*Séminaire Lotharingien de Combinatoire* **91B** (2024)
Article #45, 12 pp.

*Proceedings of the 36$^{th}$ Conference on Formal Power Series and Algebraic Combinatorics (Bochum)*

# Brewing Fubini-Bruhat Orders

Sara C. Billey*[1] and Stark Ryan†[2]

[1]*Department of Mathematics, University of Washington, Seattle, WA*
[2]*Ghost Autonomy, Mountain View, CA*

**Abstract.** The Bruhat order on permutations arises out of the study of Schubert varieties in Grassmannians and flag varieties, which have been important for over 100 years [3, 5, 8, 13, 14]. The purpose of this paper is to study variations on this theme related to subvarieties of the spanning line configurations $X_{n,k}$ as defined by Pawlowski and Rhoades [16]. These subvarieties are indexed by Fubini words, or equivalently by ordered set partitions. Three natural partial orders arise in this context; we refer to them as the decaf, medium roast, and espresso orders. The decaf order is a generalization of the weak order on permutations defined by covering relations using simple transpositions. The medium roast order is a generalization of the (strong) Bruhat order defined by the closure relationship on the subvarieties. The espresso order is the transitive closure of a relation based on intersecting subvarieties. Many properties of Schubert varieties and Bruhat order extend to one or more of the three Fubini-Bruhat orders. We examine some of the many possibilities in this work.

**Keywords:** Fubini words, ordered set partitions, Schubert varieties, permutations

# 1 Introduction

For positive integers $k \leq n$, a **Fubini word** $w = w_1 \cdots w_n$ represents a surjective map $w : [n] \to [k]$. We denote a Fubini word by its **one-line notation**, an ordered list $w = w_1 w_2 \cdots w_n$, where $w_i = w(i)$. We denote by $\mathcal{W}_{n,k}$ the Fubini words of length $n$ on the alphabet $[k]$. For $k = n$, a Fubini word $w \in \mathcal{W}_{n,n}$ is exactly a permutation in $S_n$, and the one-line notation for $w$ is the same whether $w$ is viewed as a Fubini word or a permutation. The bijection between Fubini words and ordered set partitions maps $w \in \mathcal{W}_{n,k}$ to $\mathrm{B}(w) = \mathrm{B}_1 \,|\, \mathrm{B}_2 \,|\, \ldots \,|\, \mathrm{B}_k$ where $\mathrm{B}_i = \{j \in [n] \,|\, w_j = i\}$. Hence the number of Fubini words in $\mathcal{W}_{n,k}$ is $k!S(n,k)$ where $S(n,k)$ is the Stirling number of the second kind [15, A000670, A019538].

Let $\mathcal{F}_{k \times n}(\mathbb{C})$ be the set of full rank $k \times n$ matrices with no zero columns. Such matrices have a Bruhat decomposition into orbits

*billey@math.washington.edu. The authors were partially supported by the Washington Research Foundation and DMS-1764012.

†drstarkryan@gmail.com

$$\mathcal{F}_{k\times n}(\mathbb{C}) = \bigsqcup_{w\in\mathcal{W}_{n,k}} B_-^{(k)} M_w B_+^{(n)} \tag{1.1}$$

where $M_w$ is the analog of a permutation matrix with a 1 in position $(w_j, j)$ and 0's elsewhere, $B_-$ and $B_+$ are the set of invertible lower and upper triangular matrices respectively and the superscript indicates their size. Every matrix in the double orbit $B_-^{(k)} M_w B_+^{(n)}$ can be written in many ways as a triple product, thus it can be useful to chose canonical representatives. Let $U = U_-^{(k)}$ be the set of lower unitriangular matrices in $GL_k(\mathbb{C})$, and let $T = T^{(n)}$ be the set of diagonal matrices in $GL_n(\mathbb{C})$. Pawlowski and Rhoades [16] defined the **pattern matrices** $P_w$ indexed by words $w \in \mathcal{W}_{n,k}$ to be a specific set of orbit representatives such that each $M \in B_-^{(k)} M_w B_+^{(n)}$ can be written uniquely as a product $M = XYZ$ with $X \in U, Y \in P_w$, and $Z \in T$ [16, Lem. 3.1 and Prop. 3.2]. See Section 2 for more details. Thus, we have an **efficient Bruhat decomposition**

$$\mathcal{F}_{k\times n}(\mathbb{C}) = \bigsqcup_{w\in\mathcal{W}_{n,k}} U P_w T. \tag{1.2}$$

Under right multiplication, every $T$-orbit of $\mathcal{F}_{k\times n}(\mathbb{C})$ determines an ordered list of $n$ 1-dimensional subspaces whose vector space sum is $\mathbb{C}^k$ via its ordered list of columns. The set of such "lines" in $\mathbb{C}^k$ is the $(k-1)$-dimensional complex projective space $\mathbb{P}^{k-1}$.

**Definition 1.1.** *[16, Def. 1.3] A* ***spanning line configuration*** $l_\bullet = (l_1, \ldots, l_n)$ *is an ordered $n$-tuple in the product of projective spaces $(\mathbb{P}^{k-1})^n$ whose vector space sum is $\mathbb{C}^k$. Let*

$$X_{n,k} = \mathcal{F}_{k\times n}(\mathbb{C})/T = \{l_\bullet = (l_1, \ldots, l_n) \in (\mathbb{P}^{k-1})^n \mid l_1 + \cdots + l_n = \mathbb{C}^k\} \tag{1.3}$$

*be the* ***space of spanning line configurations*** *for $1 \le k \le n$.*

In 2017, Pawlowski and Rhodes initiated the study of the space of spanning line configurations [16]. They observed and proved the following remarkable properties. The projection of $X_{n,n} = GL_n/T$ to the flag variety $GL_n/B_+^{(n)}$ is a homotopy equivalence, so they have isomorphic cohomology rings. More generally, $X_{n,k}$ is an open subvariety of $(\mathbb{P}^{k-1})^n$, hence it is a smooth complex manifold of dimension $n(k-1)$. The cohomology ring of $X_{n,k}$ may be presented as the ring

$$R_{n,k} = \mathbf{Z}[x_1, \ldots, x_n]/\langle x_1^k, \ldots, x_n^k, e_{n-k+1}, \ldots, e_n\rangle$$

defined by Haglund-Rhoades-Shimozono [12], generalizing the coinvariant algebra and Borel's theorem $H^*(GL_n/B) \cong R_{n,n}$. Here, $e_i$ is the $i^{th}$ elementary symmetric function in $x_1, \ldots, x_n$. Furthermore, there is a natural $S_n$ action on $n$-tuples of lines inducing an $S_n$ action on the cohomology ring of $X_{n,k}$, which is isomorphic to $R_{n,k}$ as a graded $S_n$-

module. See also [11] for another geometric interpretation of $R_{n,k}$. The efficient Bruhat decomposition gives rise to a cellular decomposition

$$X_{n,k} = \bigsqcup_{w \in \mathcal{W}_{n,k}} UP_w.$$

Let $C_w = UP_w$ for $w \in \mathcal{W}_{n,k}$. Let $\overline{C}_w$ be the closure of the cell $C_w$ in Zariski topology on on $X_{n,k}$. Then the cohomology classes $[\overline{C}_w]$ can be represented by variations on Schubert polynomials and these polynomials descend to a basis of $R_{n,k}$ over $\mathbb{Z}$ [16, Sec. 1, Prop 3.4]. The Poincaré polynomial for $H^*(X_{n,k}, \mathbb{Z})$ is determined by

$$\sum_{w \in \mathcal{W}_{n,k}} q^{\operatorname{codim}(C_w)} = [k]!_q \cdot \text{rev-Stir}_q(n,k), \tag{1.4}$$

where $\text{rev-Stir}_q(n,k)$ is the polynomial obtained by reversing the coefficients of a well-known $q$-analog of the Stirling numbers of the second kind [4, 17, 19].

Given the impressive results due to Pawlowski and Rhoades, we call $C_w = UP_w$ the **Pawlowski-Rhoades cell** or **PR cell** indexed by $w \in \mathcal{W}_{n,k}$. Similarly, the **PR variety** is denoted $\overline{C}_w$. The PR cells and PR varieties are natural variations on the theory of Schubert cells/varieties extending to $k \times n$ matrices, hence we believe they merit careful study of their own. We have used known theorems for Schubert varieties as inspiration for conjectures and results on PR varieties.

It follows from [16, Sec. 5] that the PR variety $\overline{C}_w$ is defined by certain bounded rank conditions. The rank conditions give rise to the ideal $I_w$ generated by the minor determinants $\Delta_{I,J} \in \mathbb{C}[x_{11}, \ldots, x_{kn}]$ for $I, J \in \binom{[n]}{h}$ with $h \in [k]$ which vanish on every matrix in $C_w = UP_w$. The zero set of these minors is well defined on the orbits in $\mathcal{F}_{k\times n}(\mathbb{C})/T$ since the right action of the diagonal matrices just rescales each such minor. Therefore, the spanning line configurations in $\overline{C}_w$ can be represented by matrices in $\mathcal{F}_{k\times n}(\mathbb{C})$ that vanish for every minor generating $I_w$.

**Definition 1.2.** *[16, Sec. 9] The* ***medium roast Fubini-Bruhat order*** *$(\mathcal{W}_{n,k}, \leq)$ is defined on Fubini words by $v \leq w$ if and only if one of the following equivalent statements is true:*

1. *$I_v \subset I_w$,*

2. *$\overline{C}_v \supseteq C_w$,*

3. *$\{(I,J) \mid \Delta_{I,J}(M) = 0 \ \forall M \in C_v\} \subset \{(I,J) \mid \Delta_{I,J}(M) = 0 \ \forall M \in C_w\}$.*

One can observe that medium roast order on Fubini words is equivalent to Bruhat order on permutations when $n = k$. As with Bruhat order, it follows by definition that $v < w$ implies $\operatorname{codim}(C_v) < \operatorname{codim}(C_w)$. However, some of the properties for Bruhat order on $S_n = \mathcal{W}_{n,n}$ do not extend to all $\mathcal{W}_{n,k}$. Specifically, if $v \leq w$ in $\mathcal{W}_{n,k}$, then

$\overline{C}_v \cap C_w \neq \emptyset$, but the converse does not necessarily hold. For example, using the third condition above and the definition of pattern matrices in Definition 2.4, one can observe that $\overline{C}_{1323}$ contains the matrix $M_{1123} \in C_{1123}$, but $C_{1323}$ and $C_{1123}$ are cells of the same dimension so 1323 and 1123 are unrelated in medium roast order. Since $\overline{C}_v \cap C_w \neq \emptyset$ is a weaker condition than $C_w \subseteq \overline{C}_v$, this suggests a refinement of the medium roast Fubini-Bruhat order, which we will denote by $\preceq$. Note that our notation for $\preceq$ is $\leq'$ in Pawlowski and Rhoades' notation. They use $\preceq$ for the dual order to $\leq$.

**Definition 1.3.** *For $v, w \in \mathcal{W}_{n,k}$, we say $C_v$ **touches** $C_w$ if $\overline{C}_v \cap C_w \neq \emptyset$, denoted $v \rightharpoonup w$.*

Pawlowski and Rhoades observe in [16, Sec. 9] that unlike the medium roast order relations, the touching relation on Fubini words is not transitive. However, they showed that the transitive closure of the touching relations is acyclic [16, Prop. 9.2], so the touching relations give rise to a poset on $\mathcal{W}_{n,k}$ first studied but not named in [16].

**Definition 1.4.** *[16, Sec. 9] The **espresso Fubini-Bruhat order** $(\mathcal{W}_{n,k}, \preceq)$ is defined by taking the transitive closure of the relations of the form $v \rightharpoonup w$ if $v$ touches $w$.*

Observe that for Fubini words $v, w \in \mathcal{W}_{n,k}$, $v \leq w$ implies $v \preceq w$. Thus, the medium roast order is a subposet of the espresso order on the same set of elements.

Pawlowski and Rhoades asked for a combinatorial description of the espresso and medium roast Fubini-Bruhat orders [16, Prob. 9.5]. We address this problem by giving two more sets of defining equations for PR varieties $\overline{C}_w$ inside $X_{n,k}$, see Theorem 1.5 and Theorem 5.4 below. Each set is typically properly contained in the set of all minors that vanish on the PR cell $C_w$, and hence "more efficient".

Let $\Delta_J$ be the **flag minor** associated to columns in $J$ and rows $1, 2, \ldots, |J|$. Such minors are used historically for the Plücker embedding of the flag variety into projective space [8]. Note that the flag minors are invariant under the left action of the unitriangular matrices. Hence, to determine the vanishing/non-vanishing flag minors of $M \in C_w = UP_w$, it suffices to consider the unique $U$-orbit representative of $M$ in $P_w$. We can partition the set of all flag minors on $k \times n$ matrices into the **sometimes, truly, and unvanishing flag minors** for $w$, by defining the indexing sets

$$S_w = \{J \in \binom{[n]}{[k]} \mid \exists\, A, B \in C_w \text{ s.t. } \Delta_J(A) = 0,\ \Delta_J(B) \neq 0\},$$

$$T_w = \{J \in \binom{[n]}{[k]} \mid \Delta_J(M) = 0\ \ \forall\, M \in C_w\}, \text{ and}$$

$$U_w = \{J \in \binom{[n]}{[k]} \mid \Delta_J(M) \neq 0\ \ \forall\, M \in C_w\}.$$

**Theorem 1.5.** *For any Fubini word $w \in \mathcal{W}_{n,k}$, the PR variety $\overline{C}_w$ is the set of spanning line configurations in $X_{n,k}$ represented by matrices such that all flag minors indexed by $T_w$ vanish, so*

$$\overline{C}_w = \{A \in X_{n,k} \mid \Delta_J(A) = 0 \,\forall\, J \in T_w\}.$$

Note, the ideal $J_w$ generated by the flag minors $\{\Delta_J \mid J \in T_w\}$ is in general not the same as $I_w$ generated by all vanishing minors for $C_w$. For example, using the definition and example of $P_w$ in Section 2, one can observe that the minor $\Delta_{\{2\},\{1\}} = x_{21}$ is not in the ideal $J_w$ for $w = 31123$, but it does vanish on all of $C_w$. Note, both $I_w$ and $J_w$ are radical ideals since determinants don't factor, so they determine different affine varieties in $\mathbb{C}^{nk}$, which agree on $X_{n,k}$.

**Corollary 1.6.** *For any two Fubini words $v, w \in \mathcal{W}_{n,k}$, we have*

1. *$v \leq w$ in medium roast Fubini-Bruhat order if and only if $T_v \subseteq T_w$, and*
2. *$v \rightharpoonup w$ if and only if $T_v \subseteq (S_w \cup T_w)$.*

Identifying vanishing flag minors of $C_w$ is more efficient than calculating all vanishing minors of $C_w$, but still cumbersome directly from the definition. In fact, we can characterize the sometimes, truly, and unvanishing flag minors via the Gale partial order on certain multisets $\alpha_J(w)$ defined below. We refer to this as the **Alpha Test**. These tests generalize Ehresmann's Criteria for Bruhat order in $S_n$ using the Gale partial order on multisets denoted $A \trianglelefteq B$. See Section 2 for more details.

**Definition 1.7.** *For any Fubini word $w \in \mathcal{W}_{n,k}$, let $\alpha_i = \alpha_i(w)$ denote the position of the initial $i$ in $w$ for each $i \in [k]$. Call $\alpha(w) = (\alpha_1, \ldots, \alpha_k)$ the **alpha vector** of $w$. We will sometimes drop the $(w)$ when it is clear from context. Observe that when $k = n$, the alpha vector coincides with the notion of $w^{-1} \in S_n = \mathcal{W}_{n,n}$. For $J \subset [n]$, define the multiset*

$$\alpha_J(w) = \{\alpha_{w(j)} \mid j \in J\}. \tag{1.5}$$

**Theorem 1.8.** ***(The Alpha Test)*** *Suppose $w \in \mathcal{W}_{n,k}$ and $J \in \binom{[n]}{[k]}$ with $|J| = h$. Then*

1. *$J \in S_w$ if and only if $\{\alpha_1, \ldots, \alpha_h\} \underset{\neq}{\triangleleft} \alpha_J(w)$,*
2. *$J \in T_w$ if and only if $\{\alpha_1, \ldots, \alpha_h\} \not\trianglelefteq \alpha_J(w)$, and*
3. *$J \in U_w$ if and only if $\{\alpha_1, \ldots, \alpha_h\} = \alpha_J(w)$.*

For example, let $w = 21231231 \in \mathcal{W}_{8,3}$ and $J = \{2, 6, 8\}$. Then $\alpha(w) = (\alpha_1, \alpha_2, \alpha_3) = (2, 1, 4)$, and $\alpha_J = \{\alpha_{w(2)}, \alpha_{w(6)}, \alpha_{w(8)}\} = \{2, 1, 2\}$. Since $\{\alpha_1, \alpha_2, \alpha_3\} = \{1, 2, 4\} \not\trianglelefteq \{1, 2, 2\} = \alpha_J(w)$ in Gale order, we know $J \in T_w$.

**Corollary 1.9.** *Let $v, w \in \mathcal{W}_{n,k}$. Then, $v \leq w$ in medium roast Fubini-Bruhat order if and only if for each $J \in \binom{[n]}{[k]}$ with $|J| = h \leq k$ such that*

$$\{\alpha_1(w), \dots, \alpha_h(w)\} \trianglelefteq \alpha_J(w) \tag{1.6}$$

*we also have*

$$\{\alpha_1(v), \dots, \alpha_h(v)\} \trianglelefteq \alpha_J(v). \tag{1.7}$$

*A similar test for $v \rightharpoonup w$ holds as well based on testing each $J$ such that $\{\alpha_1(w), \dots, \alpha_h(w)\} = \alpha_J(w)$. Therefore, if $v \leq w$ or $v \rightharpoonup w$, we have $\{\alpha_1(v), \dots, \alpha_h(v)\} \trianglelefteq \{\alpha_1(w), \dots, \alpha_h(w)\}$ for all $1 \leq h \leq k$, generalizing the Ehresmann Criterion.*

In Section 2, we briefly review our notation and key concepts from the literature. In Section 3, we indicate some of the lemmas needed to prove Theorem 1.5 and its corollaries. In Section 4, we identify certain families of covering relations and use them to define the decaf Fubini-Bruhat order. We also state an analog of the Lifting Property of Bruhat order. In Section 5, we generalize Fulton's essential set for permutations to Fubini words and show this set gives the unique minimal set of rank conditions defining a PR variety, see Corollary 5.5.

# 2 Background

For a positive integer $n$, let $[n]$ denote the set $\{1, 2, \dots, n\}$. Generalizing the notation for binomial coefficients, we let $\binom{[n]}{k}$ denote all size $k$ subsets of $[n]$ and $\binom{[n]}{[k]} = \bigcup_{h=1}^{k} \binom{[n]}{h}$. The **Gale order** on $\binom{[n]}{k}$ is given by $\{a_1 < \cdots < a_k\} \trianglelefteq \{b_1 < \cdots < b_k\}$ if and only if $a_i \leq b_i$ for all $i \in [k]$ [9]. Gale order can easily be extended to multisets of positive integers of the same size.

Let $S_n$ denote the symmetric group on $[n]$ thought of as bijections $w : [n] \to [n]$. As usual, write a permutation $w$ in **one-line notation** as $w = w_1 \cdots w_n$. Let $t_{ij}$ be the transposition interchanging $i$ and $j$, and let $s_i$ denote the simple transposition interchanging $i$ and $i+1$. The permutation $t_{ij}w$ is obtained from the one-line notation for $w$ by interchanging the values $i$ and $j$, while right multiplication $wt_{ij}$ interchanges the values $w_i$ and $w_j$. The permutation matrix $M_w$ for $w \in S_n$ is the $n \times n$ matrix with a 1 in position $(w_j, j)$ for all $j \in [n]$ and 0's elsewhere. Permutation multiplication agrees with matrix multiplication: $u = vw$ if and only if $M_u = M_v M_w$. Permutation multiplication extends to Fubini words if the corresponding matrices have the correct size.

Schubert varieties $X_w$ for $w \in S_n$ in the flag variety $GL_n / B_+^{(n)}$ are defined via bounded rank conditions on matrices coming from the associated permutation matrices [8]. The **Bruhat order** on $S_n$ is defined by reverse inclusion on Schubert varieties: $v \leq w \iff X_w \subset X_v$. This poset can be characterized as the transitive closure of the relation $w \leq t_{ij}w$

provided $i < j$ and $i$ appears to the left of $j$ in the online notation for $w$ [3]. The covering relations are given by the set of edges $w \leq t_{ij}w$ such that $t_{ij}w$ has exactly one more inversion than $w$. Ehresmann characterized Bruhat order on $S_n$ in terms of Gale order, decades prior to Gale or Bruhat's work, by the **Ehresmann Criterion** [5]

$$v \leq w \iff \{v_1, v_2, \ldots, v_i\} \trianglelefteq \{w_1, w_2, \ldots, w_i\} \; \forall i \in [n]. \tag{2.1}$$

Suppose $v \leq w$ in Bruhat order on $S_n$, $i \in [n-1]$ and $i+1$ precedes $i$ in both $v$ and $w$. Then, the **Lifting Property of Bruhat order** [3, Prop. 2.2.7] implies that $s_i v \leq s_i w$.

**Definition 2.1.** *The **Rothe diagram** of a permutation $w \in S_n$ is the subset of $[n] \times [n]$ in matrix coordinates given by $D(w) = \{(w_j, i) \mid i < j \text{ and } w_i > w_j\}$. Define the **essential set** of $w$, denoted $Ess(w)$, to include all $(i,j) \in D(w)$ such that $(i+1, j), (i, j+1) \notin D(w)$.*

The Rothe diagrams are used extensively in the theory of Schubert varieties. In particular, Fulton showed that the rank conditions coming from the coordinates $(i,j) \in Ess(w)$ determine the unique minimal set of bounded rank equations defining the Schubert variety $X_w$ [7]. Eriksson-Linusson showed that the average size of the essential set is $n^2/36$ for $w \in S_n$ [6].

Much of the notation for permutations defined above has an analog for Fubini words. For $w = w_1 \cdots w_n \in \mathcal{W}_{n,k}$, let $M_w$ be the matrix obtained from the $k \times n$ all zeros matrix by setting the $(w_j, j)$ entry to be 1 for all $j \in [n]$. Note that $M_w$ has exactly one 1 in each column and at least one 1 in each row, but it may have many 1's in any row. Recall from Definition 1.7 that $\alpha_i(w) = \alpha_i$ is the position of the first letter $i$ in $w$ for $i \in [k]$.

**Definition 2.2.** *[16, §3] For a word $w \in \mathcal{W}_{n,k}$, the **initial positions** of $w$ are the set $in(w) = \{\alpha_1, \ldots, \alpha_k\}$. A **redundant position** of $w$ is any position that is not initial. An **initial letter** is a letter appearing in an initial position, and a **redundant letter** is a letter appearing in a redundant position.*

**Definition 2.3.** *[16, §3] For $w \in \mathcal{W}_{n,k}$, the **initial permutation**, $\pi(w) \in S_k$, is obtained from $w$ by deleting the redundant letters from the one-line notation.*

**Definition 2.4.** *[16, §3] For $w = w_1 \cdots w_n \in \mathcal{W}_{n,k}$, the **pattern matrix** $P_w$ is a $k \times n$ matrix with entries 0, 1, or $\star$. Obtain $P_w$ by starting with $M_w$ and replacing the 0 by a $\star$ in each position $(w_i, j)$ such that $i \in in(w)$, $i < \alpha_{w(j)}$, $1 \leq j \leq n$, and either $j \in in(w)$ and $w_i < w_j$, or $j \notin in(w)$.*

*A matrix is said to **fit the pattern of** $w$ if that matrix can be obtained by replacing the $\star$'s in the pattern matrix of $w$ with complex numbers. We will abuse notation and consider $P_w$ both as a $k \times n$ matrix with entries in $\{0, 1, \star\}$ and as the set of all matrices fitting the pattern of $w$.*

**Definition 2.5.** *[16, Eq. (3.6)] The **dimension** of $w \in \mathcal{W}_{n,k}$, denoted $\dim(w)$, is the number $\star$'s in its pattern matrix $P_w$.*

**Example 2.6.** The pattern matrices of $v = 31422$ and $w = 31424$ in $\mathcal{W}_{5,4}$ are

$$P_{31422} = \begin{pmatrix} 0 & 1 & \star & \star & \star \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & \star & 0 & \star \\ 0 & 0 & 1 & 0 & \star \end{pmatrix} \text{ and } P_{31424} = \begin{pmatrix} 0 & 1 & \star & \star & \star \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & \star & 0 & \star \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Therefore, $\dim(31422) = 6$ and $\dim(31424) = 5$.

If $w \in \mathcal{W}_{n,k}$, then the dimension of the PR cell $C_w$ is $\dim(w) + \binom{k}{2}$. The unique largest dimensional cell in $X_{n,k}$ is $C_{123\cdots kk^{n-k}}$ and $\dim(12\cdots kk^{n-k}) = \binom{k}{2} + (n-1)(k-1)$. Hence, $X_{n,k} = \overline{C}_{12\cdots kk^{n-k}}$ has dimension $n(k-1) = 2\binom{k}{2} + (n-1)(k-1)$ and $12\cdots kk^{n-k}$ is the unique minimal element in all three Fubini-Bruhat orders. Since Fubini words are in bijection with ordered set partitions, the dimension generating function gives a natural $q$-analog of the Stirling numbers of the second kind $\sum_{w \in \mathcal{W}_{n,k}} q^{\dim(w)} = [k]!_q \cdot \mathrm{Stir}_q(n,k)$. Reversing the coefficients in this generating function gives (1.4).

# 3 Outlines of Proofs

We outline the proofs of Theorem 1.5 and Theorem 1.8. These statements form the basis from which the covering relations and other Fubini-Bruhat order properties can be proved.

**Lemma 3.1.** *Given $A \in \mathcal{F}_{k\times n}(\mathbb{C})$, the projective coordinates $P(A) = (\Delta_J(A) \mid J \in \binom{[n]}{[k]})$ determine both the unique $w \in \mathcal{W}_{n,k}$ such that $A \in UP_wT^{(n)}$ and $A' \in P_w$ such that $A \in UA'$.*

**Corollary 3.2.** *The set $T_w$ of truly vanishing flag minors on the PR cell $C_w$ determines $w \in \mathcal{W}_{n,k}$, and therefore the rank conditions defining $\overline{C}_w$ as a subset of $X_{n,k}$.*

Corollary 3.2 says there is enough information in the set $T_w$ to recover $w$. To make the relationship between $T_w$ and $\overline{C}_w$ precise, we observe several relations among minors that hold specifically on PR cells and spanning line configurations.

**Lemma 3.3.** *Suppose $w \in \mathcal{W}_{n,k}$ is a Fubini word, $J \subset [n]$, and $1 \leq h \leq k$. Let $\mathrm{rank}_w^{(h)}(J)$ be the largest value $r$ such that there exist subsets $I \subset [h]$ and $J' \subset J$ such that $r = |I| = |J'|$ and $\Delta_{I,J'}(A) \neq 0$ for some $A \in C_w$. The following conditions are equivalent.*

1. *We have $\mathrm{rank}_w^{(h)}(J) < |J|$.*
2. *For every $I \subseteq [h]$ such that $|I| = |J|$, the $(I,J)$-minor vanishes on $C_w$.*
3. *For all subsets $K \in \binom{[n]}{h}$ such that $J \subset K$, we have $K \in T_w$.*

**Corollary 3.4.** *Suppose $w \in \mathcal{W}_{n,k}$ is a Fubini word, $I \subseteq [k]$ and $J \subseteq [n]$ are sets of the same size, and $h = \max(I)$. If the $(I, J)$-minor vanishes on $C_w$, then at least one of the following hold.*

1. *For every $j \in J$, the $(I \setminus \{h\}, J \setminus \{j\})$-minor vanishes on $C_w$.*
2. *For all subsets $K$ such that $J \subseteq K \in \binom{[n]}{h}$, we have $K \in T_w$.*

Corollary 3.4 follows from Lemma 3.3. Theorem 1.5 follows by induction on the number of rows of a minor of $C_w$ using Corollary 3.4, and by Lemma 3.3.

**Lemma 3.5.** *Suppose $w \in \mathcal{W}_{n,k}$ is a Fubini word and $J \in \binom{[n]}{[k]}$ with $h = |J|$. Then, $J \in U_w$ if and only if the submatrix $M_w[[h], J]$ is a permutation matrix.*

**Lemma 3.6.** *Let $w \in \mathcal{W}_{n,k}$, $I \subseteq [k]$ and $J \in \binom{[n]}{[k]}$ such that $|I| = |J|$ and $\Delta_{I,J}(A) = 0$ for all $A$ in the PR cell $C_w$. Then $(H, J)$ indexes a vanishing minor on $C_w$ for any $H$ such that $|H| = |I|$ and $H \leq_L I$ in lex order. In particular, $\Delta_{[|I|],J}$ is a vanishing flag minor on $C_w$, so $J \in T_w$.*

Lemmas 3.5 and 3.6, together with the earlier lemmas can be used to prove Corollary 1.6. Corollary 1.6 and Lemma 3.5 imply Theorem 1.8.

# 4 Covering Relations and the Decaf Order

The following rules describe some families of covering relations for the medium roast and espresso Fubini-Bruhat orders, giving a partial answer to Problem 9.5 in [16]. The Transposition Rule and the Pushback Rule allow us to define the decaf Fubini-Bruhat order, the only ranked Fubini-Bruhat order. We also discuss a generalization of the Lifting Property from Bruhat order.

We start with two observations on covering relations that follow from the definition of medium roast order, pattern matrices, and Corollary 1.6. Let $w = w_1 \cdots w_n \in \mathcal{W}_{n,k}$ with initial permutation $\pi(w) = \pi_1 \cdots \pi_k$.

1. **The Transposition Rule.** For $1 \leq i < j \leq k$, we have $w < t_{ij}w$ in medium roast Fubini-Bruhat order if and only if $\alpha_i(w) < \alpha_j(w)$. In particular, $t_{ij}w$ covers $w$ in medium roast Fubini-Bruhat order if and only if $\pi(t_{ij}w)$ covers $\pi(w)$ in Bruhat order on $S_k$.
2. **The Pushback Rule.** Suppose $w_j = \pi_i$ is a redundant letter in $w$ for $i \in [k-1]$ and $j \in [n]$. Let $v$ be the Fubini word obtained from $w$ by replacing $w_j$ by $\pi_{i+1}$. Then, $w$ covers $v$ in medium roast Fubini-Bruhat order. See Example 2.6 for an example of $v < w$ satisfying the pushback covering relation.

**Definition 4.1.** *The **decaf Fubini-Bruhat order** on $\mathcal{W}_{n,k}$ is the transitive closure of the covering relations given by the Transposition Rule and the Pushback Rule.*

The decaf order has many nice properties. It is the product of Bruhat order for $S_k$ and the poset determined by pushbacks on the subset $\{w \in \mathcal{W}_{n,k} \mid \pi(w) = id\}$. The decaf order is a ranked poset on $\mathcal{W}_{n,k}$, and its rank generating function is the same as the Poincaré polynomial in (1.4). The medium roast and espresso orders are not ranked posets in general. For $n \geq 5$ and most values of $k$, there are covering relations in the medium roast Fubini-Bruhat order $(\mathcal{W}_{n,k}, \leq)$ with a dimension difference of 2 or more, causing the medium roast Fubini-Bruhat order to be unranked in general. For example, in $\mathcal{W}_{5,4}$, 44312 covers 41321, but 44312 has dimension 1, and 41321 has dimension 3.

**Theorem 4.2.** ***The Superpushback Rule.*** *Suppose $w \in \mathcal{W}_{n,k}$, $i \in [k-1]$, and $j \in [n]$ such that $w_j = \pi_i$ is a redundant letter in $w$. If $i + p \leq k$ and $v$ is obtained from $w$ by replacing $w_j$ by $\pi_{i+p}(w)$, then $v \rightharpoonup w$ and this is a covering relation in both espresso and medium roast orders.*

**Theorem 4.3.** ***The Lifting Property.*** *Suppose $v, w \in \mathcal{W}_{n,k}$, $i \in [k-1]$, $\alpha_{i+1}(v) < \alpha_i(v)$, and $\alpha_{i+1}(w) < \alpha_i(w)$. If $v \leq w$ in medium roast Fubini-Bruhat order, then $s_i v \leq s_i w$. Furthermore, if $v \rightharpoonup w$, then $s_i v \rightharpoonup s_i w$.*

# 5 Essential Sets

We extend the notion of a Rothe diagram from Definition 2.1 to Fubini words. This allows us to define the essential set for a Fubini word. We then show the essential set determines a minimal set of rank equations on the corresponding PR variety, generalizing Fulton's essential set for permutations and Schubert varieties [7]. This leads to an essential set characterization of $v \leq w$ in medium roast order.

**Definition 5.1.** *[16] A Fubini word $w \in \mathcal{W}_{n,k}$ is called **convex** if $h < j$ and $w_h = w_j$ implies that $w_i = w_j$ for every $i$ such that $h < i < j$. Then the **convexification** of $w$, denoted by $conv(w)$, is the unique convex word such that $\pi(conv(w)) = \pi(w)$ and the content of $w$ and $conv(w)$ are the same as multisets. The **standardization** of $w$, denoted $std(w) \in S_n$, is obtained by replacing the $n-k$ redundant letters of $w$ with $k+1, k+2, \ldots, n$ from left to right.*

Deduce from Definition 5.1 that two Fubini words $v, w \in \mathcal{W}_{n,k}$ have the same convexification, $\mathrm{conv}(v) = \mathrm{conv}(w)$, if and only if $\pi(v) = \pi(w)$ and they have the same multiset of letters.

**Definition 5.2.** *Given Fubini word $w \in \mathcal{W}_{n,k}$, define the **diagram** of $w$ to be $D(std(conv(w)))$.*

One can observe that $D(\mathrm{std}(\mathrm{conv}(w))) \subset [k] \times [n]$, as none of the bottom $n-k$ rows will contribute any elements to $D(\mathrm{std}(\mathrm{conv}(w)))$. Thus, the diagram of a Fubini word in $\mathcal{W}_{n,k}$ can be drawn as a $k \times n$ grid of dots. For example, the convexification of $w = 44253136541 \in \mathcal{W}_{11,6}$ is 44425533116, and $\mathrm{std}(44425533116) = [4,7,8,2,5,9,3,10,1,11,6]$. So the diagram for $w$ is $D([4,7,8,2,5,9,3,10,1,11,6])$. See Figure 1.

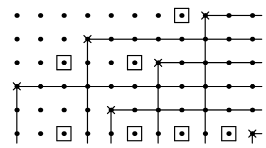

**Figure 1:** Diagram of 44253136541 with cells in the essential set boxed.

In analogy with the alpha vector, define the **beta vector** $\beta(w) = (\beta_1(w), \ldots, \beta_k(w))$ for $w \in W_{n,k}$ by $\beta_i(w) = \beta_i = \{j \in [n] \mid w_j \in \{\pi_1, \ldots, \pi_i\}\}$ where $\pi(w) = (\pi_1, \ldots, \pi_k) \in S_k$ is the initial permutation. Note that $\beta_1 \subset \cdots \subset \beta_k$. For example, if $w = 12123123 \in \mathcal{W}_{8,3}$, we observe $\beta_1 = \{1, 3, 6\}$, $\beta_2 = \{1, 2, 3, 4, 6, 7\}$, and $\beta_3 = [8]$.

Given any Fubini word $w \in \mathcal{W}_{n,k}$, define its **rank function** to be the map $r_w : [k] \times [k] \to \mathbb{Z}_{\geq 0}$ that sends $(h, i)$ to the maximum value of the rank of the submatrix $A[[h], \beta_i]$ over all $A \in C_w$. This function can be determined directly from the Fubini word $w$ as with permutations, but the statement is more complicated so we have omitted it for brevity. From the pattern matrix definition, one can observe that the jumps in the rank functions of matrices in a PR variety are determined by the sets in the beta vector.

**Definition 5.3.** *Given any Fubini word $w \in \mathcal{W}_{n,k}$, define the **ranked essential set** of $w$ to be*

$$Ess^*(w) = \{(h, \beta_i, r) \mid (h, |\beta_i|) \in Ess(std(conv(w))), r = r_w(h, i)\}.$$

**Theorem 5.4.** *A matrix $A \in \mathcal{F}_{k \times n}(\mathbb{C})$ is in the PR variety $\overline{C}_w$ if and only if the rank of the top $h$ rows of $A$ in the columns $\beta_i(w)$ is at most $r$ for each $(h, \beta_i(w), r) \in Ess^*(w)$, and no smaller set of rank conditions will suffice.*

**Corollary 5.5.** *Let $v, w \in \mathcal{W}_{n,k}$. Then $v \leq w$ if and only if for every $(m, \beta_j(v), s) \in Ess^*(v)$, there exists an $(h, \beta_i(w), r) \in Ess^*(w)$ such that $max(0, m - h) + |\beta_j(v) \setminus \beta_i(w)| \leq s - r$.*

Björner-Brenti gave an improvement on the Ehresmann Criterion for Bruhat order on permutations in [2]. Similar improvements on the Alpha Test for medium and espresso orders exist as well. Such improvements also lead to a reduction in the number of equations necessary to define a PR variety. In recent work, Gao-Yong found a minimal number of equations defining a Schubert variety in the flag variety [10]. Thus, it would be interesting to consider the following problem.

**Open Problem 5.6.** *Identify a minimal set of equations defining a PR variety.*

# Acknowledgements

This paper grew out of the Ph.D. thesis of the second author [1, 18]. Further details and more background can be found there. Many thanks to Brendan Pawlowski, Brendan Rhoades, Jordan Weaver, and the referees for insightful suggestions on this project.

# References


[1] S. C. Billey and S. Ryan. "The Fubini-Bruhat Orders". Manuscript. 2024.

[2] A. Björner and F. Brenti. "An improved tableau criterion for Bruhat order". *Electron. J. Combin.* **3**.1 (1996), Research Paper 22. Link.

[3] A. Björner and F. Brenti. *Combinatorics of Coxeter groups*. Vol. 231. Graduate Texts in Mathematics. Springer, New York, 2005, pp. xiv+363.

[4] R. Ehrenborg and M. Readdy. "Juggling and applications to $q$-analogues". *Proceedings of the 6th Conference on Formal Power Series and Algebraic Combinatorics (New Brunswick, NJ, 1994)*. Vol. 157. 1996, pp. 107–125. DOI.

[5] C. Ehresmann. *Sur la topologie de certains espaces homogènes*. NUMDAM, 1934, p. 443. Link.

[6] K. Eriksson and S. Linusson. "Combinatorics of Fulton's essential set". *Duke Math. J.* **85**.1 (1996), pp. 61–76. DOI.

[7] W. Fulton. "Flags, Schubert polynomials, degeneracy loci, and determinantal formulas". *Duke Math. J.* **65**.3 (1992), pp. 381–420. DOI.

[8] W. Fulton. *Young Tableaux*. Vol. 35. London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 1997, pp. x+260.

[9] D. Gale. "Optimal assignments in an ordered set: An application of matroid theory". *J. Combinatorial Theory* **4** (1968), pp. 176–180.

[10] S. Gao and A. Yong. "Minimal equations for matrix Schubert varieties". 2022. arXiv: 2201.06522.

[11] S. T. Griffin. "Ordered set partitions, Garsia-Procesi modules, and rank varieties". *Trans. Amer. Math. Soc.* **374**.4 (2021), pp. 2609–2660. DOI.

[12] J. Haglund, B. Rhoades, and M. Shimozono. "Ordered set partitions, generalized coinvariant algebras, and the Delta conjecture". *Adv. Math.* **329** (2018), pp. 851–915. DOI.

[13] D. Hilbert. "Mathematical problems". *Bull. Amer. Math. Soc.* **8**.10 (1902), pp. 437–479. DOI.

[14] S. L. Kleiman and D. Laksov. "Schubert calculus". *Amer. Math. Monthly* **79** (1972). DOI.

[15] OEIS Foundation Inc. "The On-Line Encyclopedia of Integer Sequences". Online. `http://oeis.org`. 2023.

[16] B. Pawlowski and B. Rhoades. "A flag variety for the Delta conjecture". *Trans. Amer. Math. Soc.* **372**.11 (2019), pp. 8195–8248. DOI.

[17] J. B. Remmel and A. T. Wilson. "An extension of MacMahon's equidistribution theorem to ordered set partitions". *J. Combin. Theory Ser. A* **134** (2015), pp. 242–277. DOI.

[18] Stark Ryan. "Flavors of the Fubini-Bruhat Order". PhD thesis. University of Washington, Seattle, 2022.

[19] E. Steingrímsson. "Statistics on ordered partitions of sets". *J. Comb.* **11**.3 (2020). DOI.