $Q = Q(G)$. Each internal face of Q is bounded by an oriented cycle. Each edge d not between two frozen boundary vertices is incident to two faces and thus part of oriented cycles c_d^+ and c_d^- bounding those faces. Say c_d^+ factors as dp_d^+ and c_d^- factors as dp_d^-. Let e_i be the empty path at the boundary vertex v_i of $Q(G)$, and let $e = \sum_{i=1}^n e_i$.

Definition 2.5. *The **dimer algebra** A_Q is $\mathbb{C}Q$ modulo the relations $p_d^+ = p_d^-$ for all edges d of Q which are not between two frozen vertices.*

Definition 2.6. *The **boundary algebra** of the positroid $\mathcal{P}_\pi$ is $B_\pi = eA_{Q(G)}e$ for any G such that $\mathcal{P} = \mathcal{P}(G)$.*

It is not obvious, but if $\mathcal{P}(G) = \mathcal{P}(G')$, then $eA_{Q(G)}e = eA_{Q(G')}e$, so this is well defined. Multiplication by e on both sides discards each path which neither originates from nor terminates at a boundary vertex. Thus, B_π can be thought of as the algebra of paths between boundary vertices, modulo the relations $p_d^+ = p_d^-$.

We define the following elements of B_π: For each $i \in [n]$, let x_i be a minimal path from v_i to v_{i+1} and let y_i be a minimal path from v_{i+1} to v_i, where all indices are taken modulo n. Let $x = \sum_{i=1}^n x_i$ and $y = \sum_{i=1}^n y_i$. Let $t = xy$; then t is central in B_π. Let p be a path in B_π between two nonadjacent vertices. Define $\tau(p)$ and $\eta(p)$ by the condition that p is directed from $v_{\tau(p)}$ to $v_{\eta(p)}$. Suppose p does not factor as a product of other paths. We then say p is a *nonadjacent arrow* and define $\text{reach}_p = \eta(p) - \tau(p)$ taken modulo n so that $\text{reach}_p \in [n]$. One can show that p satisfies $pt^{X_p} = x_{\tau(p)} x_{\tau(p)+1} \cdots x_{\eta(p)-1}$ for a suitable positive integer X_p. We represent the boundary algebra by putting the vertices v_i in clockwise order around a circle and drawing in the nonadjacent arrows. For example, the second subfigure of Figure 2 shows the representation of a boundary algebra. with a nonadjacent arrow from v_4 to v_1 in black. In this figure, the arrows x_i and y_i are shown in gray.

In forthcoming work, the first author and Khrystyna Serhiyenko prove the following two results showing how to calculate the boundary algebra B_π directly from a stabilized-interval-free permutation π. In order to state these results, we must represent the permutation π as a directed graph on vertices w_i with directed edges from w_i to $w_{\pi(i)}$ for $i \in [n]$. We will draw the permutation graph such that w_i lies between vertices v_i and v_{i+1}. We refer to edges of the permutation graph as *strands*. Figure 2 shows three examples of permutation graphs, in red.

Definition 2.7. *For $i \in [n]$, we define the **i-shifted linear order** $<_i$ on $[n]$ by $i <_i i+1 <_i \cdots <_i n <_i 1 <_i \cdots <_i i-1$. We say $(a_1 \le \cdots \le a_m) \in [n]^m$ is a **cyclic ordering** if there exists some $i \in [n]$ such that $a_1 \le_i a_2 \le_i \cdots \le_i a_m$. We will allow ourselves to replace some or all of the inequalities with strict inequalities if consecutive terms are not allowed to be equal.*

Definition 2.8. *Define $[i, j] = \{l \mid (i \le l \le j) \text{ is a clockwise ordering}\}$ to be the (closed) clockwise interval between i and j. We similarly define the clockwise intervals $(v_i, v_j]$, $[v_i, v_j)$, and (v_i, v_j) by excluding one or both of the endpoints.*

Definition 2.9. *Let v_j be a boundary vertex of Q. Consider a strand α from r to $\pi(r)$ in the permutation graph of π. We say that v_j is **to the right of** α if $j \in [\pi(r), r)$ and otherwise v_j is **to the left of** α. Let p be an arrow between nonadjacent vertices v_i and v_j. We say that α is **left-supporting** to p if $(\pi(r) \le r \le \eta(p) \le \tau(p))$ is a cyclic ordering. We say that the strand α is **right-supporting** to p if $(r \le \pi(r) \le \tau(p) \le \eta(p))$ is a cyclic ordering. In either case, we say that this strand is **supporting** to p.*

Informally, a strand left (resp. right) supports an arrow p if it points in the opposite direction of p and lies to its left (resp. right).

Theorem 2.10. *Fix a connected positroid $\mathcal{P}$ with permutation π. Fix distinct nonadjacent boundary vertices v_i and v_j. The arrow p from v_i to v_j is a nonadjacet arrow of B_π if and only if*

1. *for all $l \in (j,i)$, there is a right-supporting strand α of p with v_l to its left, and*

2. *for all $l \in (i,j)$, there is a left-supporting strand α of p with v_l to its right.*

*In this case, the relations $pt^{X_p} = x_i x_{i+1} \cdots x_{j-1}$ and $pt^{Y_p} = y_{i-1} y_{i-2} \cdots y_j$ hold in B_π, where X_p is the number of left-supporting strands and Y_p is the number of right-supporting strands of p. We call X_p the **left relation number** of p, and we call Y_p the **right relation number** of p.*

Definition 2.11. *An ideal I of a path algebra $\mathbb{C}Q$ (generated by commutation relations) is **cancellative** if, for any $a, p, q, b \in I$ with $\eta(a) = \tau(p) = \tau(q)$ and $\eta(p) = \eta(q) = \tau(b)$, we have $apb - aqb \in I \iff p - q \in I$. The **cancellative closure** of an ideal I, denoted $\mathrm{CancClos}(I)$, is the smallest cancellative ideal containing I.*

Theorem 2.12. *Let Q_π° be the quiver on v_i, for $i \in [n]$, whose arrows are x_i, y_i and the nonadjacent arrows of B_π. Let I_π° be the cancellative closure of the ideal I generated by the relations given in Theorem 2.10, the relations $xy - yx$ and $x^k - y^{n-k}$:*

$$I_\pi^\circ = \mathrm{CancClos}\left(\left\langle xy - yx, x^k - y^{n-k}, \sum_{p \text{ nonadjacent arrow}} pt^{X_p} - x_{\tau(p)} x_{\tau(p)+1} \cdots x_{\eta(p)-1} \right\rangle\right).$$

Then $B_\pi \equiv \mathbb{C}Q_\pi^\circ / I_\pi^\circ$.

Together, Theorem 2.10 and Theorem 2.12 give a way to calculate the boundary algebra of a positroid from its decorated permutation. However, this is obfuscated by the necessity of taking a cancellative closure. We address this in Section 3.3.

Example 2.13. *Let π be the permutation depicted in the middle of Figure 2. We see that there is an arrow p from v_1 to v_4. Since p has one left-supporting strand (from 6 to 4) and one right-supporting strand (from 1 to 3), it is labelled with $X_p : Y_p = 1 : 1$. By Theorem 2.12, the boundary algebra is $B_\pi \cong \mathbb{C}Q_\pi^\circ / I_\pi^\circ$, where the arrows of Q_π° are p along with the greyed-out arrows (representing x_i and y_i) and the ideal I_π° is generated up to cancellative closure by*

$$\{xy - yx,\ x^k - y^{n-k},\ pt - x_4 x_5 x_6\}.$$

3 Combinatorial construction of the boundary algebra

3.1 From Boundary Algebras to Permutations

Our work involves a new combinatorial object, which we call a *boundary chart*.

Definition 3.1. *A **boundary chart** consists of data $C = (k, n, S, X)$ as follows, where k and n are integers satisfying $1 \le k \le n$ and S is a set of arrows on vertices v_i, for $i \in [n]$, such that arrows are not between v_i and $v_{i\pm 1}$, with indices taken cyclically, and there is at most one arrow from v_i to v_j for any $i, j \in [n]$. Finally, $X \in \mathbb{Z}_{>0}^{|S|}$ gives a positive integer for each arrow.*

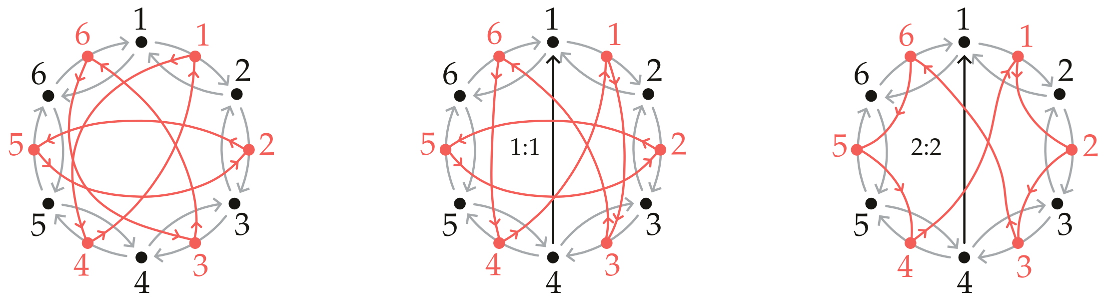

Figure 2: Three boundary charts on six vertices with $k = 3$ and their corresponding decorated permutations depicted in red. The arrows x_i and y_i are greyed out. For clarity, we use a red i in place of w_i and a black i in place of v_i.

Given a positroid, the boundary chart is precisely the data determined in Theorem 2.10. Here, we start with a connected positroid of rank k on $[n]$ with permutation π, and the set S consists of the nonadjacent arrows of B_π. The numbers X_p are as in the theorem. Observe that it is implicit in Theorem 2.10 that there is at most one arrow between any two vertices. By Theorem 2.12, this information fully determines B_π.

Definition 3.2. *A boundary chart is* ***realizable*** *if it is obtained from a positroid via this process.*

We introduce the following auxiliary piece of data for a boundary chart: Let $Y \in \mathbb{Z}_{>0}^{|S|}$ be a vector of positive integers, indexed by the arrows in S, such that $Y_p = X_p + k - \text{reach}_p$ for each $p \in S$. Using $xy = yx$ and $x^k = y^{n-k}$ from Theorem 2.12, one can show that the Y_p here coincide with the Y_p in Theorem 2.10.

We represent the data in a boundary chart by placing the vertices v_i around a circle, in clockwise order. We draw in the arrows and mark each arrow $p \in S$ with the pair of integers $(X_p : Y_p)$. We refer to these as the left and right relation numbers of p, respectively. Note that knowing X_p and Y_p suffices to recover k when the set S of nonadjacent arrows is nonempty. Three examples are illustrated in black in Figure 2 (the red and grey parts are not in the boundary chart). For the realizable boundary chart obtained from $\mathcal{P} = \mathcal{P}_\pi$, this coincides with the representation of B_π described in section Section 2.2. We will need the following terminology:

Definition 3.3. *Let p and q be two arrows in S.*

- *If $(\tau(p) < \tau(q) < \eta(p) < \eta(q))$ is a cyclic ordering, we say p and q* ***cross****.*
- *If $(\tau(p) \leq \tau(q) < \eta(q) \leq \eta(p))$ is a cyclic ordering, then p lies to the right of q. We say that p and q are* ***parallel****, with p* ***right-parallel*** *of q. Define* ***left-parallel*** *similarly.*
- *If $(\eta(p) \leq \tau(q) < \eta(q) \leq \tau(p))$ is a cyclic ordering, then p lies to the right of q. We say p and q are* ***antiparallel****, with p* ***right-antiparallel*** *of q. Define* ***left-antiparallel*** *similarly.*

Visually, crossing arrows are arrows that intersect on their interiors, like the arrow from 9 to 4 and the arrow from 7 to 2 in Figure 3. Parallel arrows are, roughly, arrows that go in the "same direction", like the arrow from 9 to 4 and the arrow from 10 to 1 in Figure 3, while antiparallel arrows are arrows which neither cross nor are parallel.

Given a realizable boundary chart $C = (k, n, S, X)$, we describe how to recover the permutation of its associated boundary algebra. We will construct the permutation as a permutation graph on vertices w_i such that w_i lies between vertices v_i and v_{i+1}. We refer to the vertices w_i as strand vertices. First, we must discuss the ideas of visibility and of influence in a boundary chart. We do this inductively. We will give an example of these definitions in Example 3.8. We start with a reformulation of the right and left relation numbers which will make the following construction easier to state.

Definition 3.4. *Let C be a boundary chart with left and right relation numbers X and Y, respectively. For $p \in S$, let $L(p)$ (resp. $R(p)$) be the set of arrows left-parallel (resp. right-parallel) to p. Then the **adjusted left relation numbers** are defined inductively by $X'_p = X_p - \sum_{q \in L(p)} X'_q$ and the **adjusted right relation numbers** are defined inductively by $Y'_p = Y_p - \sum_{q \in R(p)} Y'_q$.*

Definition 3.5. *Let α be an arrow of a realizable boundary chart $C = (k, n, S, X)$. Inductively define the following:*

1. *The **right-head-visible** strand vertices to α are those strand vertices to the right of α which are not right-head-influenced by a right-parallel arrow of α or left-head-influenced by a right-antiparallel arrow of α (this condition is vacuous if there are no arrows to the right of α). The **right-head-influenced** strand vertices of α are the Y'_α right-head-visible strand vertices most immediately clockwise of $\eta(\alpha)$.*

2. *The **left-head-visible** strand vertices of α are those strand vertices to the left of α which are not left-head-influenced by a left-parallel arrow of α or right-head-influenced by a left-antiparallel arrow of α. The **left-head-influenced** strand vertices of α are the X'_α left-head-visible strand vertices most immediately counterclockwise of $\eta(\alpha)$.*

*We will use the phrase **head-influenced** to mean either left- or right-head-influenced. We define **left-tail visible** and **right-tail-visible** strand vertices as above, swapping "head" with "tail," "$\eta(\alpha)$" with "$\tau(\alpha)$," and "clockwise" with "counter-clockwise."*

Construction 3.6. *Let $C = (k, n, S, X)$ be a realizable boundary chart. Let α be an arrow of C. Let $w_{\sigma_1}, \ldots, w_{\sigma_{X'(\alpha)}}$ be the right-head-influenced strand vertices of α, ordered clockwise so that w_{σ_1} is most immediately clockwise of $\eta(\alpha)$. Let $w_{\sigma'_1}, \ldots, w_{\sigma'_{X'(\alpha)}}$ be the right-tail-influenced strand vertices of α, ordered clockwise so that $w_{\sigma_{X'(\alpha)}}$ is most immediately counter-clockwise of $\tau(\alpha)$. Then define $\phi_\alpha(\sigma_j) = \sigma'_j$ for $j \in X'(\alpha)$. Symmetrically define ϕ_α on the left-head-influenced vertices of α. We define a function (indeed, we will see, a permutation) π on $[n]$ by*

$$\pi(j) = \begin{cases} \phi_\alpha(j) & w_j \text{ is head-influenced by an arrow } \alpha \in C \\ j - k & \text{otherwise.} \end{cases}$$

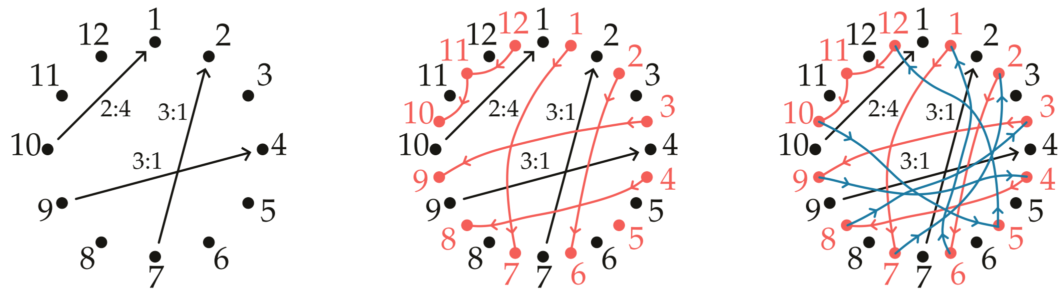

Figure 3: Application of Construction 3.6 to a boundary chart with 12 vertices and $k = 5$. For clarity, we use a red i in place of w_i and a black i in place of v_i.

It is not immediately obvious that π is well-defined; for example, some strand vertex w_j may be head-influenced by two arrows α and β. In fact, whenever this happens, $\phi_\alpha(j) = \phi_\beta(j)$.

Theorem 3.7. *The map π is a well-defined stable-interval-free permutation, and C is the boundary chart of B_π.*

This process is most easily understood visually, in an example. We denote nonadjacent arrows in the boundary algebra from v_i to v_j by $p_{i\to j}$.

Example 3.8. *We look at Figure 3. The first subfigure shows a boundary chart on 12 vertices. We have $Y'_{p_{7\to 2}} = 1$, hence the only right-head-influenced strand vertex of $p_{7\to 2}$ is w_2 and its only right-tail-influenced strand vertex is w_6 and we see that $\pi(2) = 6$. Similarly, the right-influence of $p_{9\to 4}$ induces $\pi(4) = 8$. We have $Y'_{p_{10\to 1}} = Y_{p_{10\to 1}} - Y_{p_{7\to 2}} - Y_{p_{9\to 4}} = 2$, so the right-head-influence of $p_{10\to 1}$ is $\{w_1, w_3\}$ (skipping over w_2, which is right-head-influenced by the right-parallel arrow $p_{7\to 2}$) and the right-tail-influence of $p_{10\to 1}$ is $\{w_7, w_9\}$ (skipping over w_8, which is right-tail-influenced by $p_{9\to 4}$). Then we see $\pi(1) = 7$ and $\pi(3) = 9$. The middle of Figure 3 shows in red all strands induced by the influence of an arrow; the right completes the permutation graph by adding in blue the remaining strands from w_j to w_{j-k}.*

3.2 Realizable Boundary Charts

We next classify realizable boundary charts.

Theorem 3.9. *Let $C = (k, n, S, X)$ be a boundary chart with left and right relation numbers X and Y, respectively, and with adjusted left and right relation numbers X' and Y', respectively. For $p \in S$, let $R_{\nparallel}(p)$ and $L_{\nparallel}(p)$ denote the sets of arrows right and left-antiparallel to p, respectively. Then C is realizable if and only if the following hold.*

1. *For all $p \in S$, $X_p + \sum_{q \in L_{\nparallel}(p)} Y'_q < \mathrm{reach}_p$ and $Y_p + \sum_{q \in R_{\nparallel}(p)} X'_q < n - \mathrm{reach}_p$.*

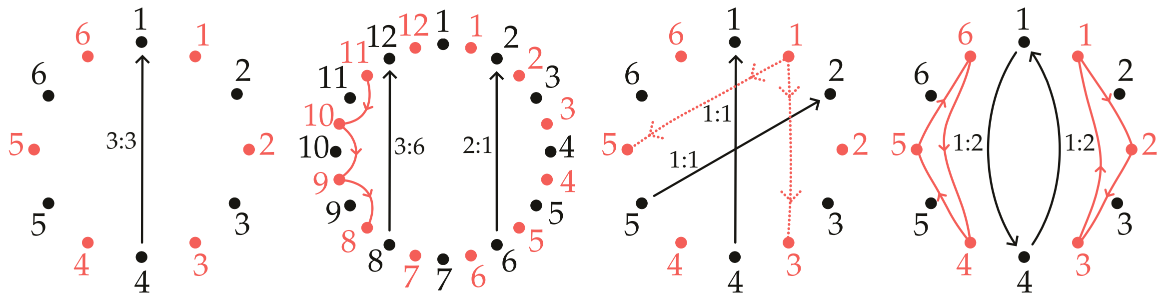

Figure 4: Four *nonrealizable* boundary charts violating the conditions of Theorem 3.9. For clarity, we use a red i in place of w_i and a black i in place of v_i.

2. *For all $p \in S$, $X'_p \geq 0$ and $Y'_p \geq 0$.*
   (a) *If $X'_p = 0$ (resp. $Y'_p = 0$), there must be crossing arrows q and r, both left (rep. right) parallel to p, such that $\tau(p) = \tau(q)$ and $\eta(p) = \eta(r)$.*

3. *Let $p, q \in S$ be crossing such that $(v_i = \eta(p) < v_j = \eta(q) < \tau(p) < \tau(q))$ is a cyclic ordering. Let A_p denote the set of arrows r right-parallel to p such that $\eta(r) \in [\eta(p), \eta(q))$. Let A_q denote the set of arrows r left-parallel to q such that $\eta(r) \in (\eta(p), \eta(q)]$. Then $Y'_p + X'_q + \sum_{r \in A_p} Y'_r + \sum_{r \in A_q} X'_r \leq j - i$, where the right side is taken modulo n so that it lies in $[n]$.*

4. *If $p, q \in S$ form an oriented digon, then $X_p + X_q + Y_p + Y_q < n$.*

One may attempt to apply Construction 3.6 to general boundary algebras. However, the conditions of Theorem 3.9 are necessary in order for the result to be a well-defined stable-interval-free permutation. For example, condition 1 ensures that there are enough strand vertices to the right of any $p \in S$ to count out Y'_p right-influenced strand vertices. In the first subfigure of Figure 4, there are not enough red vertices for $p_{4\to 1}$ to have three left-supporting strands. The second subfigure in Figure 4 violates condition 2, as $X'(p_{6\to 2}) = -1$. The left-supporting strands of $p_{8\to 12}$ also left-support $p_{6\to 2}$, so $p_{6\to 2}$ has too many left-supporting strands. Condition 3 ensures that, if p and q are arrows of C which both influence w_j, then $\phi_p(j) = \phi_q(j)$. See the third subfigure of Figure 4, where the two crossing arrows are pulling the strand starting at w_1 in different directions. Condition 4 ensures the permutation constructed in Construction 3.6 is stable-interval-free; see the fourth subfigure of Figure 4, where the permutation fixes $[1, 3]$.

The sufficiency of these conditions is more surprising. We prove sufficiency by showing that Construction 3.6 and the map of Theorem 2.10 compose to the identity on the boundary charts satisfying the conditions of Theorem 3.9. Hence, we may view the combinatorial conditions of Theorem 3.9 as a new cryptomorphism for connected positroids.

3.3 Minimal relations

Recall the presentation of the boundary algebra $B_\pi \cong \mathbb{C}Q_\pi^\circ / I_\pi^\circ$ given in Theorem 2.12, which has the drawback of the ideal I_π° being defined using a cancellative closure. In this section, we give a description of the minimal relations of the ideal I_π° using the information of the permutation π and the boundary chart $C = (k, n, S, X)$.

Definition 3.10. *Given an arrow $p \in S$ of C, let R_p (respectively T_p) be the vertex w_i of the permutation graph most immediately clockwise of $\eta(p)$ (resp. $\tau(p)$) which is the start (resp. end) of a strand which crosses p (i.e., which starts to the right of p and ends to the left, or vice versa).*

- *Two (necessarily parallel) arrows p and q are **stitch-equivalent** if $R_p = R_q$ and $T_p = T_q$.*
- *A strand is **relation-defining** if it does not travel from R_p to T_p for any arrow p.*

Example 3.11. *In Figure 3, the arrows $p_{10\to1}$ and $p_{9\to4}$ are stitch-equivalent to each other with $T_{10\to1} = T_{9\to4} = 12$ and $R_{10\to1} = R_{9\to4} = 5$, but not to the arrow from 7 to 2, with $T_{7\to2} = 8$ and $R_{7\to2} = 3$. Every strand is relation-defining with the exception of the strand from 5 to 12.*

Definition 3.12. *For v_a and v_b boundary vertices of B_π, define the **aggressive clockwise path** $\mathbf{ACL}(v_a, v_b)$ **from** v_a **to** v_b by starting at v_a and repeatedly taking the non-y_j arrow which ends most immediately counter-clockwise of v_b. Similarly define the **aggressive counter-clockwise path** $\mathbf{ACC}(v_a, v_b)$. When these paths are equivalent, we say that the **aggressive relation from** v_a **to** v_b is $[\mathrm{ACL}(v_a, v_b)] - [\mathrm{ACC}(v_a, v_b)]$.*

Theorem 3.13. *The following relations of $\mathbb{C}Q_\pi^\circ$, along with the relation $x_i y_i = y_{i-1} x_{i-1}$ for each $i \in [n]$, form a minimal generating set for I_π°:*

1. *For every relation-defining strand from w_a to w_b, take the aggressive relation from v_b to v_a.*
2. *Let $\{p_1, \ldots, p_m\}$ be a stitch-equivalence class, ordered left to right, with $T_{p_i} = T$ and $R_{p_i} = R$ for all $i \in [m]$. Define $v_{a_{m+1}} := T$, $v_{b_0} := R$, and $v_{a_i} := \tau(p_i)$, $v_{b_i} := \eta(p_i)$ for $i \in [m]$. Then, take the aggressive relation from v_{a_i} to $v_{b_{i-1}}$ for each $i \in [m+1]$.*

Example 3.14. *Consider the boundary chart of Figure 3. The strand from w_1 to w_7 is relation-defining and yields the relation $p_{7\to2} y_1 = x_7 x_8 x_9 p_{10\to1}$. All of the strands $w_j \mapsto w_{j-k}$, drawn in blue, give relations composed only of x's and y's. For example, the strand from w_5 to w_{12} gives $x_{12} x_1 \cdots x_4 = y_{11} y_{10} \cdots y_5$. There are two stitch-equivalence classes $\{p_{10\to1}, p_{9\to4}\}$ and $\{p_{7\to2}\}$. The former gives $\{y_{11} y_{10} p_{10\to1} = x_{12},\ p_{10\to1} x_1 x_2 x_3 = y_9 p_{9\to4},\ p_{9\to4} x_4 = y_8 y_7 y_6\}$ and the latter gives $\{p_{7\to2} x_2 = y_6 y_5 y_4 y_3,\ x_8 x_9 p_{10\to1} x_1 = y_7 p_{7\to2}\}$. These five relations and those given by relation-defining strands make up all minimal relations of I_π°.*

Note that Theorem 3.13 uses both the boundary chart and the permutation obtained from it by Construction 3.6. It would be hard to rephrase the theorem in terms of one or the other. This indicates that boundary charts and stabilized-interval-free permutations, while both cryptomorphisms of connected positroids, highlight different information.

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