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Wilting Theory of Flow Polytopes

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Abstract. Many important polytopes and their canonical triangulations appear as DKK triangulations of a framed directed acyclic graph (DAG) Γ. These triangulations are combinatorially modelled by cliques of routes on the framed DAG. When Γ is amply framed, the dual graph of its DKK triangulation, or DKK graph, has a lattice structure called the DKK lattice. We study the clique complex of routes which avoid an arbitrary set of "wilted" edges. This leads to various decompositions of the DKK lattice into intervals, generalizing decompositions of the Tamari lattice into ν-Tamari intervals. We further classify the framed DAGs whose DKK graphs may be understood as an interval in the DKK lattice of an amply framed DAG. We realize ν-Tamari lattices and the s-weak order as DKK lattices of such "rooted" DAGs and we extend results about shellability and h^*-polynomials from the amply framed case to the rooted case.

Keywords: flow polytopes, triangulation, ν-Tamari lattice, gentle algebras

1 Introduction

Flow polytopes, which model the space of unit *flows* on a directed acyclic graph (DAG), are a fundamental object of combinatorial optimization and have relations to many fields such as representation theory and algebraic geometry. Danilov, Karzanov, and Koshevoy [5] introduced *framed DAGs* and defined a notion of pairwise compatibility on routes. The complex of *cliques*, or sets of pairwise compatible routes, of a framed DAG Γ serves as a combinatorial model for a (regular unimodular) *DKK triangulation* of the associated flow polytope. Many important classes of polytopes and their canonical triangulations appear in this way, such as associahedra, generalized permutahedra, s-permutahedra, and many order polytopes. We refer to the dual graph of the DKK triangulation as the *DKK Graph* $\mathcal{G}_\Gamma$. An *exceptional route* is one which is in every maximal clique, and Γ is *amply framed* if every edge is in an exceptional route. It was shown in [1] that the clique complex of an amply framed DAG Γ agrees with the support τ-tilting complex of a gentle algebra as described in [3, 7]; in particular, its dual graph has a lattice stucture which we call the *DKK Lattice* $\mathcal{L}_\Gamma$.

In this abstract, we mark a set W of edges of a framed DAG as *wilted* and we study the *lush subgraph* $\mathcal{G}_{(\Gamma,W)}$ of $\mathcal{G}_\Gamma$ of maximal cliques whose nonexceptional routes avoid

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all edges of W. We show that when Γ is amply framed, this gives an interval of the DKK lattice $\mathcal{L}_\Gamma$ which we call the *lush interval* $\mathcal{L}_{(\Gamma,W)}$. We call the wilted framed DAG (Γ, W), or the set W, *viable* if the lush subgraph is nonempty. Our first result provides a complete characterization of the viable edge sets W of a framed DAG G. By choosing a set S of exceptional routes and varying W across all ways to wilt exactly one edge from each route of S, we obtain the *wilted decomposition* of $\mathcal{G}_\Gamma$ by S into lush subgraphs. When Γ is amply framed, this is a decomposition of the DKK lattice into lush intervals. Polyhedrally, we are individually restricting the DKK triangulation of the flow polytope to all codimension-$|S|$ facets which avoid all vertices of exceptional routes in S; taking the cone of these triangulations with these exceptional vertices recovers the original DKK triangulation. As an application, we realize various decompositions of the Tamari lattice, which arises as the DKK lattice of a framed DAG $\mathrm{car}(1^n)$ [9, 2], into ν-Tamari intervals as wilted decompositions.

Next, we use wilting theory to define a new class of framed DAGs which we call *rooted*. Given a rooted DAG Γ, we construct an *ample envelope* (Γ', W') of Γ such that $\mathcal{L}_{(\Gamma',W')} \cong \mathcal{G}_\Gamma$. We thus prove that rooted DAGs are precisely the framed DAGs whose DKK graphs may be understood as intervals in the DKK lattice of an amply framed DAG. As a consequence, we induce a well-defined lattice structure on DKK graphs of rooted DAGs, we prove that clique complexes of rooted DAGs are shellable, and we get a formula for the h^*-vectors of rooted flow polytopes. Rooted DAGs thus inherit many nice properties of amply framed DAGs.

In recent years, the Hasse diagrams of many prominent lattices and their generalizations have been realized as DKK graphs of framed DAGs. In particular, the Hasse diagrams of the ν-Tamari lattice and the s-weak order have been realized as the DKK graphs of $\mathrm{car}(\nu)$ and $\mathrm{oru}(s)$ DAGs. In fact, these framed DAGs are rooted, and our lattice structure realizes their DKK lattices as the ν-Tamari lattice and s-weak order.

We remark that many of our results are phrased more generally for gentle algebras, though for brevity we do not treat this generality in this extended abstract.

2 Background on DAGs and Ample Framings

We start by recalling some background on flow polytopes and amply framed DAGs. Let $G = (V, E)$ be a finite directed acyclic graph (DAG) with vertex set V and edge set E. For each $v \in V$, let $\mathrm{in}(v)$ and $\mathrm{out}(v)$ denote the set incoming and outgoing edges of v, respectively. A vertex v is called a *source* if $\mathrm{in}(v) = \emptyset$, a *sink* if $\mathrm{out}(v) = \emptyset$, and *internal* otherwise. An edge $\alpha \in E$ is directed from its *tail* $t(\alpha)$ to its *head* $h(\alpha)$. The edge α is *internal* if it is between two internal vertices, and otherwise it is a *source edge* and/or a *sink edge*. A *route* of G is a maximal (directed) path in G.

Definition 2.1. A *flow* f on a DAG G is a function $f : E \to \mathbb{R}$ which preserves flow at each internal vertex, i.e., for every internal vertex v we have $\sum_{e\in \text{in}(v)} f(e) = \sum_{e\in \text{out}(v)} f(e)$. The *flow polytope* $\mathcal{F}_1(G)$ is the space of *unit flows* on G; i.e., flows satisfying $x_e \geq 0$ for all edges $e \in E$ and $\sum_{\substack{v \text{ is a source} \\ e\in \text{out}(v)}} f(e) = 1$.

The dimension of $\mathcal{F}_1(G)$ is $\dim(\mathcal{F}_1) = |E| - \#\{v \in V \ : \ v \text{ is an inner vertex}\} - 1$. The vertices of $\mathcal{F}_1(G)$ are precisely the indicator vectors of routes of G.

Definition 2.2. Let $G = (V, E)$ be a DAG. For each internal vertex v of G, assign a linear order to the edges in $\text{in}(v)$ and assign a linear order to the edges in $\text{out}(v)$. This assignment is called a *framing* of G, which we denote by F. We use the symbol Γ to refer to a *framed DAG* (G, F). If e is less than f in the linear order for F on $\text{in}(v)$, we write $e <_{F,\text{in}(v)} f$ (and similarly for $\text{out}(v)$). We may drop one or both subscripts when clear.

In the following, assume $\Gamma = (G, F)$ is a framed DAG. To denote a framing, we often label the half-edges or edges of a DAG with integers. See Figure 1 and Figure 2 for examples. An edge of a framed DAG Γ is *tail-highest* (respectively *tail-lowest*) if α is the greatest (respectively least) element in the partial order on $\text{out}(t(\alpha))$. An edge which is neither tail-highest nor tail-lowest is *tail-middle*. Similarly, an edge may be *head-highest*, *head-lowest*, or *head-middle*. An edge which is both tail-highest and head-highest is called *highest*. We similarly may call edges *middle* or *lowest*. An edge is *steep* if it is head-highest and tail-lowest, or head-lowest and tail-highest.

Definition 2.3. A path p of Γ is *up-incompatible* to a path q if p contains $\alpha_1 R \alpha_2$ and q contains $\beta_1 R \beta_2$, for some path R and some edges α_i, β_i with $\alpha_1 >_{\text{in}(v)} \beta_1$ and $\alpha_2 <_{\text{out}(w)} \beta_2$. Two paths are *incompatible* if one is up-incompatible to the other. Otherwise, they are *compatible*. If a route p in Γ is compatible with every other route in Γ, we say that p is *exceptional*. A *clique* is a set of pairwise-compatible routes in Γ.

For example, in Figure 1, the route 121 and the route 211 are incompatible, as they share the first internal vertex but 121 enters this vertex with a higher edge and leaves with a lower edge compared to 211.

It follows that a route p of a framed DAG Γ is exceptional if and only if either every edge is highest, every edge is lowest, or p consists of a single edge. Note that an exceptional route is a route which is in every maximal clique. The *clique complex* $\mathcal{K}_\Gamma$ of Γ is the simplicial complex of cliques of Γ.

An edge α of Γ is an *idle edge* if $\text{in}(h(\alpha)) = 1$ and $h(\alpha)$ is internal, or $\text{out}(t(\alpha)) = 1$ and $t(\alpha)$ is internal. Idle edges may be contracted to obtain a new framed DAG whose clique complex and DKK graph agree with the original. Hence, we may safely assume that our DAGs have no idle edges. Γ is *amply framed* if every edge is contained in some exceptional route. In [1], it was shown that a framed DAG Γ with no idle edges is amply framed if and only if (1) Γ is full (i.e., for any internal vertex v of Γ, we have $|\text{in}(v)| = 2 = |\text{out}(v)|$), and (2) there is a map $\phi_F : E \to \{1, 2\}$ realizing the framing F (i.e., there are no steep edges in F).

2.1 Flow Polytopes and DKK Triangulations

Recall that vertices of the flow polytope $\mathcal{F}_1(\Gamma)$ are indicator vectors of routes of Γ. Through this correspondence, we may view maximal cliques of Γ as collections of vertices of $\mathcal{F}_1(G)$ which form a simplex of a regular unimodular triangulation:

Theorem 2.4 ([5]). *Let Γ be a framed DAG. The set of maximal cliques of Γ forms a regular unimodular triangulation of the flow polytope $\mathcal{F}_1(G)$.*

See Figure 1, where the top clique corresponds to the simplex whose vertices are given by (the indicator vectors of) the routes $\{111, 211, 221, 222\}$ appearing in the clique (the exceptional routes 111 and 222 are not drawn for readability). The triangulation from Theorem 2.4 is called the *DKK triangulation* of Γ. We will be particularly interested in the dual graph of a DKK triangulation (equivalently, the dual graph of the clique complex), which we refer to as the *DKK graph* $\mathcal{G}_\Gamma$.

When Γ is amply framed, $\mathcal{G}_\Gamma$ may be interpreted as the Hasse diagram of a lattice (whose lattice structure is inherited from the τ-tilting theory of an associated gentle algebra [1]), which we call the *DKK lattice* $\mathcal{L}_\Gamma$. This lattice structure may be described as follows using directional compatibility (see Figure 1 and Figure 5 for examples).

Definition 2.5 ([1, Definition 6.1]). Let $M_1 = M \cup \{p\}$ and $M_2 = M \cup \{q\}$ be adjacent maximal cliques in $\mathcal{G}_\Gamma$ for Γ amply framed. Then, without loss of generality, p is up-incompatible to q and q is not up-incompatible to p. In this case, $M_1 > M_2$ in $\mathcal{L}_\Gamma$.

3 Wilted Framed DAGs

We now *wilt* a set W of edges of a framed DAG and consider the maximal cliques whose nonexceptional routes avoid the wilted edges. In the amply framed case, these cliques form a *lush interval* in the DKK lattice $\mathcal{L}_\Gamma$. We characterize the sets of wilted edges giving nonempty lush intervals and we give a recipe to obtain canonical decompositions of $\mathcal{L}_\Gamma$ into lush intervals.

Definition 3.1. A *wilted framed DAG* (Γ, W) is a framed DAG $\Gamma = (G, F)$ along with a set W of edges of G considered as *wilted*. We say that a route of Γ is *wilted* if it contains an edge of W. Otherwise, it is *lush*. A clique is *wilted* if it contains a wilted nonexceptional route, and is otherwise *lush*. Let S be the set of exceptional routes containing an edge of W; then the *lush clique complex* $\mathcal{K}_{(\Gamma,W)}$ is the pure simplicial complex whose maximal simplices are of the form $M \backslash S$, for any lush maximal clique M. The *lush subgraph* $\mathcal{G}_{(\Gamma,W)}$ is the dual graph of $\mathcal{K}_{(\Gamma,W)}$. We say that (Γ, W) is *viable* if $\mathcal{G}_{(\Gamma,W)}$ is nonempty.

Remark 3.2. Let (Γ, W) be a viable wilted framed DAG. Let $\text{del}(\Gamma, W)$ be the framed DAG obtained by deleting all edges of W from Γ. There is a natural bijection between lush routes of (Γ, W) and routes of $\text{del}(\Gamma, W)$ which induces a bijection $\mathcal{K}_{(\Gamma,W)} \cong \mathcal{K}_{\text{del}(\Gamma,W)}$. Hence, the lush subgraph $\mathcal{G}_{(\Gamma,W)}$ is isomorphic to the DKK graph $\mathcal{G}_{\text{del}(\Gamma,W)}$.

The following result is proven representation-theoretically by considering the τ-tilting lattice of the associated gentle algebra of an amply framed DAG.

Proposition 3.3. *Let (Γ, W) be a wilted amply framed DAG. The lush subgraph $\mathcal{G}_{(\Gamma,W)}$ forms an interval in $\mathcal{L}_{(\Gamma,W)}$. We call this the* lush interval $\mathcal{L}_{(\Gamma,W)}$.

We now characterize the sets of edges which produce viable wilted framed DAGs.

Theorem 3.4. *Let (Γ, W) be a wilted framed DAG. Then (Γ, W) is viable if and only if*

1. *each edge of W is contained in an exceptional route, no exceptional route contains more than one edge of W, and*
2. *every internal vertex has an incoming and outgoing lush edge.*

3.1 Wilted Decompositions of Framed DAGs and Flow Polytopes

We define the wilted decomposition of a framed DAG with respect to a set of exceptional routes and provide an interpretation in terms of flow polytopes.

Definition 3.5. Let S be a subset of the set of exceptional routes of Γ. Define

$$\mathcal{W}_S := \{W \subseteq E \mid W \text{ consists of exactly one edge from each route of } S\}.$$

For each element W of $\mathcal{W}_S$, we obtain a wilted framed DAG (Γ, W) and a (possibly empty) lush subgraph $\mathcal{G}_{(\Gamma,W)}$. Each maximal clique M of Γ is contained in exactly one such lush subgraph $\mathcal{G}_{(\Gamma,W_M)}$. Hence, the set S gives a *wilted decomposition* of $\mathcal{G}_\Gamma$ into lush subgraphs. Conversely, given a viable wilted framed DAG (Γ, W), we may let S_W be the set of exceptional routes containing an edge of W; then $\mathcal{G}_{(\Gamma,W)}$ appears in the wilted decomposition of Γ by S_W.

When Γ is amply framed, $\mathcal{G}_\Gamma$ has a lattice structure $\mathcal{L}_\Lambda$ and Proposition 3.3 shows that each lush subgraph $\mathcal{G}_{(\Lambda,W)}$ is actually an interval $\mathcal{L}_{(\Lambda,W)} \subseteq \mathcal{L}_\Lambda$. In the future, we will see that this situation holds more generally for *rooted* DAGs. See Figure 1 or Figure 5 for an example of a decomposition of $\mathcal{L}_\Lambda$ into intervals for an amply framed DAG Γ.

Proposition 3.6. *Let Γ be a framed DAG and let S be a set of exceptional routes of Γ. The nonzero flow polytopes $\{\mathcal{F}_1(\text{del}(\Gamma, W)) : W \in \mathcal{W}_S\}$ are precisely the codimension-$|S|$ faces of $\mathcal{F}_1(\Gamma)$ containing none of the vertices given by exceptional routes in S.*

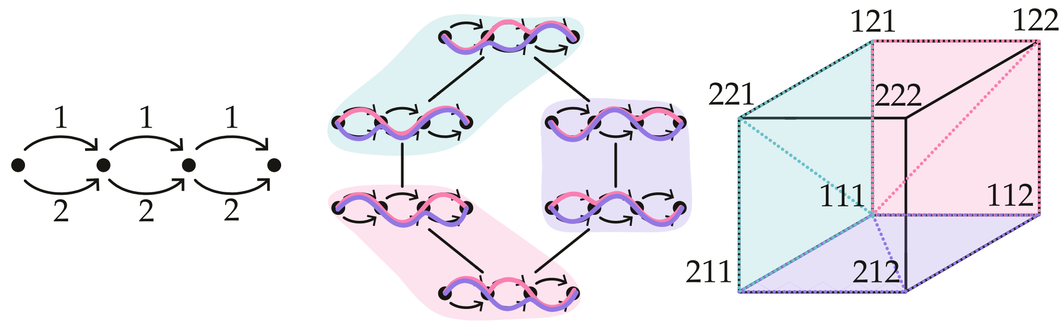

Figure 1: Shown is an amply framed DAG, the wilted decomposition of its DKK lattice by the route 222 (with no exceptional routes drawn for readability), and its flow polytope.

If S is a set of exceptional routes of Γ, then for any $W \in \mathcal{W}_S$, Proposition 3.6 shows that the lush subgraph $\mathcal{G}_{(\Gamma,W)} \subseteq \mathcal{G}_\Gamma$ is the dual graph the DKK triangulation of the codimension-$|S|$ face $\mathcal{F}_1(\text{del}(\Gamma, W))$ of $\mathcal{F}_1(\Gamma)$. By taking the DKK triangulations $\mathcal{F}_1(\text{del}(\Gamma, W))$ for all $W \in \mathcal{W}_S$ and adding the vertices corresponding to exceptional routes of S to all simplices, we recover the original DKK triangulation of $\mathcal{F}_1(\Gamma)$.

Example 3.7. Shown in Figure 1 is an amply framed DAG Γ and its flow polytope $\mathcal{F}_1(\Gamma)$, which is a cube. The vertex labelled 121, for example, corresponds to the route which first takes a 1-edge, then a 2-edge, then a 1-edge. Let $S = \{222\}$ consist only of the 2-route of Γ. The wilted decomposition of Γ by S separates $\mathcal{L}_\Gamma$ into three intervals, highlighted in different colors in the middle of Figure 1, based on which 2-edge is avoided by the nonexceptional routes. By Proposition 3.6, deleting any 2-edge of Γ restricts the triangulation to one of the three facets of $\mathcal{F}_1(\Gamma)$ not incident to the vertex corresponding to 222, which are highlighted in Figure 1. For example, wilting the sink 2-edge yields the lush interval highlighted in blue and deleting it yields the back face of the cube with vertices $\{111, 121, 211, 221\}$, highlighted in blue, with the dotted DKK triangulation. Taking the cone of these three separate DKK triangulations with the vertex corresponding to 222 gives the DKK triangulation of Γ.

4 Rooted Framed DAGs

In this section, we define a new class of framed DAGs which we call rooted. Given a rooted DAG, we obtain a wilted amply framed DAG (Γ', W') whose lush clique complex $\mathcal{K}_{(\Gamma',W')}$ is isomorphic to $\mathcal{K}_\Gamma$. As a consequence, we give a lattice structure to $\mathcal{G}_\Gamma$ and extend results about shellability and h^*-polynomials from the amply framed case.

Definition 4.1. An *exceptional segment* of a framed DAG Γ (with no idle edges) is a

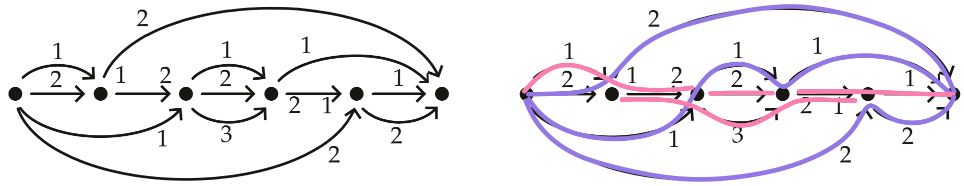

Figure 2: A framed DAG and its exceptional segments (exceptional routes in purple).

maximal path p of Γ which is compatible with every other path. An exceptional segment is *rooted* if it starts at a source vertex or ends at a sink vertex (or both). A framed DAG Γ is *rooted* if every exceptional segment of Γ is rooted.

An exceptional route is an exceptional segment which starts at a source vertex and ends at a sink vertex. Any middle edge makes up its own exceptional segment. When Γ has no idle edges, any steep edge is a part of exactly two exceptional segments, and any non-steep edge is a part of exactly one exceptional segment. See Figure 2.

Lemma 4.2. *Given a viable wilted amply framed DAG* (Γ, W), *the framed DAG* $\mathrm{del}(\Gamma, W)$ *obtained by deleting all edges of* W *from* Γ *is rooted.*

Proof. Exceptional segments of $\mathrm{del}(\Gamma, W)$ correspond to maximal lush segments of exceptional routes of (Γ, W). Theorem 3.4 shows that any exceptional route of Γ contains at most one edge of W, ensuring that each exceptional segment of $\mathrm{del}(\Gamma, W)$ either starts at a source or ends at a sink. □

We now focus on showing the converse of Lemma 4.2. More concretely, given a rooted framed DAG Γ, we wish to obtain an amply framed DAG (Γ', W') such that $\mathcal{K}_\Gamma \cong \mathcal{K}_{(\Gamma', W')}$. If a framed DAG Γ is not amply framed, then either Γ is not full (i.e., there is an internal vertex of Γ with in-degree or out-degree greater than 2), or Γ has a steep edge. We will define operations which fix these issues while preserving the lush DKK graph. We first define an operation which pulls a framed DAG closer to being full.

Definition 4.3. Let α be a tail-middle edge of a viable wilted framed DAG (Γ, W). In particular, it is necessary that $h(\alpha)$ has an in-degree greater than 2. By Theorem 3.4, α is lush. The *wilted 2-decontraction of* (Γ, W) *with respect to* α is the wilted framed DAG (Γ', W') whose vertex set is given by $V' := \{v' \ : \ v \in V\} \cup \{v_\alpha\}$ and whose edges are described as follows. For any edge $\beta : i \to j$ of Γ, there is an edge $\beta' : i' \to j'$ (if $i \neq t(\alpha)$ or if $i = t(\alpha)$ and $\beta <_{\mathrm{out}(t(\alpha))} \alpha$) or $\beta' : v_\alpha \to j'$ (else). There is an additional *connecting edge* $\delta : v' \to v_\alpha$ and there is a wilted *dummy edge* $\epsilon : v_{\mathrm{source}} \to v_\alpha$. The framing of Γ' is inherited from the framing on Γ, with the stipulation that the connecting edge δ is highest and the dummy edge ϵ is lowest. Performing a wilted 2-decontraction to the left DAG of Figure 3 at its unique tail-middle edge results in the middle DAG of Figure 3.

Note that α' is tail-lowest in Γ' and that deleting the dummy edge ϵ and contracting Γ' along the connecting edge δ recovers (Γ, W). A *wilted 1-decontraction* with respect to an edge α which is tail-middle is obtained by reversing all partial orders of the framing F, performing a wilted 2-decontraction, and reversing the partial orders again. Dually, we may define *wilted 1-decontractions* and *wilted 2-decontractions* with respect to an edge which is head-middle. Given an edge α of (Γ, W) which is head-middle or tail-middle, any wilted decontraction of α which does not create a steep edge preserves the lush clique complex. This gives us the following lemma.

Lemma 4.4. *Let α be a head-middle or tail-middle edge of a viable wilted framed DAG (Γ, W). There exists a wilted decontraction (Γ', W') of (Γ, W) with respect to α such that (Γ', W') is viable and $\mathcal{K}_{(\Gamma',W')} \cong \mathcal{K}_{(\Gamma,W)}$.*

If Γ is a rooted framed DAG with no idle edges which is not full, then it must have an edge which is head-middle or tail-middle. Then we may repeatedly apply Lemma 4.4 to obtain a full wilted DAG (Γ', W') whose lush clique complex agrees with that of Γ. The framed DAG Γ' may not be amply framed, since it may have steep edges. We now define an operation to fix this.

Definition 4.5. Let (Γ, W) be a wilted framed full DAG. Let α be an edge of Γ which is steep. Without loss of generality, suppose α is tail-highest and head-lowest; the other case is similar. We define the *amplification of* (Γ, W) *with respect to* α as the wilted framed DAG (Γ', W') as follows. See the middle and right of Figure 3 for an example. The vertex set of Γ' consists of the vertices of Γ as well as an additional vertex v_α. For any edge β of Γ other than α, there is a corresponding edge β' of Γ'. Replacing α in Γ' is an edge α'_1 from $t(\alpha)$ to v_α which is highest in F, and an edge α'_2 from v_α to $h(\alpha)$ which is lowest in F. Additionally, there is a highest wilted edge γ from a source to v_α and a lowest wilted edge β from v_α to a sink.

Lemma 4.6. *If α is a steep edge of a full viable wilted framed DAG (Γ, W), then the amplification (Γ', W') of (Γ, W) with respect to α is viable and $\mathcal{K}_{(\Gamma',W')} \cong \mathcal{K}_{(\Gamma,W)}$.*

Theorem 4.7. *If Γ is a rooted framed DAG, then there is a wilted amply framed DAG (Γ', W') such that $\mathcal{K}_\Gamma \cong \mathcal{K}_{(\Gamma',W')}$.*

Proof. We may suppose that Γ is rooted and has no idle edges. First, we repeatedly apply Lemma 4.4 until we have reached (Γ'', W''), where Γ'' is full, and then we repeatedly apply Lemma 4.6 to fix the steep edges, resulting in an amply framed (Γ', W') with $\mathcal{K}_\Gamma \cong \mathcal{K}_{(\Gamma',W')}$. For any exceptional segment p of Λ, there is an exceptional route p' of Λ' that begins with a wilted edge if and only if p begins with an internal vertex and ends with a wilted edge if and only if p ends with an internal vertex. Hence, the condition that Γ is rooted corresponds to the condition that (Γ', W') is viable by Theorem 3.4. See Figure 3 for an example of this process. □

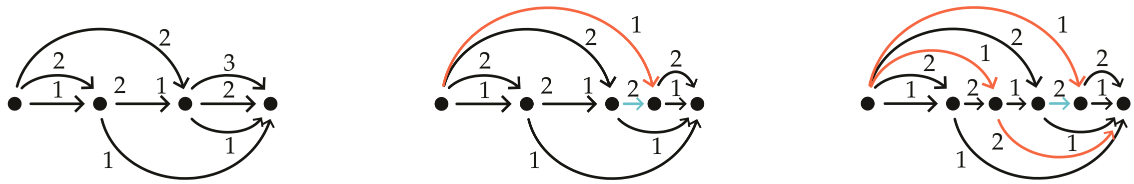

Figure 3: A framed DAG (left), a wilted 2-decontraction with respect to a its tail-middle edge (middle), and an amplification of the resulting DAG at its steep edge (right). Wilted edges are red and the connecting edge is blue.

We call (Γ', W') as in the statement of Theorem 4.7 an *ample envelope* of Γ. A consequence of the existence of ample envelopes is that the DKK graph of a rooted framed DAG has a lattice structure generalizing the amply framed case [1].

Definition 4.8. Let Γ be a rooted framed DAG. Let $M_1 = M \cup \{p\}$ and $M_2 = M \cup \{q\}$ be adjacent maximal cliques in $\mathcal{G}_\Gamma$. Then, without loss of generality, p is up-incompatible to q and q is not up-incompatible to p. In this case, we say that $M_1 > M_2$.

Corollary 4.9. *The transitive closure of the relations of Definition 4.8 gives $\mathcal{G}_\Gamma$ the structure of the Hasse diagram of a lattice, which we refer to as the* DKK lattice $\mathcal{L}_\Gamma$.

Corollary 4.9 is proven by inheriting the lattice structure of an ample envelope. Moreover, any lush subgraph of a rooted DAG may be considered as an interval in $\mathcal{L}_\Gamma$. Hence, the wilted decomposition of a rooted DAG by a set of exceptional routes (Definition 3.5) is a decomposition of $\mathcal{L}_\Gamma$ into lush intervals. The next corollary, which follows from Theorem 4.7 and Lemma 4.2, characterizes rooted DAGs as those whose DKK graphs may be understood as lush intervals of amply framed DAGs .

Corollary 4.10. *A nonempty lattice is of the form $\mathcal{L}_{(\Gamma,W)}$, where (Γ, W) is a wilted amply framed DAG, if and only if it is of the form $\mathcal{L}_{\Gamma'}$, where Γ' is a rooted framed DAG.*

It was shown in [1] that if Γ is amply framed then any linear extension of $\mathcal{L}_\Gamma$ is a shelling order for $\mathcal{K}_\Gamma$. By realizing the DKK graph of a rooted DAG as an interval in the DKK lattice of an ample envelope, we prove the following.

Theorem 4.11. *Let Γ be a rooted framed DAG. Then any linear extension of $\mathcal{L}_\Gamma$ gives a shelling order of the lush clique complex $\mathcal{K}_\Gamma$.*

Following [1, §6], we get a formula for the h^*-polynomials of flow polytopes arising from rooted framed DAGs.

Proposition 4.12. *Let Γ be a rooted DAG. The ith coefficient of the h^*-vector of $\mathcal{F}_1(\Gamma)$ is given by the number of elements in $\mathcal{L}_\Gamma$ covering exactly i elements.*

5 Motivating Example: The (ν-)Tamari Lattice

The Tamari lattice is a mathematical structure that captures the partial order of binary trees under a rotation operation. Préville-Ratelle and Viennot [8] introduced ν-Tamari lattices as a generalization of Tamari lattices and showed that the Tamari lattice has a decomposition into ν-Tamari intervals. In [4] the ν-Tamari lattice was realized as the one-skeleton of the polyhedral complex known as the ν-associahedron, and in [2] it was shown that ν-Tamari lattices arise as DKK graphs of a class of DAGs known as ν-caracol graphs. In this section, we interpret certain wilted decompositions on caracol graphs as decompositions of Tamari lattices into ν-Tamari intervals. Moreover, given a ν-Tamari lattice $\mathcal{L}_{\mathrm{car}(\nu)}$, we obtain a canonical Tamari lattice $\mathcal{L}_{\mathrm{car}(1^n)}$, which is the DKK lattice of a framed DAG $\mathrm{car}(1^n)$, and a set S of exceptional routes of $\mathrm{car}(1^n)$ such that the lattice $\mathcal{L}_{\mathrm{car}(\nu)}$ appears in the wilted decomposition of $\mathcal{L}_{\mathrm{car}(1^n)}$ by S.

Definition 5.1. Let a, b be nonnegative integers, and let $\nu := NE^{\nu_1}NE^{\nu_2}\dots NE^{\nu_a}$ be a lattice path from $(0,0)$ to (b,a) (with $\nu_i \geq 0$). The *ν-caracol graph* $\mathrm{car}(\nu)$ is the graph on the vertex set $\{0,1,\dots,a\}$, together with ν_i copies of the edge $(0,i)$ for $i = 1,\dots,a-1$, one copy of the edge (i,a) for $i = 1,\dots,a-1$, and the edges $(i,i+1)$ for $i = 0,\dots,a-1$. Give $\mathrm{car}(\nu)$ the framing such that the horizontal edges from i to $i+1$ are given the highest element in the vertex order on either side. See Figure 4 for an example. In the classical case, $\nu = 1^n = (1,\dots,1)$ for some n and the DKK lattice $\mathcal{L}_{\mathrm{car}(1^n)}$ is Tamari.

It was shown in [2, Theorem 1.2] that $\mathcal{G}_{\mathrm{car}(\nu)}$ is the Hasse diagram of the ν-Tamari lattice. In fact, the framed DAGs $\mathcal{G}_{\mathrm{car}(\nu)}$ are rooted, and our lattice structure realizes $\mathcal{L}_{\mathrm{car}(\nu)}$ as the ν-Tamari lattice. See Figure 5.

Theorem 5.2. *Let $V \subseteq [n]$. Then the wilted decomposition of* $\mathrm{car}(1^n)$ *by the set of 1-labelled routes whose internal vertices are in V is a decomposition of the Tamari lattice into ν-Tamari intervals. Any ν-Tamari lattice appears in such a decomposition, for some n and V.*

Proof. If W is a viable set of 1-edges of $\mathrm{car}(1^n)$, then deleting them and contracting yields some $\mathrm{car}(\nu)$. Conversely, it may be seen that any DAG $\mathrm{car}(\nu)$ has an ample envelope $(\mathrm{car}(1^n), W)$ where W consists of 1-edges. See the left and middle of Figure 5. □

Example 5.3. The left of Figure 5 shows the decomposition of the Tamari lattice $\mathrm{car}(T_3)$ into ν-Tamari intervals given by the first and last 1-labelled routes. The lush interval in red is the lush interval of the wilted framed DAG $\mathrm{car}(T_3)$ in the middle of Figure 4, which is the DKK graph of the $\mathrm{car}(\nu)$ DAG on the left of Figure 4 by Proposition 3.6.

The right of Figure 5 shows the wilted decomposition by the set of all 1-labelled routes, which recovers the partition introduced in [8]. This induces a wilted decomposition of $\mathcal{L}_{\mathrm{car}(\nu)}$ into two chains.

We end the extended abstract with some open questions:

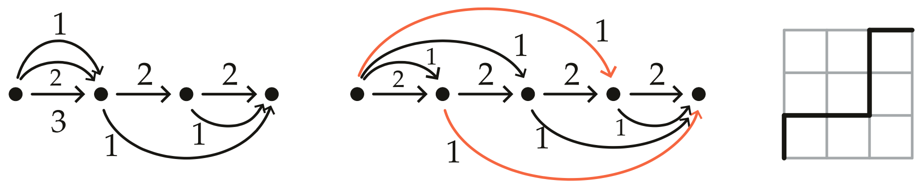

Figure 4: A ν-Caracol graph where $\nu = (0, 2, 0, 1)$, its realization as a wilted car(T_3) graph, and the corresponding lattice path.

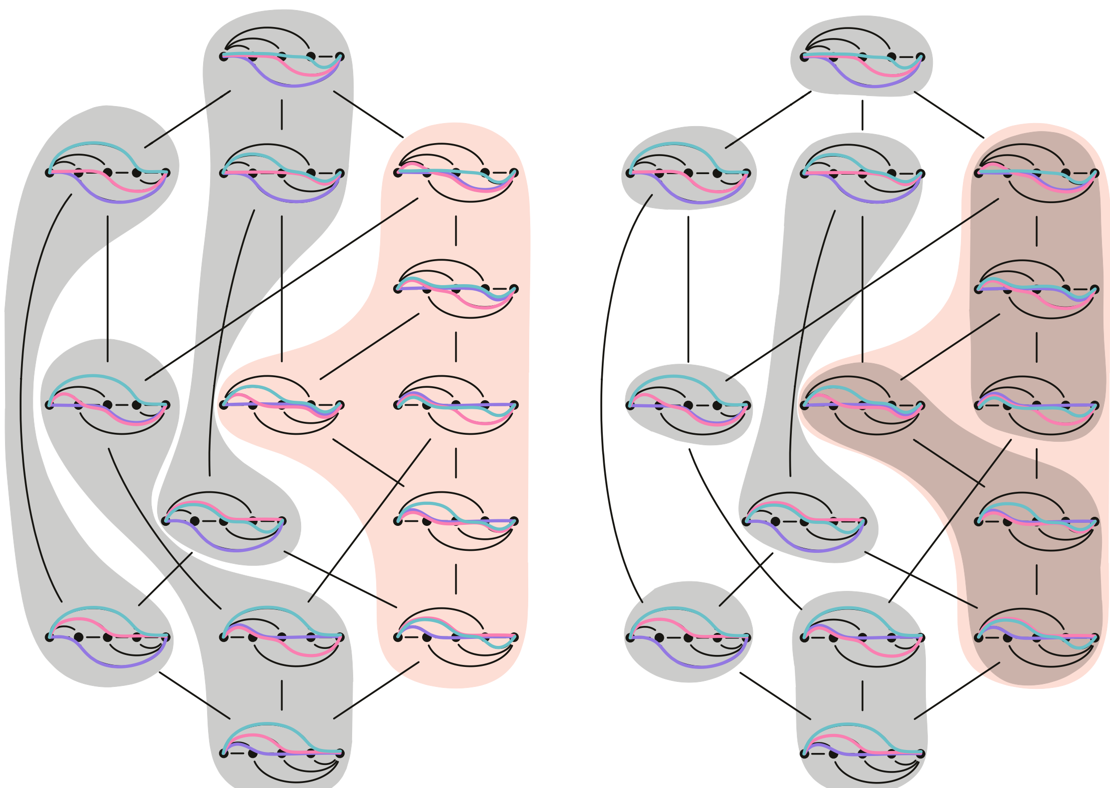

Figure 5: Shown is the wilted decomposition of the (Tamari) lattice $\mathcal{L}_{\text{car}(1^n)}$, shown in the middle of Figure 4, induced by the first and third 1-routes (left) and by the set of all 1-routes (right). For readability, exceptional routes are not drawn.

1. Rooted framed DAGs inherit nice properties from amply framed DAGs. Examples of rooted framed DAGs include ν-caracol and s-oruga [6] graphs, whose DKK lattices are the ν-Tamari lattices and the s-weak order. What other lattices may be realized as DKK lattices of rooted framed DAGs?

2. The notable class of ν-Tamari lattices may be defined as the lush intervals of $\mathrm{car}(1^n)$ in its decomposition by some set of 1-routes. Can we realize other interesting lattice decompositions using wilting theory?

3. The DKK theory of rooted framed DAGs is in some sense equivalent to the wilting theory of viable wilted amply framed DAGs. What can be gained from studying amply framed DAGs with sets of wilted edges which are *not* viable? In particular, can we realize the DKK graph of an arbitrary framed DAG as an interval in the DKK lattice of an amply framed DAG?

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